Math 535 - General Topology
Fall 2012
Homework 6, Lecture 10/1

Problem 1. Let $\mathbb{F}$ be the field $\mathbb{R}$ or $\mathbb{C}$ of real or complex numbers. Let $n \geq 1$ and denote by $\mathbb{F}[x_1, x_2, \ldots, x_n]$ the set of all polynomials in n variables with coefficients in $\mathbb{F}$.

A subset $C \subseteq \mathbb{F}^n$ of n-dimensional space will be called **Zariski closed** if it is the zero locus of some polynomials:

$$C = V(S) := \{x \in \mathbb{F}^n \mid f(x) = 0 \text{ for all } f \in S\}$$

for some $S \subseteq \mathbb{F}[x_1, \ldots, x_n]$.

Note: The zero locus $V(S)$ is sometimes called the *algebraic variety* associated to S, hence the letter V.

For example, in $\mathbb{R}^2$, the subset $V(x_1^2 + x_2^2 - 9) \subset \mathbb{R}^2$ is the circle of radius 3 centered at the origin, which is therefore a Zariski closed subset.

By convention, let's say S is not allowed to be empty, though you will show in part (a) that it doesn't matter.

a. Show that the notion of "Zariski closed" subset does define a topology on $\mathbb{F}^n$, sometimes called the **Zariski topology**.

b. Show that the Zariski topology is *strictly* coarser (i.e. smaller) and the usual metric topology on $\mathbb{F}^n$.

c. Show that the Zariski topology on $\mathbb{F}^n$ is T_1.

d. Show that the Zariski topology on $\mathbb{F}^n$ is not T_2, i.e. not Hausdorff.

e. In the one-dimensional case $n = 1$, show that the Zariski topology on $\mathbb{F}$ is the cofinite topology.