Math 535 - General Topology
Fall 2012
Homework 7, Lecture 10/8

Problem 1. Let X be a topological space and (Y, d) a metric space. A sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n\colon X \to Y$ **converges uniformly** to a function $f\colon X \to Y$ if for all $\epsilon > 0$, there is an $N \in \mathbb{N}$ satisfying

$$d(f_n(x), f(x)) < \epsilon \text{ for all } n \geq N \text{ and all } x \in X.$$

Note in particular that uniform convergence implies pointwise convergence (but not the other way around).

Assume each function $f_n\colon X \to Y$ is *continuous*, and the sequence converges *uniformly* to a function $f\colon X \to Y$. Show that f is continuous.

Problem 2. Let X be a *compact* topological space. Consider the set of all real-valued continuous functions on X

$$C(X) := \{f\colon X \to \mathbb{R} \mid f \text{ is continuous}\}$$

which is a real vector space via pointwise addition and scalar multiplication.

Consider the function $\|\cdot\|\colon C(X) \to \mathbb{R}$ defined by

$$\|f\| := \sup_{x\in X}|f(x)|.$$

a. Show that $\|\cdot\|$ is a norm on $C(X)$. (First check that $\|\cdot\|$ is well-defined.)

This norm is sometimes called the **uniform norm** or **supremum norm**.

b. Show that a sequence $(f_n)_{n\in\mathbb{N}}$ in $C(X)$ converges to f in the uniform norm (meaning $\|f_n - f\| \to 0$) if and only if the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

c. Show that $C(X)$ endowed with the uniform norm is complete (i.e. with respect to the metric induced by the norm).