Math 535 - General Topology
Fall 2012
Homework 6, Lecture 10/5

Problem 4. In this problem, we will show that a countable product of metrizable spaces is metrizable.

a. Let (X, d) be a metric space. Consider the function $\rho\colon X \times X \to \mathbb{R}$ defined by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that ρ is a metric on X.

b. Show that the metric ρ from part (a) induces the same topology on X as the original metric d.

Remark. We could also have used the formula $\rho(x, y) = \min\{d(x, y), 1\}$. The goal was just to find a metric ρ which is topologically equivalent to d and is bounded.

c. Let $\{(X_i, d_i)\}_{i\in\mathbb{N}}$ be a countable family of metric spaces, where each metric d_i is bounded by 1, i.e.

$$d_i(x_i, y_i) \leq 1 \text{ for all } x_i, y_i \in X_i.$$

Write $X := \prod_{i\in\mathbb{N}} X_i$ and consider the function $d\colon X \times X \to \mathbb{R}$ defined by

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} d_i(x_i, y_i).$$

Show that d is a metric on X. (First check that d is a well-defined function.)

d. Show that the metric d from part (c) induces the product topology on $X = \prod_{i\in\mathbb{N}} X_i$.