# Math 9052B/4152B - Algebraic Topology
# Winter 2015
# Homology with coefficients

Martin Frankland

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Given a CW-complex $X$, we know that its cellular chain complex $C_*^{\mathrm{CW}}(X)$ and singular chain complex $C_*(X)$ have isomorphic homology $H_*^{\mathrm{CW}}(X) \simeq H_*(X)$. We want to generalize this statement to homology with coefficients. Along the way, we discuss some related material from homological algebra.

## 1 Direct approach

**Proposition 1.1.** *Let $X$ be a CW-complex and $G$ an abelian group. Then there is an isomorphism of homology with coefficients $H_*^{\mathrm{CW}}(X;G) \simeq H_*(X;G)$. Moreover, this isomorphism is natural with respect to cellular maps $X \to Y$ and with respect to $G$ (and all group homomorphisms).*

*Proof.* Recall that the isomorphism $H_n^{\mathrm{CW}}(X) \simeq H_n(X)$ was obtained by showing that the two surjections illustrated in the diagram

$$\begin{array}{ccc} & H_n(X_n) & \\ \swarrow & & \searrow \\ H_n^{\mathrm{CW}}(X) & & H_n(X) \end{array}$$

have the same kernel. This was a consequence of the long exact sequences of the pairs $(X_k, X_{k-1})$, and the fact that the relative homology $H_*(X_k, X_{k-1})$ is concentrated in degree $k$. Homology with coefficients also has a (natural) long exact sequence associated to any pair, and the relative homology groups

$$\begin{aligned} H_i(X_k, X_{k-1}; G) &\cong \widetilde{H}_i(X_k/X_{k-1}; G) \\ &\cong \widetilde{H}_i(\bigvee_{k\text{-cells}} S^k; G) \\ &\cong \bigoplus_{k\text{-cells}} \widetilde{H}_i(S^k; G) \\ &\cong \begin{cases} \bigoplus_{k\text{-cells}} G & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases} \end{aligned}$$

are also concentrated in degree $k$. Therefore, the proof for the case $G = \mathbb{Z}$ works here as well.

The naturality statements follow from naturality of the diagram

$$\begin{array}{ccc} & H_n(X_n; G) & \\ \swarrow & & \searrow \\ H_n^{\text{CW}}(X; G) & & H_n(X; G) \end{array}$$

with respect to cellular maps $X \to Y$, and with respect to group homomorphisms $G \to G'$. □

## 2 Approach using chain homotopy

**Proposition 2.1.** *Let $C_*$ be a (possibly unbounded) chain complex of* free *abelian groups. Then $C_*$ is quasi-isomorphic to its homology, in fact via a quasi-isomorphism $C_* \xrightarrow{\sim} H_*(C_*)$ (as opposed to a zig-zag).*

*Proof.* Consider[1] the short exact sequence

$$0 \longrightarrow Z_n \longrightarrow C_n \xrightarrow{d} B_{n-1} \longrightarrow 0$$

which is split, since $B_{n_1}$ is a free abelian group, being a subgroup of the free abelian group $C_{n-1}$. Choosing a splitting $C_n \simeq Z_n \oplus B_{n-1}$ for each $n \in \mathbb{Z}$, the chain complex $C_*$ is

[1] Credit to Tyler Lawson for this explanation:
`http://mathoverflow.net/questions/10974/does-homology-detect-chain-homotopy-equivalence`

isomorphic (though not naturally) to the chain complex illustrated here:

$$\begin{array}{ccc}
\vdots & & \text{degree} \\
\downarrow & & \\
Z_{n+1} \oplus B_n & & n+1 \\
\downarrow & & \\
Z_n \oplus B_{n-1} & & n \\
\downarrow & & \\
Z_{n-1} \oplus B_{n-2} & & n-1 \\
\downarrow & & \\
\vdots & &
\end{array}$$

where the differential $d_n$ is given by the inclusion $B_{n-1} \hookrightarrow Z_{n-1}$. Hence, there is an isomorphicm of chain complexes $C_* \cong \bigoplus_{n \in \mathbb{Z}} C_*^{(n)}$ where $C_*^{(n)}$ denotes the tiny chain complex

$$\begin{array}{ccc}
0 & & \\
\downarrow & & \\
B_n & & n+1 \\
\big\downarrow\!\!\!\!\!\!\!\!\!\;\;\; & & \\
Z_n & & n \\
\downarrow & & \\
0 & &
\end{array}$$

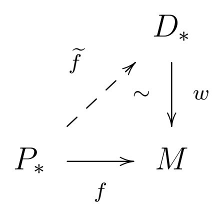
concentrated in degrees $n$ and $n+1$. Consider $H_n(C_*)$ as a chain complex concentrated in degree $n$. The map $\varphi_n \colon C_*^{(n)} \to H_n(C_*)$ given by the quotient map $Z_n \twoheadrightarrow H_n(C_*) = Z_n / B_n$ in degree $n$ is a chain map which is moreover a quasi-isomorphism. These maps assemble into a quasi-isomorphism

$$\bigoplus_{n \in \mathbb{Z}} \varphi_n \colon \bigoplus_{n \in \mathbb{Z}} C_*^{(n)} \xrightarrow{\sim} \bigoplus_{n \in \mathbb{Z}} H_n(C_*) = H_*(C_*).$$

as claimed. $\square$

Recall the following fact from homological algebra.

**Theorem 2.2** (Comparison theorem for projective resolutions)**.** *Let $\mathcal{A}$ be an abelian category, and let $M$ be an object of $\mathcal{A}$, viewed as a chain complex concentrated in degree* 0*. Let $P_*$ be a (non-negatively graded) chain complex of projective objects, with a chain map $f\colon P_* \to M$, and let $D_*$ a (non-negatively graded) chain complex with a quasi-isomorphism $w\colon D_* \xrightarrow{\sim} M$. Then $f$ admits a lift as in the diagram*




*which is unique up to chain homotopy.*

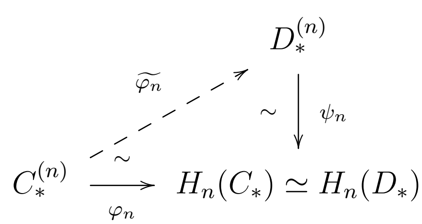
*Proof.* [1, Theorem 2.2.6]. □

**Example 2.3.** In the category $\mathcal{A} = \mathbf{Ab}$ of abelian groups, an object is projective if and only if it is a free abelian group.

**Proposition 2.4.** *Let $C_*$ and $D_*$ be (possibly unbounded) chain complexes of* free *abelian groups.*

1. *If $C_*$ and $D_*$ have isomorphic homology $H_*(C_*) \simeq H_*(D_*)$, then they are chain homotopy equivalent: $C_* \simeq D_*$.*
2. *If $f : C_* \xrightarrow{\sim} D_*$ is a quasi-isomorphism, then $f$ is a chain homotopy equivalence.*

*Proof.* 1. Consider decompositions $C_* \cong \bigoplus_{n\in\mathbb{Z}} C_*^{(n)}$ and $D_* \cong \bigoplus_{n\in\mathbb{Z}} D_*^{(n)}$ as in the proof of 2.1. For each $n \in \mathbb{Z}$, consider the diagram of chain complexes




where a lift $\widetilde{\varphi_n}\colon C_*^{(n)} \to D_*^{(n)}$ exists, by Theorem 2.2. Reversing the roles of $C_*$ and $D_*$, there also exists a lift $\widetilde{\psi_n}\colon D_*^{(n)} \to C_*^{(n)}$. Uniqueness of lifts up to chain homotopy shows that $\widetilde{\psi_n}$ is chain homotopy inverse to $\widetilde{\varphi_n}$. Therefore, the chain map

$$\bigoplus_{n\in\mathbb{Z}} \widetilde{\varphi_n}\colon \bigoplus_{n\in\mathbb{Z}} C_*^{(n)} \xrightarrow{\simeq} \bigoplus_{n\in\mathbb{Z}} D_*^{(n)}$$

is a chain homotopy equivalence, which chain homotopy inverse $\bigoplus_{n\in\mathbb{Z}} \widetilde{\psi_n}$.

2. For each $n \in \mathbb{Z}$, consider the diagram of chain complexes

$$\begin{array}{ccc} D_*^{(n)} & \overset{\psi_n}{\dashrightarrow} & C_*^{(n)} \\ \downarrow & & \downarrow \sim \\ H_n(D_*) & \xrightarrow{H_n(f)^{-1}} & H_n(C_*) \end{array}$$

where there exists a lift $\psi_n \colon D_*^{(n)} \to C_*^{(n)}$ (unique up to chain homotopy), by Theorem 2.2. These chain maps define a chain map $\psi \colon D_* \to C_*$ via the diagram

$$\begin{array}{ccc} \bigoplus_{n\in\mathbb{Z}} D_*^{(n)} & \xrightarrow{\bigoplus_{n\in\mathbb{Z}} \psi_n} & \bigoplus_{n\in\mathbb{Z}} C_*^{(n)} \\ \cong \Big| & & \Big| \cong \\ D_* & \xrightarrow{\quad \psi \quad} & C_*. \end{array}$$

One readily checks that the restriction $f|_{C_*^{(n)}} \colon C_*^{(n)} \to D_*$ is chain homotopic to the composite

$$C_*^{(n)} \xrightarrow{f|_{C_*^{(n)}}} D_* \xrightarrow{\text{proj}} D_*^{(n)} \xhookrightarrow{\text{inc}} D_*$$

and that $\psi \colon D_* \to C_*$ is chain homotopy inverse to $f \colon C_* \to D_*$. □

**Proposition 2.5.** *The relation of chain homotopy is compatible with the tensor product of chain complexes. In other words, if the chain maps $\varphi, \psi \colon C_* \to D_*$ are chain homotopic and $\varphi', \psi' \colon C'_* \to D'_*$ are chain homotopic, then the chain maps*

$$\varphi \otimes \varphi', \psi \otimes \psi' \colon C_* \otimes C'_* \to D_* \to D'_*$$

*are chain homotopic.*

*Proof.* Using the factorizations illustrated in the diagram

$$\begin{array}{ccccc} & & D_* \otimes C'_* & & \\ & \overset{\varphi\otimes \mathrm{id}_{C'_*}}{\nearrow} & & \overset{\mathrm{id}_{D_*}\otimes\varphi'}{\searrow} & \\ C_* \otimes C'_* & & \xrightarrow{\qquad \varphi\otimes\varphi' \qquad} & & D_* \otimes D'_* \\ & \underset{\mathrm{id}_{C_*}\otimes\varphi'}{\searrow} & & \underset{\varphi\otimes\mathrm{id}_{D'_*}}{\nearrow} & \\ & & C_* \otimes D'_* & & \end{array}$$

it suffices to show that $\varphi \otimes \mathrm{id}_{C'_*}$ is chain homotopic to $\psi \otimes \mathrm{id}_{C'_*}$. Let $h\colon C_n \to D_{n+1}$ be a chain homotopy from $\varphi$ to $\psi$, i.e., such that the equation $\psi - \varphi = dh + hd$ holds.

Let us check that $h \otimes \mathrm{id}_{C'_*}\colon (C_* \otimes C'_*)_n \to (D_* \otimes C'_*)_{n+1}$ is a chain homotopy from $\varphi \otimes \mathrm{id}_{C'_*}$ to $\psi \otimes \mathrm{id}_{C'_*}$. For any $x_i \in C_i$ and $x'_j \in C'_j$, with $i + j = n$, we have

$$\begin{aligned} d(h \otimes \mathrm{id}_{C'_*})(x_i \otimes x'_j) &+ (h \otimes \mathrm{id}_{C'_*})d(x_i \otimes x'_j) \\ &= d\left(hx_i \otimes x'_j\right) + (h \otimes \mathrm{id}_{C'_*})\left(dx_i \otimes x'_j + (-1)^{|x_i|} x_i \otimes dx'_j\right) \\ &= dhx_i \otimes x'_j + (-1)^{|hx_i|} hx_i \otimes dx'_j + hdx_i \otimes x'_j + (-1)^{|x_i|} hx_i \otimes dx'_j \\ &= dhx_i \otimes x'_j + (-1)^{i+1} hx_i \otimes dx'_j + hdx_i \otimes x'_j + (-1)^{i} hx_i \otimes dx'_j \\ &= dhx_i \otimes x'_j + hdx_i \otimes x'_j \\ &= (dh + hd)x_i \otimes x'_j \\ &= (\psi - \varphi)x_i \otimes x'_j \\ &= \psi x_i \otimes x'_j - \varphi x_i \otimes x'_j. \end{aligned}$$

Therefore the equation

$$d(h \otimes \mathrm{id}_{C'_*}) + (h \otimes \mathrm{id}_{C'_*})d = \psi \otimes \mathrm{id}_{C'_*} - \varphi \otimes \mathrm{id}_{C'_*}$$

holds. □

**Corollary 2.6.** *If $\varphi\colon C_* \xrightarrow{\simeq} D_*$ and $\varphi'\colon C'_* \xrightarrow{\simeq} D'_*$ are chain homotopy equivalences, then their tensor product*

$$\varphi \otimes \varphi'\colon C_* \otimes C'_* \xrightarrow{\simeq} D_* \otimes D'_*$$

*is a chain homotopy equivalence.*

*Proof.* Let $\alpha\colon D_* \to C_*$ and $\alpha'\colon D'_* \to C'_*$ be chain homotopy inverses of $\varphi$ and $\varphi'$ respectively. Then

$$\alpha \otimes \alpha'\colon D_* \otimes D'_* \to C_* \otimes C'_*$$

is a chain homotopy inverse of $\varphi \otimes \varphi'$. □

The following proposition says that "any chain complex of free abelian groups will do", as long as it has the correct homology (with coefficients in $\mathbb{Z}$).

**Proposition 2.7.** *Let $X$ be a space and $C_*$ a chain complex of free abelian groups whose homology is isomorphic to the singular homology of $X$, i.e., $H_n(C_*) \simeq H_n(X)$ holds for all $n$. Then for any abelian group $G$ and any $n$, there are isomorphisms $H_n(C_* \otimes G) \simeq H_n(X; G)$.*

*Proof.* The assumption is that the homology $C_*$ is isomorphic to the homology of the singular chain complex $C_*(X)$. By Proposition 2.4, there is a chain homotopy equivalence $\varphi\colon C_* \xrightarrow{\simeq} C_*(X)$. By Corollary 2.6, the chain map

$$\varphi \otimes \mathrm{id}_G\colon C_* \otimes G \xrightarrow{\simeq} C_*(X) \otimes G$$

is a chain homotopy equivalence, in particular a quasi-isomorphism. $\square$

**Example 2.8.** Let $X$ be a $\Delta$-complex, and $C_*^\Delta(X)$ the associated simplicial chain complex. Then there are isomorphisms $H_n^\Delta(X;G) \simeq H_n(X;G)$. Naturality with respect to $\Delta$-maps $X \to Y$ does not follow directly from the first part of Proposition 2.7.

However, recall that the isomorphism $H_n^\Delta(X) \simeq H_n(X)$ is induced at the chain level by a quasi-isomorphism $\theta\colon C_*^\Delta(X) \xrightarrow{\sim} C_*(X)$, which is natural with respect to $\Delta$-maps $X \to Y$. By the second part of Proposition 2.4, $\theta$ is in fact a chain homotopy equivalence. By Corollary 2.6, the chain map $\theta \otimes \mathrm{id}_G\colon C_*^\Delta(X) \otimes G \xrightarrow{\simeq} C_*(X) \otimes G$ is also a chain homotopy equivalence, and in particular induces isomorphisms $H_n^\Delta(X;G) \simeq H_n(X;G)$. These isomorphisms are natural with respect to $\Delta$-maps $X \to Y$, since the chain map $\theta$ is.

**Example 2.9.** Let $X$ be a CW-complex, and $C_*^{\mathrm{CW}}(X)$ the associated cellular chain complex. Then there are isomorphisms $H_n^{\mathrm{CW}}(X;G) \simeq H_n(X;G)$. Naturality with respect to cellular maps $X \to Y$ does not follow from Proposition 2.7.

# 3 Approach using the universal coefficient theorem

Recall the following fact.

**Theorem 3.1** (Universal coefficient theorem)**.** *Let $C_*$ be a chain complex of* free *abelian groups, and $G$ an abelian group. Then for each $n \in \mathbb{Z}$, there is a short exact sequence*

$$0 \longrightarrow H_n(C_*) \otimes G \xrightarrow{\times} H_n(C_* \otimes G) \longrightarrow \mathrm{Tor}\,(H_{n-1}(C_*), G) \longrightarrow 0$$

*which is natural in $C_*$ and $G$. Moreover, the sequence is split, though the splitting is not natural.*

Here, the map $\times\colon H_n(C_*) \otimes G \to H_n(C_* \otimes G)$ sends $[\alpha] \otimes g$ to the homology class $[\alpha \otimes g]$. The functor Tor denotes $\mathrm{Tor}_1^{\mathbb{Z}}$, just like our tensor product $\otimes$ denotes the tensor product $\otimes_{\mathbb{Z}}$ over the integers.

**Corollary 3.2.** *If $f\colon C_* \xrightarrow{\sim} D_*$ is a quasi-isomorphism between chain complexes of* free *abelian groups, then the map $f \otimes \mathrm{id}_G\colon C_* \otimes G \to D_* \otimes G$ is a quasi-isomorphism.*

*Proof.* By the universal coefficient theorem, for each $n \in \mathbb{Z}$, $f$ induces a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_n(C_*) \otimes G & \xrightarrow{\times} & H_n(C_* \otimes G) & \longrightarrow & \mathrm{Tor}\,(H_{n-1}(C_*), G) & \longrightarrow & 0 \\
 & & {\scriptstyle H_n(f)\otimes G}\downarrow \simeq & & {\scriptstyle H_n(f\otimes \mathrm{id}_G)}\downarrow \therefore\simeq & & {\scriptstyle \mathrm{Tor}(H_{n-1}(f),\mathrm{id}_G)}\downarrow \simeq & & \\
0 & \longrightarrow & H_n(D_*) \otimes G & \xrightarrow{\times} & H_n(D_* \otimes G) & \longrightarrow & \mathrm{Tor}\,(H_{n-1}(D_*), G) & \longrightarrow & 0
\end{array}$$

where the rows are exact. By assumption, $H_n(f)$ and $H_{n-1}(f)$ are isomorphisms, thus so are the downward maps $H_n(f) \otimes G$ and $\operatorname{Tor}(H_{n-1}(f), \mathrm{id}_G)$. By the 5-lemma, the downward map in the middle $H_n(f \otimes \mathrm{id}_G)$ is also an isomorphism. □

**Example 3.3.** Corollary 3.2 provides an alternate proof that the chain map

$$\theta \otimes \mathrm{id}_G \colon C_*^{\Delta}(X) \otimes G \xrightarrow{\sim} C_*(X) \otimes G$$

is a quasi-isomorphism, as discussed in Example 2.8.

What if an isomorphism in homology does not come from a chain map, as in the cellular homology theorem? Then we can still argue as follows.

**Proposition 3.4.** *If two (possibly unbounded) chain complexes of* free *abelian groups $C_*$ and $D_*$ have isomorphic homology $H_*(C_*) \simeq H_*(D_*)$, then the chain complexes $C_* \otimes G$ and $D_* \otimes G$ have isomorphic homology.*

*Proof.* Using the splitting in the universal coefficient theorem, we have (non-natural) isomorphisms:

$$\begin{aligned} H_n(C_* \otimes G) &\simeq H_n(C_*) \otimes G \oplus \operatorname{Tor}(H_{n-1}(C_*), G) \\ &\simeq H_n(D_*) \otimes G \oplus \operatorname{Tor}(H_{n-1}(D_*), G) \\ &\simeq H_n(D_* \otimes G). \end{aligned}$$

□

*Alternate proof.* By Proposition 2.4, there exists a quasi-isomorphism $\varphi \colon C_* \xrightarrow{\sim} D_*$ (which is in fact a chain homotopy equivalence). By Corollary 3.2, the chain map $\varphi \otimes \mathrm{id}_G \colon C_* \otimes G \xrightarrow{\sim} D_* \otimes G$ is also a quasi-isomorphism. □

*Remark* 3.5. Section 3 is essentially doing the same thing as Section 2, from a more computational perspective. A key step for proving the universal coefficient theorem is to choose splittings of the short exact sequences

$$0 \longrightarrow Z_n \longrightarrow C_n \xrightarrow{d} B_{n-1} \longrightarrow 0$$

like we did in the proof of Proposition 2.1.

# References

[1] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324 (95f:18001)