Stanley decompositions and Hilbert depth in the Koszul complex

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Kingston, September 27, 2010
Joint work with

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to appear in J. Commut. Algebra (Fröberg volume)
The framework

We consider

- \( R = K[X_1, \ldots, X_n], K \) a field, with the
  - standard grading, \( \deg X_i = 1 \), indicated by subscript 1,
  - the multigrading, \( \deg X_i = e_i \in \mathbb{Z}^n \), indicated by subscript \( n \),

- a finitely generated graded \( R \)-module.

The Hilbert function of \( M \) is denoted by

\[
H(M, k) = \dim_K M_k, \quad k \in \mathbb{Z}^m, \quad m = 1 \text{ or } m = n.
\]
The Hilbert series are defined by

\[ H_M(T) = \sum_{k \in \mathbb{Z}} H(M, k) T^k, \]

\[ H_M(T_1, \ldots, T_n) = \sum_{k \in \mathbb{Z}^n} H(M, k) T_1^{k_1} \cdots T_n^{k_n}, \]

They are rational functions:

\[ H_M(T) = \frac{Q_M(T)}{(1 - T)^n}, \]

\[ H_M(T_1, \ldots, T_n) = \frac{Q_M(T_1, \ldots, T_n)}{(1 - T_1) \cdots (1 - T_n)}, \]

where \( Q_M(T) \in \mathbb{Z}[T^{\pm 1}] \) and \( Q_M(T_1, \ldots, T_n) \in \mathbb{Z}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}] \).
A *Stanley decomposition* of $M$ is a finite family

$$\mathcal{D} = (S_i, x_i)_{i \in I}$$

in which $x_i$ is a homogeneous element of $M$ and $S_i$ is a graded $K$-algebra retract of $R$ for each $i \in I$ such that $S_i \cap \text{Ann } x_i = 0$, and

$$M = \bigoplus_{i \in I} S_i x_i \quad (= \mathcal{D})$$

as a graded $K$-vector space.

The *Stanley depth* $\text{Stdepth } M$ of $M$ is the maximal depth of a Stanley decomposition of $M$. 
Definition

A Hilbert decomposition is a finite family

\[ \mathcal{H} = (S_i, s_i)_{i \in I} \]

such that \( s_i \in \mathbb{Z}^m \) (where \( m = 1 \) or \( m = n \)), \( S_i \) is a graded \( K \)-algebra retract of \( R \) for each \( i \in I \), and

\[ M \cong \bigoplus_{i \in I} S_i(-s_i) \quad (= \mathcal{H}) \]

as a graded \( K \)-vector space.

Definition

The Hilbert depth \( \text{Hdepth} M \) of \( M \) is the maximal depth of a Hilbert decomposition of \( M \).
Suppose $\mathcal{H} = (S_i, s_i)_{i \in I}$ is a Hilbert decomposition. Then

$$H_M(T_1, \ldots, T_n) = \sum_i H_{S_i}(-s_i)(\ldots) = \sum_i \frac{T_1^{s_{i1}} \cdots T_n^{s_{in}}}{\prod_{j \in U_i} (1 - T_j)}.$$ 

Thus $\text{depth } \mathcal{H} \geq d \iff |U_i| \geq d$ for all $i$.

This amounts to the following for the numerator polynomial: It can be written as a sum of terms

$$\text{monomial} \cdot \prod_{k \in V_i} (1 - T_k)$$

with $U_i \cup V_i = \{1, \ldots, n\}$, $U_i \cap V_i = \emptyset$.

Thus $\text{depth } \mathcal{H} \geq d \iff |U_i| \leq n - d$ for all $i$. 
Stanley depth and Hilbert depth

Conjecture (Stanley)

\[ \text{Stdepth} \ M \geq \text{depth} \ M \]

Since a Stanley decomposition is a Hilbert decomposition:

\[ \text{Stdepth}_n \ M \leq \begin{cases} 
\text{Hdepth}_n \ M \\ 
\text{Stdepth}_1 \ M 
\end{cases} \leq \text{Hdepth}_1 \ M \]

Strategy:

- compute a Hilbert decomposition,
- convert it into a Stanley decomposition.

Alas: does not help in the case of fine gradings (\( \dim M_k \leq 1 \) for all \( k \)).

Question

\[ \text{Hdepth} \ M \geq \text{depth} \ M \quad ?? \]
Stanley’s conjecture was a theorem in the standard graded case for $|K| = \infty$ (Baclawski and Garsia). Moreover, Hilbert depth can be computed “easily” in the standard graded case:

**Theorem (Uliczka)**

Then the following numbers coincide:

1. $\max\{\text{depth } N : H_M(T) = H_N(T)\}$,
2. $H_{\text{depth}_1} M$,
3. the maximum $d$ such that $H_M(T) = \sum_{e=d}^{n} \frac{Q_e(T)}{(1 - T)^e}$, $Q_e(T) \in \mathbb{Z}_+[T, T^{-1}]$,
4. $\max\{p : (1 - T)^p H_M(T) \text{ positive}\}$,
5. $n - \min\{q : Q_M(T)/(1 - T)^q \text{ positive}\}$.
The Koszul complex

\[ \mathcal{K}(X_1, \ldots, X_n; R) : 0 \to \bigwedge^n R^n \xrightarrow{\partial} \bigwedge^{n-1} R^n \xrightarrow{\partial} \ldots \xrightarrow{\partial} R^n \xrightarrow{\partial} R \to 0 \]

resolves \( K \cong R/\mathfrak{m}, \mathfrak{m} = (X_1, \ldots, X_n) \).

Let \( M(n, k) \) be the \( k \)-th syzygy module of \( K \) \hspace{1em} (M(n, 1) = \mathfrak{m}).

The basis vector \( e_{i_1} \wedge \cdots \wedge e_{i_k} \) of \( \bigwedge^k R^n \), \( i_1 < \cdots < i_k \), has multidegree \( X_{i_1} \cdots X_{i_k} \) and standard degree \( k \).

**Lower half:** \( 1 \leq k < \lfloor n/2 \rfloor \). \hspace{1em} **Upper half:** \( \lfloor n/2 \rfloor \leq k < n \).
Theorem (Biró et al.)

\[ \text{Hdepth}_1 \ m = \text{Hdepth}_n \ m = \text{Stdepth}_n \ m = \left\lfloor \frac{n + 1}{2} \right\rfloor. \]

Induction arguments yield

Corollary

\[ \text{Stdepth}_n \ M(n, k) \geq \left\lfloor \frac{n + k}{2} \right\rfloor. \]

However,

Theorem

In the upper half

\[ \text{Hdepth}_1 \ M(n, k) = \text{Hdepth}_n \ M(n, k) = n - 1. \]
Proof.

Multigraded numerator polynomial of Hilbert series of $M(n, k)$ is

$$Q(n, k) = \sigma_{n,k} - \sigma_{n,k+1} + \cdots + (-1)^{n-k}\sigma_{n,n}$$

where $\sigma_{n,k}$ is the $k$-th elementary symmetric polynomial in $T_1, \ldots, T_n$.

For $j \geq \lfloor n/2 \rfloor$ one has an injective map from the monomials in $\sigma_{n,j+1}$ to those in $\sigma_{n,j}$ that maps each monomial to a divisor! Apply this with $j = k, k + 2, \ldots$ Consequence:

$Q(n, k)$ can be written as a sum of terms of type

- monomial
- $(1 - T_i) \cdot$monomial
The case \( k = n - 3 \)

For \( n = 5, k = 2 \) we have realized the Hilbert decomposition as a Stanley decomposition. By induction

**Proposition**

\[
\text{Stdepth}_n M(n, n - 3) = n - 1.
\]

**Question**

Is \( \text{Stdepth} M(n, k) = n - 1 \) in the upper half?

It is enough to do the case \( n \) odd, \( k = (n - 1)/2 \).
The numerator polynomial of the standard Hilbert series of $M(n, k)$

\[
\binom{n}{k} T^k - \binom{n}{k+1} T^{k+1} + \cdots + (-1)^{n-k} T^n
\]

How often do we have so sum (i.e. divide by $1 - T$) to get a positive power series? At least

\[
v = \left\lfloor \binom{n}{k+1} / \binom{n}{k} \right\rfloor \text{ times}
\]

**Proposition**

\[
\text{Hdepth}_1 M(n, k) \leq n - \left\lfloor \frac{n - k}{k + 1} \right\rfloor.
\]

Is this naive bound sharp? Yes... for $n \leq 22$, but not for $n = 23$, $k = 3, 4, 5$. (It is sharp for the powers of $m$.)
The numerator polynomial of the standard Hilbert series of $M(n, k)$

$$\binom{n}{k} T^k - \binom{n}{k+1} T^{k+1} + \cdots + (-1)^{n-k} T^n$$

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The numerator polynomial of the standard Hilbert series of $M(n, k)$

$\binom{n}{k} T^k - \binom{n}{k+1} T^{k+1} + \cdots + (-1)^{n-k} T^n$

How often do we have so sum (i.e. divide by $1 - T$) to get a positive power series? At least

$v = \left\lceil \frac{n}{k+1} \right\rceil / \binom{n}{k}$ times

**Proposition**

$$Hdepth_1 M(n, k) \leq n - \left\lfloor \frac{n-k}{k+1} \right\rfloor.$$  

Is this **naive bound** sharp? Yes... for $n \leq 22$, but not for $n = 23$, $k = 3, 4, 5$. (It is sharp for the powers of $m$.)
Proposition

Let $Q_{n,k}$ be the numerator polynomial of the $\mathbb{Z}$-graded Hilbert series of $M(n, k)$. Then

$$
\frac{Q_{n,k}}{(1 - T)^s} = \sum_{j=0}^{\infty} \left( (-1)^j \binom{n - s}{k + j} + \sum_{t=1}^{s} \binom{n - t}{k - 1} \binom{s - t + j}{s - t} \right) T^{j+k}
$$

Hypergeometric sum . . . E-mail to Krattentahler: help!!!!

Answer: not summable since of type $3F_2$. No explicit bound possible.
Proposition

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Answer: not summable since of type $3F_2$. No explicit bound possible.
Asymptotic estimate I

**Theorem**

For a fixed positive integer $k$, we have

$$H_{\text{depth}_1} M(n, k) = \frac{1}{2} n + \frac{1}{2} \sqrt{(k - 1)n \log n}$$

$$+ \frac{1}{4} \sqrt{\frac{(k - 1)n}{\log n}} \log \log n + o \left( \sqrt{\frac{n}{\log n}} \log \log n \right),$$

as $n \to \infty$.

Without the $o$-term an upper bound for $k \geq 4$, $n \gg 0$.

We have another asymptotic result for $n, k \to \infty$, $n/k$ fixed.
The quality of the asymptotic estimate is indicated by the following numerical results:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$k$</th>
<th>$\left\lfloor \frac{n + k}{2} \right\rfloor$</th>
<th>Hdepth</th>
<th>asympt bd</th>
<th>$n - \left\lfloor \frac{n-k}{k+1} \right\rfloor$</th>
</tr>
</thead>
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<td>6</td>
<td>28</td>
<td>41</td>
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<td>6</td>
<td>503</td>
<td>602</td>
<td>605</td>
<td>858</td>
</tr>
</tbody>
</table>
Now we let $n$ and $k$ go to infinity simultaneously with $n/k$ fixed:

**Theorem**

Let $\beta$ be a positive real number with $\beta \leq 1/2$. For $k = \beta n + o(n)$, we have

$$Hdepth_1 M(n, k) = (1 - \gamma)n + o(n), \quad \text{as } n \to \infty,$$

where $\gamma$ is the smallest nonnegative solution of the equation

$$\frac{(\alpha + \gamma)(\alpha + \beta)^{\alpha + \beta} (1 - \alpha - \beta - \gamma)^{1-\alpha - \beta - \gamma}}{\alpha^\alpha \beta^\beta \gamma^\gamma (1 - \beta)^{1-\beta} (1 - \gamma)^{1-\gamma}} = 1,$$

with

$$\alpha = \frac{1}{4}(1 - 2\beta - 2\gamma + \sqrt{(1 - 2\beta - 2\gamma)^2 - 8\beta \gamma}).$$
Again, the quality of the asymptotic estimate is surprising:

<table>
<thead>
<tr>
<th>n</th>
<th>k</th>
<th>(\frac{(n + k)}{2})</th>
<th>Hdepth</th>
<th>asympt bd</th>
<th>(n - \left\lfloor \frac{(n-k)}{(k+1)} \right\rfloor)</th>
</tr>
</thead>
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<tr>
<td>1000</td>
<td>250</td>
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<td>960</td>
<td>962</td>
<td>997</td>
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