Products of Borel fixed ideals of maximal minors

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Products of Borel fixed ideals of maximal minors (arXiv:1601.03987)

Linear resolutions of powers and products (arXiv:1602.07996)
Northeast ideals of maximal minors

Let \( K \) be a field, \( R = K[x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n] \), \( X = (x_{ij}) \).

For \( a + t \leq n + 1 \), the ideal \( I_t(a) \) is generated by the \( t \)-minors of the northeast submatrix

\[
X_t(a) = (x_{ij} : 1 \leq i \leq t, \ a \leq j \leq n).
\]

\( I_t(a) \) is a (row oriented) northeast ideal of maximal minors.
All $t$-minors involved have the first $t$ rows of $X$ as their rows. Therefore we only need the column indices to denote them:

$$[a_1 \ldots a_t] = \det(X_{ia_i} : i = 1, \ldots, t).$$

We assume $a_1 < \ldots a_t$.

For example, $[1 \ldots t]$ is just the determinant of the $t \times t$-submatrix in the left upper corner of $X$. $l_t(a)$ is generated by all minors $[a_1 \ldots a_t]$ with $a_1 \geq a$.

Products of minors will be called tableaux.

The essential point in working with ideals of minors is to single out a class of $K$-linearly independent tableaux. Usually one chooses the standard (bi)tableaux, but they are not sufficient for our results.
Borel fixedness, diagonal monomial orders

\[ B_m \subset \text{GL}(m.K) \text{ Borel subgroup of lower triangular matrices}, \]
\[ B'_n \subset \text{GL}(n, K) \text{ Borel subgroup of upper triangular matrices}. \]

\[ I_t(a) \text{ fixed by the action of } B_m \times B'_n \text{ on } R \text{ via linear substitution: it is a Borel fixed ideal of maximal minors}. \]

Why northeast? Because we want to work with a diagonal monomial order for which the leading monomial of a minor is the product of the diagonal elements, for example the lex order induced by

\[ X_{11} > \cdots > X_{1n} > X_{21} > \cdots > X_{2n} > \cdots > X_{mn}. \]

(Used in proofs.) We fix a diagonal monomial order.

In a diagonal monomial order we have

\[ \text{in}([a_1 \ldots a_t]) = X_{1a_1} \cdots X_{ta_t}. \]
The main theorem

Theorem

Let $I_{t_1}(a_1), \ldots, I_{t_w}(a_w)$ be northeast ideals of maximal minors, and let $I$ be their product. Then

1. $I$ has a linear resolution.

2. $\text{in}(I) = \text{in}(I_{t_1}(a_1)) \cdots \text{in}(I_{t_w}(a_w))$, and the natural generators of $I$ form a Gröbner basis.

3. $I$ is integrally closed, and it has a primary decomposition by powers of ideals $I_t(a)$ for various values of $t$ and $a$.

Moreover, the multi-Rees algebra associated to the family of ideals $I_t(a)$ is Koszul, Cohen-Macaulay and normal.

In view of statement (1) we say, that the family $(I_t(a))$ has linear products.
The archetype

**Theorem**

1. *The powers of $I_m(X)$ have a linear resolution* (Akin-Buchsbaum-Weyman): $I_m(X)$ has **linear powers**.
2. They are primary and integrally closed (Trung).
3. $\text{in}(I_m(X)^k) = \text{in}(I_m(X))^k$ for all $k$, and the natural generators of $I_m(X)^k$ form a Gröbner basis (Conca).
4. *The Rees algebra of $I_m(X)$ is Cohen-Macaulay normal* (Eisenbud-Huneke) and Koszul.

This is everything we want to prove for the powers of a single $I_t(a)$.

Intermediate step: products $I_{t_1}(1) \cdots I_{t_w}(1)$ (Berget-B-Conca).

Roughly speaking, **standard bitableaux** and the **KRS correspondence** are enough for this case. The general case of northeast ideals is combinatorially harder.
Remarkable classes of ideals

All the following classes generalize the class of principal stable monomial ideals:

1. polymatroidal (monomial) ideals (Herzog, Hibi, Vladoiu)
2. ideals generated by linear forms (Conca-Herzog)
3. northeast ideals of maximal minors

They share the following properties:

1. linear products
2. “good” primary decompositions of products by ordinary powers of primes (with multiplicities given by valuations)
3. normal and Cohen-Macaulay multi-Rees algebras of “fiber type” defined by low degree relations.
The crucial intersection formulas

Let $S = ((t_1, a_1), \ldots, (t_w, a_w))$. Set $J_t(a) = \text{in}(l_t(a))$,

$$l_S = l_{t_1}(a_1) \cdots l_{t_w}(a_w) \quad \text{and} \quad J_S = \text{in}(l_S).$$

Note: $b \leq a_i$ and $u \leq t_i \iff l_{t_i}(a_i) \subset l_u(b)$. Set

$$e_{ub}(S) = |\{i : b \leq a_i \text{ and } u \leq t_i\}|.$$

**Theorem**

$$J_{t_1}(a_1) \cdots J_{t_w}(a_w) = J_S = \bigcap_{u,b} J_u(b)^{e_{ub}(S)}$$

(1)

$$l_S = \bigcap_{u,b} l_u(b)^{e_{ub}(S)}.$$  

(2)

Equation (2) gives a primary decomposition of $l_S$. The ideals $l_S$ and $J_S$ are integrally closed.
The crucial inclusion is

$$\bigcap_{u,b} J_u(b)^{e_{ub}(S)} \subset J_{t_1}(a_1) \cdots J_{t_w}(a_w).$$

Everything else follows from easy arguments.

The inclusion means: a monomial that contains $e_u(b)$ diagonals

$$X_{1j_1} \cdots X_{uj_u}, \quad b \leq j_1 < \cdots < j_u \leq n,$$

of “type $(u, b)$” for all $u$ and $b$ can be factored in a “NE-canonical” way (depending on $S$!) that fits the decomposition $J_{t_1}(a_1) \cdots J_{t_w}(a_w)$.

It leads to a “NE canonical” representation “of pattern $S$” of elements in $I_{t_1}(a_1) \cdots I_{t_w}(a_w)$, generalizing the straightening law. Standard tableaux are generalized to “NE-canonical tableaux of pattern $S$”.

Winfried Bruns

Products of Borel fixed ideals of maximal minors
The NE canonical factorization lets us find a NE-canonical tableaux $\Delta \in I_S$ for a given monomial $M$ such that $M = \text{in}(\Delta)$.

Consider $M = x_{11}x_{12}x_{13}x_{23}x_{24}x_{25}x_{35}$, symbolized by the table

```
•   •   •   •   •   •
   •   •   •   •   •
   •   •   •   •   •
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It depends on the pattern $S$ which $S$-canonical tableau has $M$ as its initial monomial.

1. For $S = ((2, 1), (3, 2), (2, 2))$ the canonical tableau with initial monomial $M$ is

$$[13][245][35].$$

2. For $S = ((2, 1), (2, 2), (3, 3))$ it is

$$[13][25][345].$$
The NE straightening law

**Theorem**

Let $S = ((t_1, a_1), \ldots, (t_w, a_w))$ be a NE-pattern and $x \in I_S$. then there exist uniquely determined $S$-canonical NE-tableaux $M_i \Gamma_i$, $i = 0, \ldots, p$, and coefficients $\lambda_i \in K$ such that

$$x = \lambda_0 M_0 \Gamma_0 + \lambda_1 M_1 \Gamma_1 + \cdots + \lambda_p M_p \Gamma_p$$

and

$$\text{in}(x) = \text{in}(M_0 \Gamma_0) > \text{in}(M_1 \Gamma_1) > \cdots > \text{in}(M_p \Gamma_p).$$
The multi-Rees algebra

The natural object for the simultaneous investigation of the products $I_{t_1}(a_1) \cdots I_{t_w}(a_w)$ is the multi-Rees algebra defined by the ideals $I_t(a)$:

$$\mathcal{R} = R(I_t(a) : \text{all } (t, a))$$

$$= R[I_t(a)T_{ta} : \text{all } (t, a)] \subset R[T_{ta} : \text{all } (t, a)]$$

It is naturally $\mathbb{Z}^{1+N}$-graded, $N = \#\{\text{all } (t, a)\}$. Since every $I_t(a)$ is generated in a single degree, it is also naturally $\mathbb{Z}$-graded.

It is useful to define partial Castelnuovo-Mumford regularities with respect to the $1+N$ partial degrees. We are mainly interested in the 0-th partial degree coming from $R$ and the corresponding regularity $\text{reg}_0$.
The theorems of Blum and Römer

**Theorem**

Let $R$ be a standard graded polynomial ring over the field $K$. The family $I_1, \ldots, I_w$ of ideals in $R$ has linear products if and only if $\text{reg}_0(R(I_1, \ldots, I_w)) = 0$.

Implication $\Leftarrow$ due to Römer, $\Rightarrow$ by B-Conca-Varbaro.

**Theorem**

Let $R = K[X_1, \ldots, X_n]$ and $I_1, \ldots, I_w$ ideals of $R$ such that $R(I_1, \ldots, I_w)$ is Koszul. Then the family $I_1, \ldots, I_w$ has linear products.

Due to Blum. His theorem actually says more. Roughly speaking, diagonal submodules over diagonal subalgebras of multigraded Koszul algebras have linear resolutions.
Extend the monomial order from $R$ to

$$R[T_{ta} : \text{all } (t, a)] = K[X, T_{ta} : \text{all } (t, a)].$$

As a subalgebra of this polynomial ring, $\mathcal{R}$ has a well-defined initial subalgebra $\text{in}(\mathcal{R})$ (generated by a Sagbi basis).

Recall that

$$\text{in}(l_{t_1}(a)) \cdots \text{in}(l_{t_w}(a_w)) = \text{in}(l_{t_1}(a)) \cdots l_{t_w}(a_w))$$

This implies

$$\text{in}(\mathcal{R}) = R(\text{in}(l_t(a))) : \text{all } (t, a))$$

**Theorem**

1. $\text{in}(\mathcal{R})$ and $\mathcal{R}$ are normal.
2. Both are Cohen-Macaulay.
Linear resolutions

Write $\mathcal{R}$ (and/or $\text{in}(\mathcal{R})$) as a residue class ring of a polynomial ring $\mathcal{S}$ over $K$:

$$\Phi : \mathcal{S} \to \mathcal{R}.$$ 

The “NE straightening law” fits a monomial order on $\mathcal{S}$ that is lifted from $\mathcal{R}$ via $\Phi$ with a reverse-lexicographic “tie breaker”.

Thus we get

Theorem

1. $\text{in}(\mathcal{R})$ and $\mathcal{R}$ are defined by Gröbner bases of quadrics.
2. Both are Koszul algebras.
3. All products $I_{t_1}(a_1) \cdots I_{t_w}(a_w)$ and their initial ideals have linear resolutions.

The essential point: the rewriting of the initial of a tableau in NE canonical decomposition can be done in steps representing degree 2 relations.
Theorem

Let \( I_{t_1}(a_1) \subset I_{t_2}(a_2) \subset \cdots \subset I_{t_p}(a_p) \) such that \( \text{ht} \ I_{t_1}(a_1) = 1 \) or \( 2 \) and \( \text{ht} \ I_{t_i}(a_i) = 1 + \text{ht} \ I_{t_{i-1}}(a_{i-1}) \) for \( i = 2, \ldots, p \).

Then the multi-Rees algebra \( R(I_{t_1}(a_1), \ldots, I_{t_p}(a_p)) \) is Gorenstein and normal with divisor class group \( \mathbb{Z}^{p-1} \) or \( \mathbb{Z}^p \), depending on whether \( a_1 = n - t_1 \) or \( a_1 = n - t + 1 \).


Theorem

Let \( t_1 < \cdots < t_p \) and \( a_1 \geq \cdots \geq a_p \) and \( I_i = I_{t_i}(a_i) \) for \( i = 1, \ldots, p \). Then the multi-fiber ring \( F(I_1, \ldots, I_p) \) is factorial and therefore Gorenstein.