Math 535 - General Topology
Fall 2012
Homework 5, Lecture 9/28

Problem 5. (Munkres Exercise 27.2) Let X be a metric space and $A \subseteq X$ a subset. Define the **ϵ-neighborhood** of A as the set

$$B_\epsilon(A) := \{x \in X \mid d(x, A) < \epsilon\}.$$

a. Show that the ϵ-neighborhood of A is

$$B_\epsilon(A) = \bigcup_{a \in A} B_\epsilon(a)$$

i.e. the union of all open balls of radius ϵ around points $a \in A$.

b. Assume A is *compact* and let $U \subseteq X$ be an open set containing A. Show that some ϵ-neighborhood of A is contained in U, i.e. $B_\epsilon(A) \subseteq U$ for some $\epsilon > 0$.

Problem 6. In all parts of this problem, let $f \colon X \to Y$ be a *uniformly* continuous map between *metric* spaces.

a. Show that f sends Cauchy sequences to Cauchy sequences. In other words, if $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in X, show that $(f(x_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in Y.

b. Assuming moreover that f is a homeomorphism and Y is complete, show that X is complete.

c. Find an example where f is a homeomorphism and X is complete, but Y is *not* complete. (Don't forget to show that your example f is uniformly continuous.)

Remark. Part (b) implies that if two metric spaces are uniformly equivalent, then one is complete if and only if the other is complete. In other words, completeness depends on more than just the topology, but at most on the uniform type.