Math 9052B/4152B - Algebraic Topology
Winter 2015
Homework 6

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Due date: Friday April 10

Problem 1. (Hatcher # 2.2.2) Let $n \geq 1$.

(a) For any map $f\colon S^{2n} \to S^{2n}$, show that there is a point $x \in S^{2n}$ satisfying either $f(x) = x$ or $f(x) = -x$.

(b) Show that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point.

(c) Construct a map $\mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ without any fixed point. Justify your answer.

Problem 2. Let $n \geq 1$.

(a) Let J be a set and consider the wedge of spheres $\bigvee_{j \in J} S^n$. Show that for any homology class $a \in H_n\left(\bigvee_{j \in J} S^n\right)$, there exists a map $f\colon S^n \to \bigvee_{j \in J} S^n$ such that the induced map $H_n(f)\colon H_n(S^n) \to H_n\left(\bigvee_{j \in J} S^n\right)$ sends a generator of $H_n(S^n)$ to a.

(b) Let G be an abelian group . Show that there exists a space X whose reduced homology is

$$\widetilde{H}_i(X) = \begin{cases} G & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

Problem 3. (Variant of Hatcher # 2.2.13) Consider the circle S^1 with its CW-structure having one 0-cell and one 1-cell. Let X be the CW-complex obtained from S^1 by attaching two 2-cells via attaching maps of degrees 5 and 7 respectively.

(a) For every (non-empty) subcomplex A of X, compute the homology groups $H_*(A)$ and $H_*(X/A)$ in all degrees.

(b) Show that the only subcomplex A of X for which the quotient map $X \twoheadrightarrow X/A$ is a homotopy equivalence is the trivial subcomplex, consisting only of the 0-cell.

(c) Bonus part, for extra credit. Show that X is homotopy equivalent to S^2. (Feel free to use the homology Whitehead theorem: Hatcher Corollary 4.33.)

Problem 4. (Hatcher # 2.2.28) Compute the homology groups $H_*(X)$ of the space X obtained from $\mathbb{R}P^2$ by attaching a Möbius band M via a homeomorphism of its boundary circle $\partial M \cong S^1$ to the standard $\mathbb{R}P^1$ inside $\mathbb{R}P^2$.