Betti numbers and Bass numbers

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Betti numbers

R is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue class field $k = R/\mathfrak{m}$. Modules $M, N, \ldots$ are finitely generated

An R-module M has a minimal free resolution:

$$F_\bullet : \cdots \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to M \to 0$$

exact with $\varphi_i(F_i) \subset \mathfrak{m}F_{i-1}$.

$\Rightarrow \quad \operatorname{rank} F_i = \dim_k \operatorname{Tor}_i^R(k, M)$ independent of $F_\bullet$

$\beta_i(M) = \dim_k \operatorname{Tor}_i^R(k, M)$ $\quad$ i th Betti number of M

Bass numbers

An R-module M has a minimal injective resolution:

$$E^\bullet : 0 \to M \to E^0 \xrightarrow{\psi_0} E^1 \to \cdots \to E^n \xrightarrow{\psi_n} E^{n+1} \to \cdots$$

$E^0 = E(M)$, $E^{i+1} = E(\operatorname{Coker}\psi_{i-1})$, $E(\dots) =$ injective envelope.

E injective R-module $\Rightarrow E = \displaystyle\bigoplus_{\mathfrak{p}\in \operatorname{Spec} R)} E(R/\mathfrak{p})^{\varepsilon(E,\mathfrak{p})}$

Uniqueness of injective env. $\Rightarrow \varepsilon(\mathfrak{p}, E^i)$ uniquely determined by M

$$\mu^i(\mathfrak{p}, M) = \varepsilon(\mathfrak{p}, E^i) = \dim_{k(\mathfrak{p})} \operatorname{Ext}^i_R(k(\mathfrak{p}), M_\mathfrak{p}), \quad k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$$

$$\mu^i(M) = \mu^i(\mathfrak{m}, M) = \dim_k \operatorname{Ext}^i_R(k, M)$$ ith Bass number of M.

Projective dimension and Betti numbers

$$F_\bullet : \cdots \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to M \to 0$$

minimal free resol. of M Clearly, $\operatorname{proj\,dim} M = \sup\{i : F_i \neq 0\}$

$\Rightarrow \beta_i(M) \neq 0$ for $0 \leq i \leq \operatorname{proj\,dim} M$: no gaps!

Theorem 1 (Auslander-Buchsbaum). $\operatorname{proj\,dim} M < \infty \Rightarrow$

$$\operatorname{proj\,dim} M = \operatorname{depth} R - \operatorname{depth} M$$

and all values between 0 *and* $\operatorname{depth} R$ *are attained.*

A module M has a rank if $M \otimes Q(R)$ is free ($Q(R) =$ total ring of fractions), and in this case

$$\operatorname{rank} M = \operatorname{rank} M \otimes Q(M).$$

If $\operatorname{proj\,dim} M < \infty$, then

$$\operatorname{rank} M = \sum (-1)^i \beta_i(M).$$

Injective dimension and Bass numbers

Clearly, $\operatorname{inj\,dim} M = \inf\{i : \exists \mathfrak{p} \text{ with } \mu^i(\mathfrak{p}, M) \neq 0\}$. But:

Theorem 2.

$$\operatorname{depth} M = \inf\{i : \mu^i(M) \neq 0\}$$

$$\operatorname{inj\,dim} M = \sup\{i : \mu^i(M) \neq 0\}$$

Theorem 3 (Bass). *Suppose* $\operatorname{inj\,dim} M < \infty$. *Then*

$$\dim M \leq \operatorname{inj\,dim} M = \operatorname{depth} R.$$

Question. Can there be gaps? Is it possible that

$$\mu^i(M) \neq 0 \qquad \text{for some } i, \quad \operatorname{depth} M < i < \operatorname{inj\,dim} M?$$

Proposition 4 (Bass). $\mu^i(M) > 0, \quad \dim M \leq i \leq \operatorname{inj}\dim M.$

Proof. Apply $\Gamma_{\mathfrak{m}}$ to $E^\bullet(M)$:

$$0 \to E(k)^{\mu^0} \to \cdots \to E(k)^{\mu^i} \to E(k)^{\mu^{i+1}} \to E(k)^{\mu^{i+2}} \to \cdots$$

$$\Rightarrow \qquad H^i\big(\Gamma_{\mathfrak{m}}(E^\bullet(M))\big) = H^i_{\mathfrak{m}}(M)$$

$\mu^i = 0,\ \mu^{i+1} \neq 0 \quad \Rightarrow \quad H^{i+1}_{\mathfrak{m}}(M) \neq 0$ – apply $\operatorname{Hom}_R(\ldots, E(k))$

Theorem 5 (Fossum-Foxby-Griffith-Reiten).

$$\mu^i(M) > 0, \quad \operatorname{depth} M \leq i \leq \operatorname{inj}\dim M.$$

Previous results by Foxby (R Cohen-Macaulay) and Peskine-Szpiro ($\operatorname{inj}\dim M < \infty$)

Ranks of syzygy modules

Let

$$F_\bullet : \cdots \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to M \to 0$$

be a minimal free resolution of M. Then

$$\mathrm{syz}_i(M) = \mathrm{Im}\,\varphi_i \cong \mathrm{Coker}\,\varphi_{i+1}$$

is the ith syzygy module of $M = \mathrm{syz}_0(M)$.

$$0 \to \mathrm{syz}_{i+1}(M) \to F_i \to \mathrm{syz}_i(M) \to 0 \quad \text{exact}$$

$$\Rightarrow \quad \mathrm{rank}\, F_i = \mathrm{rank}\,\mathrm{syz}_{i+1}(M) + \mathrm{rank}\,\mathrm{syz}_i(M)$$

Bounds for ranks of syzygy modules $\Rightarrow$ bounds for Betti numbers

Shrinking the ranks of syzygy modules

M an i-th syzygy if $M = \operatorname{syz}_i(N)$ for some N

Lemma 6. *M i th syzygy,* $\operatorname{rank} M \geq i + 1$, $\operatorname{proj\,dim} M < \infty$

$\Rightarrow$ M/Rx *i th syzygy* *for $x \in M$ generic*

Proof. Auslander-Buchsbaum $\Rightarrow$ $M_{\mathfrak{p}}$ free for all $\mathfrak{p}$, $\operatorname{depth} R_{\mathfrak{p}} \leq i$.

Eisenbud-Evans basic element theory $\Rightarrow$

$(M/Rx)_{\mathfrak{p}}$ free for all $\mathfrak{p}$ with $\operatorname{depth} R_{\mathfrak{p}} \leq \min(i, \operatorname{rank}(M) - 1) = i$.

$\operatorname{depth}(M/Rx)_{\mathfrak{p}} \geq \min(\operatorname{depth} R_{\mathfrak{p}} - 1, \operatorname{depth} M_{\mathfrak{p}}) \geq i$ else

$\Rightarrow$ M/Rx has "enough depth" to be i th syzygy

Corollary 7. M *i th syzygy,* $\operatorname{proj\,dim} M < \infty$, $\operatorname{rank} M > i \Rightarrow$ *there exists a free submodule* F *such that*

$$M/F \quad i\text{ th syzygy}, \quad \operatorname{rank} M/F = i\,.$$

Corollary 8. *"Every" resolution is the free resolution of a three generated ideal.*

Suppose $\operatorname{rank} M = 1$, $\operatorname{proj\,dim} M < \infty$, M torsionfree.

MacRae: $M \cong I \subset R$, $\operatorname{grade} I \geq 2$.

M 2nd syzygy $\Rightarrow \quad \operatorname{Ass}(R/I) = \emptyset \Rightarrow \quad I = R$

Question. M i th syzygy, $\operatorname{proj\,dim} M < \infty$, $\operatorname{rank} M < i \Rightarrow M$ free ?

Theorem 9 (Evans-Griffith). *Suppose R contains a field.*
M *i th syzygy*, $\operatorname{proj\,dim} M < \infty$, $\operatorname{rank} M < i \Rightarrow M$ *free*

Corollary 10. $p = \operatorname{proj\,dim} M < \infty$,

$$F_\bullet\, 0 \to F_p \to \cdots \to F_1 \to F_0 \to M \to 0$$

minimal free resolution. Then

$$\beta_i(M) \geq \begin{cases} 2i+1, & i = 0, \ldots, p-2, \\ p, & i = p-1, \\ 1, & i = p. \end{cases}$$

Proof. $\beta_i(M) = \operatorname{rank} \operatorname{syz}_{i+1}(M) + \operatorname{rank} \operatorname{syz}_i(M)$

$F_\bullet : 0 \to F_p \xrightarrow{\varphi_p} F_{p-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to 0$ satisfies (BCM) if

$$\operatorname{codim} I_{r_i}(\varphi_i) = \dim R - \dim R/\ldots \geq i, \qquad i = 1, \ldots, p$$

where $\quad r_i = \sum_{j=i}^{p} (-1)^{j-i} \operatorname{rank} F_i =$ expected rank of φ_i.

Lemma 11. *$F_\bullet$ acyclic $\Rightarrow$ $F_\bullet$ has (BCM) $\Longleftrightarrow$ $F_\bullet \otimes C$ acyclic for a balanced big CM module C of R.*

Proof. $\operatorname{grade}(I_{r_i}(\varphi_i), C) = \operatorname{codim} I_{r_i}(\varphi_i)$.

Apply Buchsbaum-Eisenbud acyclicity criterion

Theorem 12 (Hochster). *R contains a field $\Rightarrow$ R has a balanced big CM module.*

Slight generalization of Evans-Griffith syzygy theorem:

Theorem 13. *Suppose R contains a field and $F_\bullet$ satisfies (BCM). Then $r_i \geq i$ for $i = 1, \dots, p-1$.*

...and of their **improved new intersection theorem:**

Theorem 14. *Suppose R contains a field and $F_\bullet$ satisfies (BCM). Moreover, let $e \in F_i$, $e \notin \mathfrak{m} F_i$*

$$\Rightarrow \quad (R\bar{e})_{\mathfrak{p}} \quad \textit{free direct summand of } (\operatorname{Coker} \varphi_{i+1})_{\mathfrak{p}}$$

for all prime ideals $\mathfrak{p}$ such that $\operatorname{codim} \mathfrak{p} \geq i-1$.

$\Rightarrow$ induct. proof of Theorem 12 by rank reduction: pass from $F_\bullet$ to

$$0 \to F_p \to F_{p-1} \to \cdots \to F_{i+1} \to (F_i/Re) \to F'_{i-1} \to \cdots \to F'_1$$

The Evans-Griffith bound is the best possible on free resolutions in general. But:

Problem. (Buchsbaum-Eisenbud/Horrocks) Suppose M is a module of finite length. Is

$$\beta_i(M) \geq \binom{\dim R}{i} \ ?$$

Positive answers:

(a) $F_\bullet$ has an algebra structure (Buchsbaum-Eisenbud)

(b) $M = R[X_1, \ldots, X_d]/I$, I a monomial ideal (Charalambous, generalization by M. Brun and Römer)

(c) other special cases: Hochster-Richert, . . .

From Betti to Bass

In order to analyze the Bass numbers $\mu^i(M)$ we apply $\Gamma_\mathfrak{m}$ to the minimal injective resolution

$$E^\bullet : 0 \to M \to E^0 \xrightarrow{\psi_1} E^1 \to \cdots \to E^n \xrightarrow{\psi_n} E^{n+1} \to \cdots$$

It yields $\quad \Gamma_\mathfrak{m}(E^\bullet) : 0 \to E(k)^{\mu^0} \to \cdots \to E(k)^{\mu^n} \to \cdots$

We may assume R to be complete $\Rightarrow$ Matlis duality

Since $\mathrm{Hom}_R(E(k), E(k)) = R$ we obtain a minimal complex

$$G_\bullet : 0 \to R^{\mu^t} \to R^{\mu^{t+1}} \to \cdots \to R^{\mu^{d-1}} \to R^{\mu^d}$$

where $t = \mathrm{depth}\, M$, $d = \dim R$

Theorem 15. *Suppose R contains a field. Then*

(a) $G_{\bullet}$ *satisfies (BCM). Hence*

$$\mu^i(M) \geq \begin{cases} 1, & i = t, \\ d - t, & i = t+1, \\ 2(d-i)+1, & i = t+2, \ldots, d. \end{cases}$$

(b) (Foxby) if $t < d = \dim M$, then $\mu^d(M) \geq 2$.

Corollary 16 (Foxby; Roberts). *(Suppose R contains a field.) If $\mu^d(R) = 1$, then R is a Gorenstein ring.*

Corollary 17 (Peskine-Szpiro; Roberts). *(Suppose R contains a field.) If M has a module of finite injective dimension, then R is Cohen-Macaulay.*

A bound for the nonfree locus

What if $\operatorname{proj\,dim} M = \infty$?

There is no lower bound on the rank of syzygy modules: R may have maximal Cohen-Macaulay modules of rank 1 (and any Krull dimension).

Theorem 18.

$$\operatorname{codim} \operatorname{Nonfree}(M) \leq \operatorname{rank}(M) + \operatorname{rank}(\operatorname{syz}_2(M)) + 1.$$

Corollary 19. *Suppose R has an isolated singularity and M is a nonfree maximal Cohen-Macaulay module. Then*

$$\operatorname{rank} M + \operatorname{rank}(\operatorname{syz}_2(M)) \geq \dim R - 1.$$

Corollary 20. *Suppose R is a hypersurface ring with an isolated singularity and M is a nonfree maximal Cohen-Macaulay module. Then*

$$\operatorname{rank} M \geq (\dim R + 1)/2.$$

Eisenbud: $M \cong \operatorname{syz}_2(M)$.

Corollary 21. *R hypersurface. If R has a rank 1 nonfree maximal Cohen-Macaulay module, then $\operatorname{codim} \operatorname{Sing}(R) \leq 3$.*

Grothendieck: If R is a complete intersection such that $\operatorname{codim} \operatorname{Sing} R \geq 4$, then R is factorial.

Phil's most recent paper: *Approximate liftings in local algebra and a theorem of Grothendieck.* J. Pure Appl. Algebra **196**, 185–202 (2005).

References

The general reference is

W. Bruns and J. Herzog, *Cohen-Macaulay rings, rev. ed.* Cambridge University Press 1998

In particular, see sections 9.5 and 9.6.

The results on pp. 18 and 19 are based on

W. Bruns, *The Eisenbud-Evans generalized principal ideal theorem and determinantal ideals*. Proc. Amer. Math. Soc. **83**, 19–24 (1981).