Normal lattice polytopes

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Joint work with

Joseph Gubeladze (San Francisco, since 1995)

and

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A lattice polytope in $\mathbb{R}^d$ is the convex hull of finitely many lattice points:

We set $L(P) = P \cap \mathbb{Z}^d$.

$$cP = P + \cdots + P \quad (c \text{ summands})$$

$P$ is normal if

$$L(cP) = L(P) + \cdots + L(P) \quad (c \text{ summands})$$

This equation means: normal lattice polytopes are the discrete analog of convex polytopes.

Other terminology: $P$ has the integer decomposition property, $P$ is integrally closed.
The cone $C(P)$ is generated by $P' = P \times \{1\} \subset \mathbb{R}^{d+1}$:

The monoid $C(P) \cap \mathbb{Z}^{d+1}$ is finitely generated by Gordan’s lemma.

**Proposition**

$P$ is normal $\iff$ $L(P')$ generates $C(P) \cap \mathbb{Z}^{d+1}$.

Easy observation: $P = Q_1 \cup \cdots \cup Q_n$, $Q_i$ normal $\implies$ $P$ is normal

Normality $\implies$ $\mathbb{Z}L(P') = \mathbb{Z}^{d+1}$.
A lattice \textit{d-simplex} is a lattice polytope \(S\) of dimension \(d\) with \(d + 1\) vertices \(v_0, \ldots, v_d\): a triangle, a tetrahedron, \ldots

\(S\) is unimodular \(\iff\) \(v_1 - v_0, \ldots, v_d - v_0\) are a \(\mathbb{Z}\)-basis of \(\mathbb{Z}^d\)
\(\iff\) \(\text{vol}(S) = 1/d!\).

A unimoduler simplex is evidently normal.

\(S\) is \textit{empty} if the vertices are the only lattice points of \(S\).

\textbf{Theorem}

\begin{enumerate}
\item An empty 2-simplex is unimodular.
\item Every lattice 2-polytope is normal.
\end{enumerate}

\textbf{Proof.}

(1) By Pick’s formula, an empty 2-simplex has area 1/2.
(2) Every lattice polytope has a triangulation into empty simplices.

In general \((d - 1)P\) is always normal, \(d = \dim P\).
Quantum jumps and partial order

Let $\text{NPol}(d)$ be the set of normal lattice $d$-polytopes.

A pair $(P, Q)$, $P, Q \in \text{NPol}(d)$, is a quantum jump if $P \subset Q$ and $\#L(Q) = 1 + \#L(P)$.

$P < Q \iff$ there are $Q_0, \ldots, Q_n$ in $\text{NPol}(d)$ such that $P = Q_0, \ldots, Q_n = Q$ and $(Q_i, Q_{i+1})$ a quantum jump for every $i$.

**Question**

*Is $<$ the same as $\subset$? Do there exist nontrivial minimal and/or maximal elements in $\text{NPol}(d)$?*

**Theorem**

*In $\text{NPol}(2)$ one has $P < Q \iff P \subset Q$.*

In dim $> 3$ there exist nontrivial minimal elements: $< \neq \subset$.

Dim 3: nontrivial minimal elements?? Nevertheless $< \neq \subset$. 

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Normal lattice polytopes
Characterizations of normality?

**Definition**

A lattice $d$-polytope $P$ has

(UC) $\iff$ $P$ is the union of unimodular simplices (Unimodular Cover);

(ICP) $\iff$ $\mathbb{Z}L(P') = \mathbb{Z}^{d+1}$ and for every $x \in \mathbb{Z}_+ L(P')$ there exist $x_1, \ldots, x_{d+1} \in L(P')$, such that $x = a_1 x_1 + \cdots + a_{d+1} x_{d+1}$, $a_i \in \mathbb{Z}_+$ (Integral Carathéodory Property).

Clearly: (UC) $\implies$ (ICP) & normality.

More difficult: (ICP) $\implies$ normality (B.-Gubeladze)

**Question (Sebő)**

Does normality imply (UC) or at least (ICP) ?
No characterization of normality

**Theorem**

Suppose $P$ is a counterexample of minimal dimension $d$ to (UC) or (ICP). Then there exists a descending chain in $\text{NPol}(d)$ down from $P$ to a minimal counterexample with respect to $<$.  

So, to find a counterexample, start from some “randomly” chosen polytope, descend from $P$ in $\text{NPol}(d)$ and hope to end at a counterexample. This search strategy was indeed successful:

There exist normal 5-polytopes $P_5$ and $Q_5 \supset P_5$ with 10 and 12 lattice points, resp., both minimal in $\text{NPol}(5)$, such that

- (B.-Gubeladze, 1998) $P_5$ fails (UC),
- (Henk-Martin-Weismantel) $P_5$ fails (ICP),
- (B.,2006) $Q_5$ fails (UC), but has (ICP).
Open problems on covering and normality

The coordinates of $P_5$:

$$z_1 = (0, 1, 0, 0, 0, 0), \quad z_6 = (1, 0, 2, 1, 1, 2),$$
$$z_2 = (0, 0, 1, 0, 0, 0), \quad z_7 = (1, 2, 0, 2, 1, 1),$$
$$z_3 = (0, 0, 0, 1, 0, 0), \quad z_8 = (1, 1, 2, 0, 2, 1),$$
$$z_4 = (0, 0, 0, 0, 1, 0), \quad z_9 = (1, 1, 1, 2, 0, 2),$$
$$z_5 = (0, 0, 0, 0, 0, 1), \quad z_{10} = (1, 2, 1, 1, 2, 0).$$

**Question**

1. Does (UC) or at least (ICP) hold in dimension 3 or 4?
2. Does every counterexample inherit the failure of (UC) or (ICP) from $P_5$?
**The height of quantum jumps**

**Question**

*Do maximal polytopes exist?*

Let \((P, Q)\) be a quantum jump, \(z \in L(Q) \setminus L(P)\). Then \(z\) is also called a quantum jump over \(P\). How far can it be from \(P\)? How close is the nearest jump?

Let \(F\) be a facet of \(P\). Then \(ht_F : \mathbb{Z}^d \rightarrow \mathbb{Z}\) is the unique surjective affine linear function that vanishes on \(F\) and is \(\geq 0\) on \(L(P)\).

\(F\) is **visible** from \(z\) if \(ht_F(z) < 0\).

\[ ht_P(z) = \max_F \text{ visible} \{|ht_F(z)|\} \]

\[ \text{width}_F P = \max_{x \in P} \{ht_F(x)\} \]
In dimension 2 the situation is again simple, at least globally:

**Proposition**

1. If $\text{ht}_P(z) = 1$, then $z$ is a jump over $P$.
2. The converse holds in dimension 2.

Locally the situation is more complicated, even in dimension 2. Let us say that vertex $x \in P$ is **dark** if it is not visible from a jump.

**Proposition**

*For every $n$ there exists a 2-polytope $P$ with $n$ adjacent dark vertices.*
The dashed lines indicate ht $-1$ over the facets parallel to them. A point illuminating the origin must have coordinates $(-1, m)$ or $(m, -1)$. Each of them is excluded by one of the other facets.
Theorem

For every \( n \) there exists a normal 3-polytope \( P \) such that

1. there is no lattice point of height \( < n \) over \( P \);
2. there exists a jump \( z \) of height \( n \) over \( P \).

One can take the cross-polytope with half axes \( n, n + 1, n^2 + n + 1 \): 

![Cross-polytope diagram](image-url)
Theorem

$P \subset Q$ lattice 3-polytopes, $P \in \text{NPol}(3)$, $\#L(Q) = \#L(P) + 1$, $z \in L(Q) \setminus L(P)$. Then the following are equivalent:

1. $z$ is a quantum jump over $P$.
2. For each facet $F$ of $P$ visible from $z$, $P_{z,F}$ contains exactly $\mu(F)$ lattice points $y$ such that $ht_F(y) = j$, $1 \leq j < |ht_F(z)|$.

$\mu(F) = \text{multiplicity of } F$
$= \text{lattice normalized volume}$
The height bound

**Theorem**

Let $z$ be a quantum jump over $P$. Then

$$| \text{ht}_F(z) | \leq 1 + (d - 2) \text{width}_F P$$

for every facet $F$ of $P$ that is visible from $z$.

**Theorem**

For all $d \geq 2$ and $w \geq 1$ there exists a quantum jump $(P, Q)$ in $\text{NPol}(d)$ such that:

1. $z \in \text{L}(Q) \setminus \text{L}(P)$ is visible from exactly one facet $F \subset P$,
2. $\text{width}_F P = w$,
3. $| \text{ht}_F(z) | = 1 + (d - 2)w$. 
A jump of extreme height

\[ P = \text{conv}(0, \mathbf{e}_1, \ldots, \mathbf{e}_{d-1}, -w\mathbf{e}_d), \quad z = (1, \ldots, 1, (d - 2)w + 1) \]

\[ d = 3, \ w = 1: \]
Maximal polytopes

We are not sure in dimension 3, but there is a good chance that maximal polytopes do not exist:

**Theorem**

*No simplex in dimension 3 is maximal.*

We have no construction that produces a maximal polytope in dimension $d + 1$ from one in dimension $d$, but there is no doubt that maximal polytopes exist in dimensions $\geq 4$:

**Theorem**

*There exist maximal polytopes, even simplices, in dimensions 4 and 5.*

The theorem is based on examples found by a computer search.
A maximal 4-simplex

The simplex with vertices

\[(0, 3, 2, 0) \ (1, 1, 3, 2) \ (2, 3, 0, 4) \ (4, 0, 0, 2) \ (4, 4, 4, 2)\]

is maximal. In order to verify maximality one computes all 125852 lattice points satisfying the height bound, and checks that none of them is a jump. This takes about 2 minutes.

Our program quantum does the search and verifies that a potentially maximal polytope is indeed maximal. It uses the library interface of Normaliz.
Search strategies

After long experimenting we found two successful strategies:

1. Start from a “random” normal polytope and extend it successively in such a way that the new polytope has a chance to be maximal. Stop when a maximal polytope is reached or some size bound is reached, and start again.

2. Make a “random” simplex and test it for maximality. If it is not maximal, test the next simplex.

It was a complete surprise that (2) works. We tried it after (1) had found a maximal 5-simplex in dimension 5. One must test MANY polytopes and (2) is fast: mass production beats sophistication.

For (1), all “pure” extension strategies have failed. The following has turned out optimal: if $P$ allows a height 1 jump, take it. If not take the jump for which the new polytope maximizes the average facet multiplicity. Also successful: maximize $\text{vol}(Q \setminus P)$.
Final questions

**Question**

*Do there exist maximal and nontrivial minimal elements in \(\text{NPol}(3)\)?*

**Question**

*Do there exist isolated points in \(\text{NPol}(d)\), i.e., polytopes that are both minimal and maximal?*