Math 9052B/4152B - Algebraic Topology
Winter 2015
Homework 2

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Due date: Wednesday February 4

Problem 1. Let X and Y be spaces, and let $p \colon X \times Y \to X$ be the projection onto the first factor.

(a) Show that p has the path lifting property.

(b) Show that p has the *unique* path lifting property if and only if the path components of Y consist of single points. (This holds in particular when Y is discrete.)

Problem 2. (Variant of Hatcher #1.1.16) In each case below, show that the subspace A is *not* a retract of the space X.

(a) The solid torus $X = S^1 \times D^2$ and its boundary torus $A = \partial X = S^1 \times S^1$.

(b) The torus $X = S^1 \times S^1$ and the wedge $A = S^1 \vee S^1 = (S^1 \times \{y_0\}) \cup (\{x_0\} \times S^1)$.

Problem 3. (Variant of Hatcher #0.12) Let $\mathrm{Conn}(X)$ denote the set of connected components of a space X.

(a) Describe the function $\mathrm{Conn}(f) \colon \mathrm{Conn}(X) \to \mathrm{Conn}(Y)$ induced by a (continuous) map $f \colon X \to Y$ and check that it is well-defined. Show that the resulting construction defines a functor

$$\mathrm{Conn} \colon \mathbf{Top} \to \mathbf{Set}$$

from the category of topological spaces to the category of sets.

(b) Show that Conn is a *homotopy functor* in the following sense: If f and g are homotopic maps, then they induce the same function $\text{Conn}(f) = \text{Conn}(g)$.

(c) Deduce that a homotopy equivalence between spaces induces a bijection between their sets of connected components. (In particular, being connected is a homotopy invariant.)

Now let $\pi_0(X)$ denote the set of path components of a space X. The same arguments as above show that $\pi_0 : \mathbf{Top} \to \mathbf{Set}$ is a homotopy functor. (Do **not** show this. It really is the same as above.)

(d) Consider the function $\eta_X : \pi_0(X) \to \text{Conn}(X)$ sending a path component C to the connected component containing C. Show that these functions η_X define a natural transformation $\eta : \pi_0 \to \text{Conn}$.

(e) Deduce that if X and Y are homotopy equivalent spaces, and X is such that its connected components coincide with its path components, then the same holds for Y.

Problem 4. (Hatcher #1.2.4) Let A be the union of n lines through the origin in $\mathbb{R}^3$. Compute $\pi_1(\mathbb{R}^3 \setminus A)$.