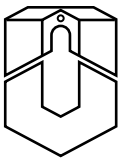
Computational Physics

MATLAB

Peter Hertel



University of Osnabrück, Germany

MATLAB is a compact, but powerful language for numerical computations. Matlab is concerned with matrices: rectangular patterns of integer, real, or complex numbers. A number itself is a 1×1 matrix, a row vector a $1 \times n$ pattern, a column an $n \times 1$ matrix. Most MATLAB operations affect the entire matrix. MATLAB contains and makes available the software treasure of more than 50 years, originally coded in Fortran. MATLAB programs, if they avoid repetitive statements, are highly efficient. This introductory crash course addresses physicists who are used to learn by studying well-chosen examples.

February 18, 2009

Contents

1 Getting startet

MATLAB is an interpreter. It reads the lines you type und executes them immediately. Here is an example:

```
» x=1
```

» is the MATLAB prompt. `x=1` says: if there is none, create a variable `x` and assign it the value 1. Since the line does not end with a semicolon, the effect of the command is echoed. Convince yourself of the difference

```
» x=2;
```

Now type

```
» x
```

With

```
» x1=[1,2,3];
```

you may create a row vector with three elements. Type

```
» x1
```

```
» x2=[4,5,6];
```

generates another row vector. You may combine them to a 2×3 Matrix z by

```
z=[x1;x2];
```

Summary

MATLAB variables begin with a letter. They need not be declared. The assignment operator is `=`. Square brackets `[...]` define a rectangular collection of data, a matrix. The comma operator `,` assembles data from left to right, the semicolon operator `;` from top to bottom. Lines are echoed unless delimited by a semicolon.

Exercises

1. What is wrong with

   ```
   » x=[1,2,3; 4, 5, 6, 7];
   ```

2. What will be the result of

   ```
   » det([1, 2, 3; 4, 5, 6; 7, 8, 9])
   ```

   `det(...)` stands for the determinant of a square matrix. Think before trying. Invoke

```
» help det
```

or

```
» helpdesk
```

for further details.

3. Try

   ```
   » x=pi
   » x=i
   ```

4. The same with

   ```
   » format long;
   ```

5. Compare

   ```
   » a=[[1;2],[3;4],[5;6]]
   ```

   with

   ```
   » b=[[1,3,5];[2,4,6]]
   ```

2 Beijing-Frankfurt

We want to work out the distance of two locations on the globe. For this purpose, we create a file `globe.m`. If we type

```
» globe
```

at the MATLAB prompt, the commands in `globe.m` are executed. MATLAB searches for the file along its search path which you must adept to your needs.

A location on the globe is specified by its latitude and longitude. The equator is 0 degrees latitude, the poles are +90 degrees (north) and -90 degrees (south). Greenwich (a suburb of London/UK) has 0 degrees longitude, Beijing is located at +116.58 degrees (east), New York city is found at -73.95 degrees (west).

Here is the program. The line numbers do not belong to it.

```
 1  % this file is globe.m
 2  % find shortest distance (in km) between start/end points
 3  % latitude, longitude in degrees, east and north are positive
 4  R=40000/2/pi; % km, earth radius
 5  lat_s=input('latitude of start point (degrees)  : ')/180*pi;
 6  lon_s=input('longitude of start point (degrees) : ')/180*pi;
 7  lat_e=input('latitude of end point (degrees)    : ')/180*pi;
 8  lon_e=input('longitude of end point (degrees)   : ')/180*pi;
 9  stav=[cos(lat_s)*sin(lon_s),cos(lat_s)*cos(lon_s),sin(lat_s)];
10  endv=[cos(lat_e)*sin(lon_e),cos(lat_e)*cos(lon_e),sin(lat_e)];
11  cos_a=stav*endv';
12  alpha=acos(cos_a);
13  distance=R*alpha;
14  fprintf('distance is %.1f km\n', distance);
```

MATLAB ignores text following a percentage symbol up to the end of a line. The `input` command writes a prompt (a string, characerized by `'...'`) and reads a number from the keyboard. In our case this number is divided by 180 and multiplied by `pi`, such that angles are in radians. We calculate two unit vectors for the start and end points of a flight, work out the angle `alpha` between these vectors and print the corresponding distance in C style.

Summary

You have learned that commands can be collected in `.m`-files. The text may be commented. We have made use of the sine, cosine, arcus cosine, and

fprintf functions. The scalar product of two row vectors `a` and `b` is denoted by `a*b'`; the dash operator denoting transposing (interchanging the meaning of rows and columns).

Exercises

1. Rewrite the above program such that the input can be specified in degrees and minutes.
2. Find out the reaction to erroneous input.
3. Play with the `fprintf` statement.
4. Replace the fourfold `/180*pi` statement by something better.
5. Calculate the shortest distance between Beijing = [40.05, 116.58] and Frankfurt/Main = [50.03, 8.55].

3 Avoid loops!

Assume you want to plot the sine function. In a traditional programming language, you would write something like

```
1   N=1023;
2   x,y = ARRAY [0..N] of REAL;
3   step:=2*pi/N;
4   for i:=0 to N do
5     x[i]:=i*step;
6     y[i]:=sin(x[i]);
7   end;
```

The same in Matlab:

```
» x=linspace(0,2*pi,1024);
» y=sin(x);
```

We refrain from showing a dull sine curve which you may obtain by typing

```
» plot(x,sin(x));
```

We consider `x` to be a variable within the interval $[0, 2\pi]$, but since we work on a machine with a finite number of states only, the variable must be characterized by representative values, here 1024. The function `linspace` generates a row vector of equally spaced values. The first two arguments denote the begin and end of the interval, the third the number of representative points. If `x` is a matrix, then `sin(x)` denotes an array of the same dimension, with each component of `x` being replaced by its sine.

The operators `+` and `-` work as expected, they work on matrices componentwise. However,

```
» x=[1,2,3;4,5,6];
» y=[7,8;9,10;11,12];
» z=x*y
```

produces `[58,64;139,154]` which is obviously the result of matrix multiplication.

However, if you have two matrices of the same format, you may multiply them componentwise by the `.*` operator (dot multiplication). The same holds true for component-wise division by the `./` operator.

You should avoid `for`-loops whenever you can. To demonstrate this, we generate two matrices of normally distributed random numbers and multiply them, first in the MATLAB style.

```
1   a=randn(100,100);
2   b=randn(100,100);
3   tic;
4   c=a*b;
5   toc;
```

And now the same in the old style:

```
1   a=randn(100,100);
2   b=randn(100,100);
3   tic;
4   for i=1:100
5     for k=1:100
6       sum=0;
7       for j=1:100
8         sum=sum+a(i,j)*b(j,k);
9       end;
10      c(i,k)=sum;
11    end;
12  end;
13  toc;
```

Although formally correct, the second version requires 50 seconds (on my laptop), the first is done within 0.11 seconds. You see the difference?

Summary

Most MATLAB operators or functions work on entire matrices. `for`-loops can usually be avoided.

Exercises

1. What are the execution times of the above two matrix multiplication programs on your machine?

2. Generate a column vector `r=randn(10000,1)` of 10000 normally distributed random numbers and generate its sum by `sum(r)`. What do you expect? Note that N normally distributed numbers tend to add up to zero. The deviation is of order $\sqrt{N}$. Try once more, or even better: `r=randn(10000,10)` and `sum(r)`.

3. Generate two 2×2 matrices and demonstrate the difference between `c=a*b` and `c=a.*b`.

4. Generate two row vectors vectors `a` and `b` of normally distributed random number and work out their scalar product `s=a*b'`. Explore `a*b` and `a'*b` as well.

5. Call the Matlab help system to find out the difference between `rand` and `randn`.

6. Make yourself acquainted with `tic` and `toc`.

4 True, false and NaN

There is an if-else construct in MATLAB. You might say

```
if a(i,j)>=1
  b(i,j)=1
elseif a(i,j)<=-1
  b(i,j)=-1
else
  b(i,j)=0
end
```

All this within an `i,j` loop which you should avoid.

Much better is

```
» b=(a>=1)-(a<=-1)
```

If `a` is any matrix, then `(a>0)` is a matrix of the same dimension, but with 1 where the condition holds true and 0 where it is false.

This feature of matrix-wise comparisons allows short and powerful code. If you want to know the number of positive entries into a matrix, just say

```
» sum(sum(a>0))
```

`a>0` is a matrix with ones and zeros, `sum` performs a summation over columns, and the second sum produces a sum over the resulting row vector.

We next show how to solve a common problem: division of zero by zero. For example, $f(x) = \sin(x)/x$ is ill-defined at $x = 0$. However, f can be made continuous by setting $f(0) = 1$ since $\sin(x) = x + \ldots$ for small x.

In MATLAB the statement `y=sin(x)./x` will produce `NaN` (not a number) if one `x`-value vanishes. All operations with `NaN` yield `NaN`, and after a few more program lines almost all results will be `NaN`.

Let us first create an x-axis,

```
» N=127
```

```
» x=linspace(-pi,pi,N)
```

Old-fashioned programming would use a for-loop with an if-else construct, such as

```
1   for j=1:N
2     if x(j)==0
3       y(j)=1;
4     else
```

```
5       y(j)=sin(x(j))/x(j);
6     end;
7   end;
```

Good MATLAB practice is to find all indices where `x` differs from zero:

```
1   y=ones(x);
2   k=find(x~=0);
3   y(k)=sin(x(k))./x(k);
```

The first line creates a matrix `y` which has the same dimension as `x`, and assigns it the value 1. The second line creates a vector `k` of position indices where `x` differs from zero. The third line re-assigns to all these indices the proper expression.

Even shorter is[1]

```
y=(x==0)+sin(x)./(x+(x==0));
```

Note that a matrix is stored internally as a long vector, the first column first, the second column next, and so on. The matrix element $a_{mn} = a(m, n)$ of an $M \times N$ matrix is to be found at the position index $r = m + M * (n - 1)$. Thus, $m = 1, 2, \ldots M$, $n = 1, 2, \ldots N$ and $r = 1, 2, \ldots MN$. In the above example, `k` is a vector of such position indices.

Summary

MATLAB provides all conventional program control structures (but no `goto`). `for` loops should be avoided. Logical decisions pertain to all elements of a matrix and result in boolean matrices with 0 (false) and 1 (true). You have also learned how to address a matrix elementwise or blockwise by a vector of position indices. Watch out for `NaN`: such nonsense numbers indicate errors.

Exercises

1. Try the if-elseif-else construct (above) with `a=5*randn(10)`. Note that MATLAB matrix indices run over 1,2,..., in contrast to C or C++ where they start with 0.

2. Try `a=(a>=3)-(a<=-3)+0*(abs(a)<3)` with `a=5*randn(10)`.

3. Make yourself acquaited with the `if`, `while`, `break`, and `for` commands.

[1] Thanks to Andreas Müller

4. Generate two matrices `a` and `b` and play with `~` (not), `==` (equal), `<=` (smaller than or equal), `>` (larger than), `~=` (not equal), `&` (and), `|` (or) etc.

5. Work out $y = \sin(x)/x$ by

   - `» y=sin(x)./x;`
   - `» y=(x==0)+(x~=0).*sin(x)./x;`
   - with `for` and `if-else`
   - using the `find` function

   and check for `NaN`.

6. What do you expect?

   ```
   » a=randn(4);
   » k=find(a==nan)
   ```

7. Convince yourself that a number, as assigned by

   ```
   » x=7
   ```

   can be addressed as

   ```
   » x(1,1)
   ```

   or

   ```
   » x(1)
   ```

   or

   ```
   » x
   ```

   The two index form is correct since everything in MATLAB is a matrix. The one index form refers to the storage method. The no index form is a for ease of usage.

5 Functions

Assume a continuous real valued function $f = f(x)$ depending on one real variable. The definite integral $\int_a^b f(x)dx$ is well defined. Here we are not concerned with numerical quadrature, or integration methods. The problem is how to formulate a function in MATLAB. There is a large group of functions which take functions as an argument. In the help system they are grouped as 'funfun', a funny pun.

In order to be specific, we concentrate on the Planck formula for the spectral density of radiation. With T as temperature of the black body, k the Boltzmann constant, $\hbar$ the Planck constant and ω the angular frequency, one defines $x = \hbar\omega/kT$ as the photon energy in terms of the reference energy kT. Simple dimensional considerations show that the spectral density depends on x only. In fact, in 1900 Max Planck has published its famous formula

$$S(x) = \frac{15}{\pi^4}\,\frac{x^3}{e^x - 1} \tag{1}$$

(in modern wording). The factor in front assures $\int_0^\infty dx\, S(x) = 1$, i.e. S is a probability distribution. Is it? We check this numerically with MATLAB.

For this purpose we write an `.m`-file:

```
1   % this file is planck.m
2   function s=planck(x);
3   s=(15/pi^4)*x.^3./(exp(x)-1);
```

The reserved word `function` introduces a return value, a function name, and a parameter list. Since we want to use our function for an integration routine, it should be defined for an array of x-values. This explains the `.^` power-to operator and the `./` division operator. They act component-wise.

On the Matlab prompt we may now type

```
» quad('planck',0,20)
```

demanding the integration of the 'planck' function from 0 to 20, by a rather accurate integration procedure. MATLAB will complain: the function has to be avaluated at $x = 0$ which amounts to dividing zero by zero the result of which is `NaN` (not a number).

A dirty trick helps. Any floating point system has a number `eps` (the so-called maschine-epsilon) which is the smallest value for which `1` and `1+eps` are different. So we try

```
» quad('planck',eps,20)
```

and obtain 1.0000 (with `format short`).

Functions may return more than one value, and they may have more than one parameter, which can be matrices. Here is an example

```
1   % this file is distance.m
2   function [d,t]=distance(s,e,v)
3   % positions [longitude,latitude] in degrees
4   % s is start position, e is destination;
5   % v is average cruising speed in km/h
6   % return distance in km and flight time in hours
7   R=40000/2/pi; % earth radius
8   s=s/180*pi;
9   ss = [cos(s(1))*sin(s(2)),cos(s(1))*cos(s(2)),sin(s(2))];
10  e=e/180*pi;
11  ee = [cos(e(1))*sin(e(2)),cos(e(1))*cos(e(2)),sin(e(2))];
12  c=ss*ee'; % cosine of angle between start/end vectors
13  d=R*acos(c); % distance
14  t=d/v; % flight time
```

Summary

Functions are special m-files beginning with the `function` keyword. They specify the return arguments, a suggestive name, and the input arguments. Note that a function is identified by its file name, not by the function identifier. The file name extension must be `.m`, but you call it without the extension.

Exercises

1. Create the distance.m file and calculate

   ```
   » Beijing=[40.05,116.58];
   » Frankfurt=[50.03,8.55];
   » [kms,hours]=distance(Beijing,Frankfurt,920)
   ```

2. Query the help system for `eps`, `nan`, `inf` (machine epsilon, not-a-number, infinity) and play with them!

3. Reformulate the `planck` function such that it is well defined for $x = 0$.

4. Plot the Planck function for $0 \leq x \leq 8$ and `print` it into an Encapsulated Postscript file `planck.eps`.

6 Ordinary differential equations

Let us study the motion of a planet in the gravitational field of the sun. Because angular momentum is conserved, the planet moves in a plane, the `x1,x2`-plane for example. Let us choose units of length and time such that the sun's mass and the universal gravitational constant becomes unity. The state of the planet is described by a vector with four components: two coordinates and the corresponding two velocities, `x=[x1,x2,v1,v2]`. The independent variable is t, the time. We have to solve a system of four differential equations of first order:

$$\frac{dx_j}{dt} = f_j(t,x) \quad . \tag{2}$$

Let us write a MATLAB function which describes the dynamics of planetary motion:

```
1  % this file is kepler.m
2  function f=kepler(t,x);
3  f=zeros(4,1);
4  r=sqrt(x(1)^2+x(2)^2);
5  f(1)=x(3);
6  f(2)=x(4);
7  f(3)=-x(1)/r^3;
8  f(4)=-x(2)/r^3;
```

Line1 is the usual comment. Line 2 says that two arguments `(t,x)` are required to calculate the return value `f`. In line 3 we announce that the object to be returned is a column vector with four components. The statement is there for efficiency reasons only. Next we work out the distance `r` between the sun and the planet. Line 5 says that the third component of the state vector is the first derivative of the first component, line 6 likewise. Lines 7 and 8 finally express Newtons law of graviation. On the left hand side, one finds the time derivatives of the velocities, i.e. the acceleration or force. It points towards the sun, hence the `-x(1)/r` and `-x(2)/r` factors. The force is inversely proportional to the square of the distance which introduces another two powers of the distance.

MATLAB contains various programs for solving ordinary differential equations (ODE). In most cases, `ode45` is a good first choice.

On the MATLAB prompt we may type in

```
» x0=[1,0,0,0.8];
```

```
» [t,x]=ode45('kepler',[0,10],x0);
```

`x0` is the start vector. At `t=0` our planet starts at `(x1,x2)=(1,0)` with velocity `(v1,v2)=(0,0.8)`. This is the aphel, the position of largest distance. The next line calls `ode45` with the minimum number of arguments: the differential equation to solve (an m-file), the time span, and the start vector. The function returns a vector t of time points and an array of corresponding positions. The semicolon suppresses output to the command window.

Note that the returned `x` is a matrix. For each time in `t`, there is a row vector in `x`, the corresponding state. This is an inconsistency (less politely: an error). The m-file requires the state to be a column vector, but the ODE solver returns rows, one for each time.

We extract the first and second column which are the planet's coordinates:

```
» x1=x(:,1);
```

```
» x2=x(:,2);
```

The colon operator `:` is short for "all components". If you now order

```
» plot(x1,x2)
```

you should see a few orbits. Because of inaccuracies they do not lie on one and the same ellipse.

We insert before the `ode45` command

```
» options=odeset('RelTol',1e-6);
```

and change the `ode45` command into

```
» [t,x]=ode45('kepler',[0:0.05:10],x0,options);
```

These are two modifications. First, the time span has been detailed. Second, the relative integration tolerance is lowered. Now all orbits are one and the same ellipse, in accordance with the findings of Kepler and Newton.

Summary

Integrating ordinary differential equations is a simple problem, in principle. First, you have to rewrite your problem into a system of coupled first-order equations. Second, an m-file must be created which describes the drivative in terms of the independet variable and the state vector. Third, you start with the standard ODE solver `ode45`. If you are not yet satisfied, refine the time span and set options. If this does not resolve your problems, study the MATLAB help system for alternative ODE solvers, or write one yourself.

Exercises

1. Implement the crude integration, as outlined above.
2. Work out from the returned set of states the angular momentum `l=x(:,1).*x(:,4)-x(:,2).*x(:,3)` and the total energy (likewise).
3. How constant are they? Do the same after the integration procedure has been refined.
4. Study the options record returned by odeset.
5. Modify the `kepler.m` program in such a way that the force is inverse proportional to the first power of the distance, instead of the second power.
6. Create a random matrix a=rand(3,5) and study the effect of
   - `» a(:,1)`
   - `» a(1,:)`
   - `» a([1,2],[1,2])`
   - `» a([1:3],:)`
   - `» a(1,[1:2:5])`
7. Write an essay 'MATLAB: How to extract submatrices' of not more than two pages.

7 Reading and writing data

As we know, MATLAB is concerned with matrices. `s='Matlab'` for instance, a string, is a 1×6 character array requiring 12 bytes of storage. `r=[1,2;3,4;5,6]`, a 3×2 matrix of numbers, uses 48 bytes, `c=r+i` needs 96 bytes.

There is a command `who` listing all currently active variables. Another command, `whos`, adds size information.

A simple command like

```
» x=randn(2000)+i*randn(2000);
```

may consume a lot of memory, 64 MB in this case. If you do not need `x` any longer, say

```
» clear x;
```

We recommend freeing memory whenever possible. Note that variables which are local to a function are destroyed automatically.

If you type

```
» save;
```

the entire workspace is saved to a disk file.

```
» clear all;
```

will clear the entire workspace. But it is not lost. Just type

```
» load;
```

MATLAB knows which file to read and how to restore the original data.

You save a single matrix by

```
» save x;
```

A file `x.mat` is created which contains the data in binary form. We recommend

```
» save x.dat x -ascii -double;
```

The first argument is the file to be created, the second the matrix to be saved. The `-ascii` option provides for plain text, an additional `-double` avoids loss of precision. You may inspect this file with your favourite editor.

```
» load x.dat
```

opens the file, creates a matrix `x` of proper size and assigns to it the values stored in the file.

Summary

The collection of currently available variables, the workspace, can be queried and cleared. The entire workspace or single matrices may be saved to files and loaded on request. The default is in binary format, but it is much safer to insist on Ascii which can be fed to other application programs.

Exercises

1. Inspect

```
>> s='Matlab';
>> r=[1,2;3,4;5,6];
>> c=r+i;
>> whos
```

2. Now execute

```
>> save;
>> clear all;
>> whos
>> load;
>> whos
```

3. Try

```
>> clear all;
>> x=rand(5);
>> y=x;
>> save x.out x -ascii;
>> clear x;
>> load x.out;
>> norm(y-x)
```

4. Same as previous, but with

```
>> save x.out x -ascii -double
```

8 Linear Algebra

Linear algebra was the primary goal of MATLAB and still is. Very many problems of science, technology and business administration may be reduced to linear algebra. Mappings of one space into another, if they are smooth, can be approximated, at least locally, as linear mappings. Linear mappings between finite dimensional spaces are represented by matrices, and working (in Latin: laborare) with matrices gave rise to the name of the program package which we are exploring here.

Whatever you have learned in your linear algebra course: it is available in MATLAB, but probably much more. We can mention just a few examples.

Let us set up a random matrix to play with,

```
» A=randn(10);
```

It determinant

```
» d=det(A);
```

will not vanish, almost certainly. Therefore, the inverse matrix

```
» B=inv(A)
```

may be calculated.

Check

```
» format long;
```

```
» det(A)*det(B)
```

Satisfied? Now a tougher problem:

```
» norm(A*B-eye(10))
```

Whatever the norm is, it should be linear, $||\alpha A|| = |\alpha|||A||$, fulfill the triangle inequality, $||A+B|| \leq ||A||+||B||$, and warrant $||AB|| \leq ||A||||B||$. Moreover, $||A|| = 0$ implies $A = 0$.

Let us now solve the standard problem of linear algebra: a system $Ax = y$ of linear equations. The matrix A and the right hand side y is given, which vector x solves this equation?

```
1   >> A=randn(10);
2   >> y=randn(10,1);
3   >> x=A\y;
4   >> norm(y-A*x)
```

Note the pseudo division operator (backslash) for solving a system of linear equations. And wonder about the speed!

Let us now generate a symmetric matrix of normally distributed random numbers. We generate a random matrix and copy its upper-right half into the lower-left part.

```
1   >> A=randn(4);
2   >> A=diag(diag(A))+triu(A,1)+triu(A,1)'
3   >> norm(A-A')
```

This looks rather tricky. `v=diag(A)` extracts the diagonal of a matrix, which is a vector. `diag(v)` with `v` a vector, generates a diagonal matrix with the components of `v` on the main diagonal. Consequently, `A=diag(diag(A))` is `A` with its off-diagonal elements removed. `triu(A,1)` extracts from `A` the upper triangular matrix, beginning with 1 location away from the diagonal. `triu(A,1)'` transposes it, so that it may become the lower triangular part. The result is a matrix the diagonal and upper diagonal part are choosen at random while the lower triagonal part mirrors its upper part. This matrix should be Hermitian,

```
» norm(A-A')
```

tests this.

A Hermitian matrix A may be written such that $A = UDU'$ holds true where D is a diagonal matrix and U a unitary matrix. The diagonal elements of D are the eigenvalues of A, the column vectors of U are the corresponding eigenvectors, being normalized and mutually orthogonal. The command

```
» [U,D]=eig(A)
```

actually does this.

Summary

Matlab is tailored for working with matrices. Solving systems of linear equations and diagonalization are just highlights.

Exercises

1. Check

   ```
   1   >> a=randn(10);
   2   >> b=randn(1);
   3   >> norm(a)+norm(b)-norm(a+b)
   4   >> norm(a)*norm(b)-norm(a*b)
   ```

2. Try this m-file:

```
1   % this file is test.m
2   n=1000;
3   A=randn(n);
4   y=randn(n,1);
5   tic;
6   x=A\y;
7   toc;
```

3. Generate a Hermitian matrix `A` of normally distributed numbers, diagonalize it and check that `A` and `U*D*U'` are indeed the same.

4. Try

   ```
   » eig(A)
   ```

5. Explain why

   ```
   1   >> [U,D]=eig(A);
   2   >> D(1,1)=0;
   3   >> B=U*D*U';
   4   >> C=inv(B);
   ```

   produces an error message or a warning.

9 Their logo

What is now their logo, was a challenge to programmers for a long time: solve the wave equation

$$\Delta u + \lambda u = 0 \tag{3}$$

on an L-shaped domain $\Omega = [-1,1] \times [-1,1] - [-1,0) \times [-1,0)$.

u shall vanish on the boundary $\partial\Omega$.

We approximate the x-axis by $x_j = jh$ with integer j and the y-axis by $y_k = hk$ with integer k. The field u at (x_j, y_k) is denoted by $u_{j,k}$. We approximate Δu at (x_j, y_k) by

$$(\Delta u)_{j,k} = \frac{u_{j+1,k} + u_{j,k+1} + u_{j-1,k} + u_{j,k-1} - 4u_{j,k}}{h^2} \quad . \tag{4}$$

Interior points of Ω—which are labelled by $a = 1, 2, \ldots$—correspond to field variables u_a, and the Laplacien is represented by a matrix L_{ba}. We look for a non-vanishing field u_a such that $\sum_b L_{ba} u_b + \lambda u_a = 0$ holds, and λ shall be the smallest value for which this is possible.

The following main program describes the domain Ω by a rectangular matrix D which assumes values 1 (interior point, variable) and 0 (field vanishes). A subprogram calculates the sparse Laplacien matrix `L` and the variable mapping: $j = J(a)$ and $k = K(a)$ yield the coordinates (x_j, y_k) for the variable u_a. We then look for one (1) eigenvalue (closest to 0.0) of the sparse matrix L by `[u,d]=eigs(L,1,0.0)`; the field u and the eigenvalue d is returned. We then transform u_a into $u_{j,k} = u(x_j, y_k)$. A graphical mesh representation is produced and saved into an encapsulated postscript file.

```
1   % this file is logo.m
2   N=32; % resolution
3   x=linspace(-1,1,N);
4   y=linspace(-1,1,N);
5   [X,Y]=meshgrid(x,y);
6   D=(abs(X)<1)&(abs(Y)<1)&((X>0)|(Y>0));
7   [L,J,K]=laplace(D);
8   [u,d]=eigs(L,1,0.0);
9   U=zeros(N,N);
10  for a=1:size(u)
11    U(J(a),K(a))=u(a);
12  end;
13  mesh(U);
14  print -depsc 'logo.eps'
```

The main program calls a function program Laplace:

```
% this file is laplace.m
function [L,J,K]=laplace(D);
[Nx,Ny]=size(D);
jj=zeros(Nx*Ny,1);
kk=zeros(Nx*Ny,1);
aa=zeros(Nx,Ny);
Nv=0;
for j=1:Nx
  for k=1:Ny
     if D(j,k)==1
        Nv=Nv+1;
        jj(Nv)=j;
        kk(Nv)=k;
        aa(j,k)=Nv;
     end;
  end;
end;
L=sparse(Nv,Nv);
for a=1:Nv
   j=jj(a);
   k=kk(a);
   L(a,a)=-4;
   if D(j+1,k)==1
      L(a,aa(j+1,k))=1;
   end;
   if D(j-1,k)==1
      L(a,aa(j-1,k))=1;
   end;
   if D(j,k+1)==1
      L(a,aa(j,k+1))=1;
   end;
   if D(j,k-1)==1
      L(a,aa(j,k-1))=1;
   end;
end;
J=jj(1:Nv);
K=kk(1:Nv);
```

The program is certainly suboptimal. It makes use of the `for` statement and resorts to `if` statements within loops. This can be repaired, although the code will become less readable. Also note that a rather impressive 3D graphics has been produced by a command as simple as `mesh(U)`.

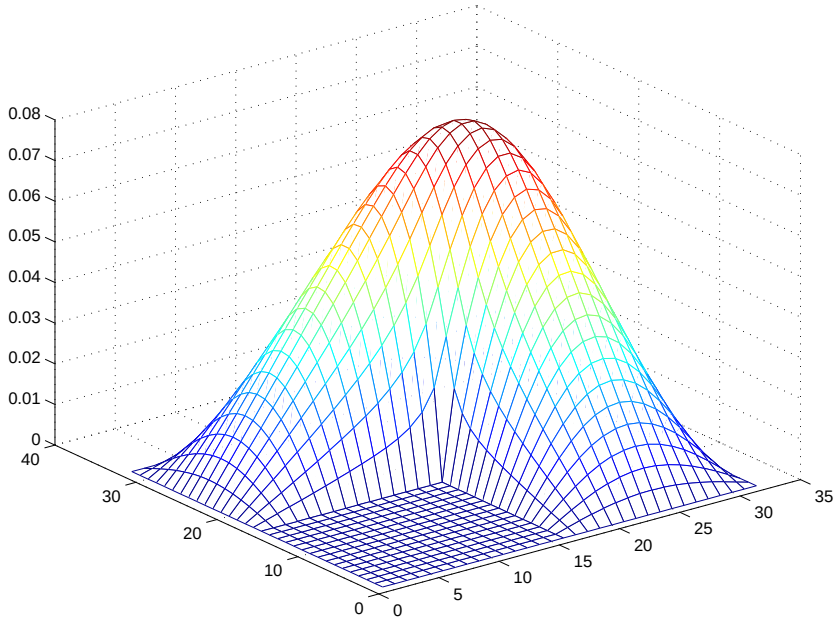


Figure 1: The MATLAB logo

A matrix is sparsely populated, or sparse, if there are very many zero entries which need not be stored. This situation is encountered in particular when partial differential equations are to be solved.

Summary

As a case study, we have solved a partial differential eigenvalue problem by resorting to MATLAB's sparse matrix facilities. If you have mastered this problem, you need no more teaching.

Exercises

1. Reproduce the MATLAB logo as presented here.
2. Imitate the MATLAB logo more closely by interactively rotating the mesh plot.
3. Produce a contour plot of the MATLAB logo.
4. Visualize the first excited mode.
5. Modify the domain by `D=((X.^2+Y.^2)<1)&((X>0)|(Y>0))`, a three-quarter spherical disk.