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Additions to the Theory of Algebras with Straightening Law

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In this article we want to supplement the theory of algebras with straightening law, ASLs for short, by two additions. The first addition concerns the arithmetical rank of ideals generated by an ideal of the poset underlying the ASL. (The arithmetical rank is the least number of elements generating an ideal up to radical.) It turns out that there is a general upper bound only depending on the combinatorial data of the poset. In particular we discuss ideals generated by monomials and show that the ideas leading to the general bound can be used to derive sharper results in this special case. On the other hand there exists a class of ASLs, called symmetric, in which the general bound is always precise. This class includes the homogeneous coordinate rings of Grassmannians, and perhaps the most prominent result in this context is the determination of the least number of equations defining a Schubert subvariety of a Grassmannian.

As a second addition we introduce the notion of a module M with straightening law over an ASL A. Such a module has a partially ordered set of generators, a basis of standard elements each of which is a product of a standard monomial and a generator, and finally the multiplication $A \times M \to M$ satisfies a straightening law similar to the straightening law in A. We discuss some examples, among them the powers of certain ideals generated by poset ideals and the generic modules. Remarkable facts: (i) the first syzygy of a module with straightening law has itself a straightening law, and (ii) the existence of natural filtrations which, for example, lead to a lower bound on the depth of a module with straightening law. In the last part we introduce a natural strengthening of the axioms which under special conditions leads to a straightening law on the symmetric algebra of the module. The most interesting examples to which we will apply this result are the generic modules.

The theory of ASLs has been developed in [**Ei**] and [**DEP.2**]; the treatment in [**BV.1**] also satisfies our needs. For the readers convenience we have collected the definition, results relevant for us, and some significant examples in the first section. Matsumura's book [**Ma**] may serve as a source for the commutative ring theory needed in this article.

1. Algebras with Straightening Laws.

An algebra with straightening law is defined over a ring B of coefficients. In order to avoid problems of secondary importance in the following sections we will assume throughout that B is a noetherian ring.

DEFINITION. Let A be a B-algebra and $\Pi \subset A$ a finite subset with partial order $\leq$. A is an *algebra with straightening law on* Π *(over* B*)* if the following conditions are satisfied:
(ASL-0) $A = \bigoplus_{i \geq 0} A_i$ is a graded B-algebra such that $A_0 = B$, Π consists of homogeneous elements of positive degree and generates A as a B-algebra.
(ASL-1) The products $\xi_1 \cdots \xi_m$, $m \geq 0$, $\xi_1 \leq \cdots \leq \xi_m$ are a free basis of A as a B-module. They are called *standard monomials.*
(ASL-2) (*Straightening law*) For all incomparable $\xi, \upsilon \in \Pi$ the product $\xi\upsilon$ has a representation

$$\xi\upsilon = \sum a_\mu \mu, \qquad a_\mu \in B, a_\mu \neq 0, \quad \mu \quad \text{standard monomial,}$$

satisfying the following condition: every μ contains a factor $\zeta \in \Pi$ such that $\zeta \leq \xi$, $\zeta \leq \upsilon$. (It is of course allowed that $\xi\upsilon = 0$, the sum $\sum a_\mu \mu$ being empty.)

In [**Ei**] and [**BV.1**] B-algebras satisfying the axioms above are called graded ASLs, whereas in [**DEP.2**] they figure as graded ordinal Hodge algebras.

In terms of generators and relations an ASL is defined by its poset and the straightening law:

(1.1) PROPOSITION. *Let A be an ASL on Π. Then the kernel of the natural epimorphism*

$$B[T_\pi \colon \pi \in \Pi] \longrightarrow A, \qquad T_\pi \longrightarrow \pi,$$

is generated by the relations required in (ASL-2), *i.e. the elements*

$$T_\xi T_\upsilon - \sum a_\mu T_\mu, \qquad T_\mu = T_{\xi_1} \cdots T_{\xi_m} \quad \textit{if} \quad \mu = \xi_1 \cdots \xi_m.$$

See [**DEP.2**], 1.1 or [**BV.1**], (4.2).

(1.2) PROPOSITION. *Let A be an ASL on Π, and $\Psi \subset \Pi$ an ideal, i.e. $\psi \in \Psi$, $\phi \leq \psi$ implies $\phi \in \Psi$. Then the ideal $A\Psi$ is generated as a B-module by all the standard monomials containing a factor $\psi \in \Psi$, and*

$A/A\Psi$ is an ASL on $\Pi \setminus \Psi$ ($\Pi \setminus \Psi$ being embedded into $A/A\Psi$ in a natural way.)

This is obvious, but nevertheless extremely important. First several proofs by induction on $|\Pi|$, say, can be based on (1.2), secondly the ASL structure of many important examples is established this way.

For an element $\xi \in \Pi$ we define its rank by

$$\operatorname{rk} \xi = k \quad \iff \quad \begin{array}{l} \text{there is a chain } \xi = \xi_k > \xi_{k-1} > \cdots > \xi_1,\ \xi_i \in \Pi, \\ \text{and no such chain of greater length exists.} \end{array}$$

For a subset $\Omega \subset \Pi$ let

$$\operatorname{rk} \Omega = \max\{\operatorname{rk} \xi \colon \xi \in \Omega\}.$$

The preceding definition differs from the one in [**Ei**] and [**DEP.2**] which gives a result smaller by 1. In order to reconcile the two definitions the reader should imagine an element $-\infty$ added to Π, vaguely representing $0 \in A$.

(1.3) PROPOSITION. *Let A be an ASL on Π. Then*

$$\dim A = \dim B + \operatorname{rk} \Pi \quad \textit{and} \quad \operatorname{ht} A\Pi = \operatorname{rk} \Pi.$$

Here of course $\dim A$ denotes the Krull dimension of A and $\operatorname{ht} A\Pi$ the height of the ideal $A\Pi$. A quick proof of (1.3) may be found in [**BV.1**], (5.10).

In the context of ASLs A we denote the length of a maximal M-sequence in $A\Pi$, M a finitely generated A-module, by $\operatorname{depth} M$.

We list three important examples of ASLs to which we will pay special attention in the following sections.

(1.4) EXAMPLES. (a) In order to study ideals generated by square-free monomials in the indeterminates of the polynomial ring $B[X_1, \ldots, X_n]$ one chooses Π as the set of all square-free monomials ordered by:

$$\xi \leq \upsilon \quad \iff \quad \upsilon \text{ divides } \xi.$$

(ASL-0) is satisfied for trivial reasons, and (ASL-1) holds since the standard monomials correspond bijectively to the ordinary monomials in $X_1, \ldots, X_n$. The straightening law is given by

$$\xi\upsilon = (\xi \sqcap \upsilon)(\xi \sqcup \upsilon)$$

where $\xi \sqcap \upsilon$ is the greatest common divisor and $\xi \sqcup \upsilon$ the least common multiple of ξ and υ. If $\Omega \subset \Pi$ is an arbitrary subset, then Ω and the smallest ideal $\Psi \supset \Omega$, $\Psi = \{\psi\colon \psi \le \omega$ for some $\omega \in \Omega\}$, generate the same ideal in $B[X_1, \dots, X_n]$, so the ideals generated by square-free monomials belong to the class covered by (1.2).

For a given poset Σ the *discrete ASL on* Σ is constructed as follows: One makes the polynomial ring $B[T_\sigma\colon \sigma \in \Sigma]$ an ASL as just described and passes to the residue class modulo the ideal generated by all products $T_\sigma T_\tau$, σ, τ incomparable. Thus one obtains an ASL on Σ in which the straightening law takes the special form $\sigma\tau = 0$ for all incomparable $\sigma, \tau \in \Sigma$.

(b) Let X be an $m \times n$ matrix of indeterminates over B, and $\mathrm{I}_t(X)$ denote the ideal generated by the t-minors (i.e. the determinants of the $t \times t$ submatrices) of X. For the investigation of the ideals $\mathrm{I}_t(X)$ and the residue class rings $\mathrm{R}_t(X) = B[X]/\mathrm{I}_t(X)$ one makes $B[X]$ an ASL on the set $\Delta(X)$ of all minors of X. Denote by $[a_1, \dots, a_t | b_1, \dots, b_t]$ the minor with row indices $a_1, \dots, a_t$ and column indices $b_1, \dots, b_t$. The partial order on $\Delta(X)$ is given by

$$[a_1, \dots, a_u | b_1, \dots, b_u] \le [c_1, \dots, c_v | d_1, \dots, d_v] \iff u \ge v \quad \text{and} \quad a_i \le c_i,\ b_i \le d_i,\ i = 1, \dots, v.$$

Then $B[X]$ is an ASL on $\Delta(X)$; cf. [**BV.1**], Section 4 for a complete proof. Obviously $\mathrm{I}_t(X)$ is generated by an ideal in the poset $\Delta(X)$, so $\mathrm{R}_t(X)$ is an ASL on the poset $\Delta_{t-1}(X)$ consisting of all the i-minors, $i \le t-1$.

(c) In the situation of (b) assume that $m \le n$. Then the B-subalgebra $\mathrm{G}(X)$ generated by the m-minors of X is a sub-ASL in a natural way, its poset being given by the set $\Gamma(X)$ of m-minors. This result is essentially due to Hodge. Again we refer to [**BV.1**], Section 4 for a proof. In denoting an m-minor we omit the row indices.

If $B = K$ is a field, then $\mathrm{G}(X)$ is the homogeneous coordinate ring of the Grassmannian $\mathrm{G}_m(K^n)$ of m-dimensional subspaces of the vector space K^n. The (special) Schubert subvariety $\Omega(a_1, \dots, a_m)$ of the Grassmannian is defined by the ideal generated by

$$\{\, \delta \in \Gamma(X)\colon \delta \not\ge [n+1-a_m, \dots, n+1-a_1] \,\},$$

an ideal in $\Gamma(X)$.

(d) Another example needed below is given by "pfaffian" rings. Let X_{ij}, $1 \le i < j \le n$, be a family of indeterminates over B, $X_{ji} = -X_{ij}$, $X_{ii} = 0$. The pfaffian of the alternating matrix ($X_{i_u i_v}$: $1 \le u,\ v \le t$), t even, is denoted by $[i_1, \dots, i_t]$. The polynomial ring $B[X]$ is an ASL on the set $\Phi(X)$ of the pfaffians $[i_1, \dots, i_t]$, $i_1 < \cdots < i_t$, $t \le n$. The pfaffians are partially ordered in the same way as the minors in (b). The residue class ring $\mathrm{P}_{r+2}(X) = B[X]/\operatorname{Pf}_{r+2}(X)$, $\operatorname{Pf}_{r+2}(X)$ being generated by the $(r+2)$-pfaffians, inherits its ASL structure from $B[X]$ according to (1.2). The poset underlying $\mathrm{P}_{r+2}(X)$ is denoted $\Phi_r(X)$. Note that the rings $\mathrm{P}_{r+2}(X)$ are Gorenstein rings over a Gorenstein B—in fact factorial over a factorial B, cf. [**Av.1**], [**KL**]. —

2. The Arithmetical Rank of a Poset Ideal.

Let V be an affine or projective algebraic variety, and W a closed subvariety of V. In general it is difficult to determine the smallest number w of hypersurfaces H_i in the ambient affine or projective space such that

$$W = V \cap H_1 \cap \cdots \cap H_w.$$

In more general and algebraic terms the problem above amounts to the determination of the *arithmetical rank* $\operatorname{ara} I$ of an ideal I in a commutative (noetherian) ring R, the arithmetical rank being defined to be the smallest number t for which there are elements $x_1, \dots, x_t \in R$ such that

$$\operatorname{Rad} I = \operatorname{Rad} \sum_{i=1}^{t} R x_i.$$

(In the projective situation one of course requires the x_i to be homogeneous.) In this section we obtain an upper bound for the arithmetical rank of an ideal I generated by an ideal Ω of the poset Π underlying an ASL:

(2.1) PROPOSITION. *Let A be an ASL on Π over B, $\Omega \subset \Pi$ an ideal, and $I = A\Omega$. Then there are homogeneous elements $x_1, \dots, x_r \in I$, $r = \operatorname{rk} \Omega$, such that* $\operatorname{Rad} I = \operatorname{Rad} \sum_{i=1}^{r} A x_i$. *In particular* $\operatorname{ara} I \le \operatorname{rk} \Omega$.

PROOF: Let m be the least common multiple of the degrees of the elements $\xi \in \Omega$, and $e(\xi) = m/\deg \xi$. We put

$$x_i = \sum_{\substack{\xi \in \Omega \\ \operatorname{rk} \xi = i}} \xi^{e(\xi)}, \qquad i = 1, \dots, r.$$

Let υ be a minimal element of Ω (and, hence, of Π). Then $\upsilon\zeta = 0$ for every different minimal element $\zeta \in \Omega$, and one concludes

$$\upsilon x_1 = \upsilon^{e(\upsilon)+1} \in \operatorname{Rad} \sum_{i=1}^{r} A x_i.$$

Now an induction argument finishes the proof: Let $\widetilde{\Omega}$ be the set of minimal elements of Ω, $\overline{\Omega} = \Omega \setminus \widetilde{\Omega}$, $\overline{A} = A/A\widetilde{\Omega}$, $\overline{\Pi} = \Pi \setminus \widetilde{\Omega}$. The data $\overline{A}, \overline{\Pi}, \overline{\Omega}$ satisfy the hypotheses of the proposition, and it follows that

$$\operatorname{Rad} A\Omega \subset \operatorname{Rad}(A\widetilde{\Omega} + \sum_{i=2}^{r} A x_i) = \operatorname{Rad} \sum_{i=1}^{r} A x_i. \text{ —}$$

In the next section we shall see that the bound of (2.1) is sharp under special circumstances. A first specialization:

(2.2) COROLLARY. *Let X be an $m \times n$ matrix of indeterminates over B. Then*

$$\operatorname{ara} \mathrm{I}_t(X) \leq mn - t^2 + 1.$$

Obviously $[m-t+1, \ldots,, m|n-t+1, \ldots,, n]$ is the only maximal element of the poset ideal generating $\mathrm{I}_t(X)$, and one easily computes its rank. Cf. (1.4),(b) for the ASL structure on $B[X]$.

(2.3) REMARKS. (a) In general the bound given by (2.1) is not sharp: Consider the ideal generated by X_1, say, under the hypotheses of (1.4),(a), or the ideal generated by $[1|1]$, $[2|1]$, and $[1\,2|1\,2]$ under the hypotheses of (2.2), $m = n = 2$. Admittedly, none of these counterexamples is completely convincing: If one first takes the ideals in their "natural" rings $B[X_1]$ and $B[X_{11}, X_{21}]$ resp. and then extends the ideal to $B[X_1, \ldots, X_n]$ or $B[X]$, the precise bounds are obtained. It is quite clear that $\operatorname{ara} I$ in general cannot be determined from the combinatorial data given by Π and Ω. In (2.5) below we will note an improvement of (2.1) for a specific ASL which depends on the form of the straightening relations.

(b) In [Ne], p. 180, Example (i),(a) Newstead showed that the bound in (2.2) is precise for $t = 2$, B a field of characteristic 0. As Cowsik told us, Newstead's argument goes through for every t and can be transferred to characteristic $p > 0$ via the use of étale cohomology. There is of course no restriction in assuming that B is a field; otherwise one factors by a maximal ideal first.

For the case $t = \min(m, n)$ Hochster has given an invariant-theoretic argument which shows that $\operatorname{ara} \mathrm{I}_t(X) = mn - t^2 + 1$ in characteristic 0. Suppose $m \le n$. Then (in all characteristics) $\mathrm{G}(X)$ is the ring of invariants of the $\mathrm{SL}(m, B)$-action induced by the substitutions $X \to TX$, $T \in \mathrm{SL}(m, B)$, on $B[X]$. In characteristic 0 the group $\mathrm{SL}(m, B)$ is linearly reductive. This implies that $\mathrm{G}(X)$ is a direct $\mathrm{G}(X)$-summand of $B[X]$, $B[X] = \mathrm{G}(X) \oplus C$. Let $I = \mathrm{I}_m(X) \subset B[X]$, and $J = I \cap \mathrm{G}(X)$. Then $I = JB[X]$, and

$$\mathrm{H}_I^d(B[X]) = \mathrm{H}_J^d(B[X]) = \mathrm{H}_J^d(\mathrm{G}(X)) \oplus \mathrm{H}_J^d(C);$$

here H_I denotes cohomology with support in I, cf. [**Ha**]. Taking $d = nm - m^2 + 1$, one concludes $\mathrm{H}_I^d(B[X]) \neq 0$ since $d = \dim \mathrm{G}(X)$. By [**Ha**], p. 414, Example 2, $\operatorname{ara} I \ge d$.

The preceding argument breaks down in positive characteristic since $\mathrm{H}_I^d(B[X]) = 0$ then, provided $n > m$: $\mathrm{H}_I^i(B[X]) = 0$ for all $i > \operatorname{ht} I = n - m + 1$ according to [**PS**], p. 110, Proposition (4.1). It likewise fails for $t < \min(m, n)$ since the subalgebra of $B[X]$ generated by the t-minors has the same dimension as $B[X]$, cf. [**CN**] or [**BV.1**], Section 10. —

Specializing (2.1) to the example (1.4),(a) we obtain the following result of Gräbe ([**Gr**], Theorem 1):

(2.4) COROLLARY. *Let $X_1, \ldots, X_n$ be indeterminates over B, and I an ideal generated by square-free monomials $f_1, \ldots, f_m$. Let p be the smallest number of factors occuring among the f_i. Then*

$$\operatorname{ara} I \le n - p + 1.$$

The ASL A considered in the preceding corollary has a very special property: If Ω is an ideal in its underlying poset, then the maximal elements of Ω generate $A\Omega$. This allows a slight improvement of (2.1) which we only give under the hypotheses of (2.4), a result almost obtained by Gräbe ([**Gr**], Theorem 2). The straightening relations in (2.4) are the equations

$$\xi v = (\xi \sqcup v)(\xi \sqcap v),$$

cf. (1.4),(a), and $(\xi \sqcup v) < \xi, v$. Therefore it is enough to consider the smallest subset of Ω which contains the generators of I and is closed under taking least common multiples, i.e. the subset $\widetilde{\Omega}$ formed by the least common multiples of the subsets of $\{f_1, \ldots, f_m\}$.

(2.5) COROLLARY. *Under the hypotheses of* (2.4) *one has*

$$\operatorname{ara} I \leq \operatorname{rk} \widetilde{\Omega}$$

(where of course $\operatorname{rk}\widetilde{\Omega}$ *is measured by chains in* $\widetilde{\Omega}$*). In particular, if* q *is the smallest number of factors occuring in any of the least common multiples of two of the elements* $f_1, \ldots, f_m$*, then*

$$\operatorname{ara} I \leq n - q + 2.$$

PROOF: One takes

$$x_i = \sum_{\substack{\xi \in \widetilde{\Omega} \\ \operatorname{rk} \xi = i}} \xi^{e(\xi)}, \qquad i = 1, \ldots, \operatorname{rk} \widetilde{\Omega},$$

and argues as in the proof of (2.1). The second part follows since the submaximal elements of $\widetilde{\Omega}$ are given as $\xi \sqcup \upsilon$, $\xi, \upsilon \in \{f_1, \ldots, f_m\}$, and $\operatorname{rk}_{\widetilde{\Omega}} \xi \sqcup \upsilon \leq \operatorname{rk}_{\Omega} \xi \sqcup \upsilon \leq n - q + 1$. —

Let $n = 5$, $f_1 = X_2X_4$, $f_2 = X_1X_3X_4$, $f_3 = X_1X_3X_5$, $f_4 = X_2X_3X_5$. Then (2.4) gives the trivial bound $\operatorname{ara} I \leq 4$ whereas (2.5) yields $\operatorname{ara} I \leq 3$, and $\operatorname{ara} I = 3$, as shown in [**Gr**], 5., Beispiel 4. An example for which (2.5) fails to give the precise value: Take $n = 5$, $f_1 = X_2X_4$, $f_2 = X_2X_5$, $f_3 = X_1X_4$, $f_4 = X_1X_3X_5$ ([**Gr**], 5., Beispiel 2). It is easily seen that $x_1 = X_2X_4X_5$, $x_2 = X_1X_3X_5 + X_1X_2X_4$, $x_3 = X_1X_4 + X_2X_5 + X_2X_4$ generate the ideal up to radical. The algorithm by which Gräbe finds x_1, x_2, x_3 can be described in the following way: One chooses subsets $\Psi_1, \ldots, \Psi_r \subset \Omega$ such that

(i) Ψ_i consists of incomparable elements of Ω, $i = 1, \ldots, r$,

(ii) the least common multiple of every pair $\psi_1, \psi_2 \in \Psi_i$, $\psi_1 \neq \psi_2$, is in the ideal generated by $\Psi_1 \cup \cdots \cup \Psi_{i-1}$,

(iii) $\Psi_1 \cup \cdots \cup \Psi_r$ contains the maximal elements of Ω.

Then it follows that the elements $x_i = \sum_{\omega \in \Psi_i} \omega$, $i = 1, \ldots, r$ generate I up to radical. (2.4) and (2.5) reflect two special choices of $\Psi_1, \ldots, \Psi_r$ which work for all Ω.

3. Symmetric ASLs.

In this section we introduce a special class of ASLs which, though certainly very small, contains some important examples.

DEFINITION. Let A be an ASL on Π over B. A is called *symmetric* if it is also a graded ASL with respect to the reverse order on Π.

Note that this is only a condition concerning (ASL-2): the standard monomials in the reverse order are the same as those with respect to the given one.

(3.1) EXAMPLES. (a) If A is a symmetric ASL and $\Omega \subset \Pi$ an ideal or a coideal (i.e. the complement of an ideal), then $A/A\Omega$ is a symmetric ASL.

(b) The discrete ASLs are symmetric.

(c) $G(X)$, the homogeneous coordinate ring of a Grassmann variety, is a symmetric ASL. This is stated in [**Ho**], Lemma 2.1 and [**BV.1**], (4.6), and can in fact be seen very easily: The automorphism of $B[X]$ which reverses the order of the columns of X, induces an automorphism of $G(X)$ which (up to sign) permutes the maximal minors of X and reverses the order of $\Gamma(X)$. It follows from (a) that the homogeneous coordinate rings of the Schubert subvarieties are symmetric ASLs, too.

(d) More generally than (c), the multihomogeneous coordinate ring of a flag variety is a symmetric ASL, cf. [**Ei**], Example (5). It can be described in the following way. Let X be an $n \times n$ matrix of indeterminates over B, and $n_1 > n_2 > \cdots > n_k$ a sequence of integers, $n_1 \leq n$, $n_k \geq 1$. Then one considers the B-subalgebra generated by the n_i-minors of the first n_i rows of X, $i = 1, \ldots, k$. It is a sub-ASL of $B[X]$ in a natural way since in a standard representation

$$[1, \ldots, i|a_1, \ldots, a_i][1, \ldots, j|b_1, \ldots, b_j] = \sum a_\mu \mu$$

every standard monomial μ is of the form $[1, \ldots, i| \ldots][1, \ldots, j| \ldots]$: first it has at most two factors, and secondly every row index appears in μ with the same multiplicity as on the left side, cf. [**DEP.1**], Theorem 2.1 or [**BV.1**], (11.3). Now one can again apply the automorphism argument from (c). In this case the automorphism does not completely reverse the order on the poset; nevertheless the argument goes through as the reader may check.

(e) Let $(L, \sqcap, \sqcup)$ be a finite lattice, K a field, and A the residue class ring of the polynomial ring $K[X_\alpha : \alpha \in L]$ modulo the ideal generated by all the polynomials

$$X_\alpha X_\beta - X_{\alpha\sqcap\beta} X_{\alpha\sqcup\beta}, \qquad \alpha, \beta \quad \text{incomparable.}$$

In [**Hi**], Theorem, Hibi has shown that the following conditions are equivalent: (i) A is a graded ASL on L (relative to the embedding $L \to A$, $\alpha \to \overline{X}_\alpha$), (ii) A is an integral domain, (iii) L is distributive. Of course A is a symmetric ASL if L is distributive.

Moreover Hibi shows on p. 103 of [**Hi**] that a lattice L must be distributive if there is a symmetric ASL on L which is a domain and in which the standard monomials in the straightening relations all have exactly two factors. —

In symmetric ASLs the arithmetical rank on an ideal $A\Omega$ is always given by the rank of Ω, and sometimes this holds under more general circumstances:

(3.2) PROPOSITION. *Let A be an ASL on Π over B, $\Omega \subset \Pi$ an ideal, and $I = A\Omega$. Let $C(\Omega)$ denote the B-submodule generated by all the standard monomials which have a factor in $\Pi \setminus \Omega$. Suppose that one of the following hypotheses is satisfied:*
(i) $B[\Omega] \cap C(\Omega) = 0$ and $C(\Omega)$ is a $B[\Omega]$-submodule of A,
(ii) $C(\Omega)$ is an ideal in A.
Then $\operatorname{ara} I = \operatorname{rk} \Omega$.

(3.3) COROLLARY. *Let A be a symmetric ASL on Π over B, $\Omega \subset \Pi$ an ideal, and $I = A\Omega$. Then* $\operatorname{ara} I = \operatorname{rk} \Omega$.

The corollary follows immediately from the proposition since $C(\Omega)$ is the ideal generated by $\Pi \setminus \Omega$ if A is symmetric. By the way, one easily finds examples which demonstrate that none of the hypotheses (i) or (ii) in (3.2) implies the other one.

PROOF OF (3.2): In view of (2.1) we may first factor out a maximal ideal of B and assume that B is a field.

If hypothesis (i) is satisfied, $B[\Omega]$ is the B-module generated by all the standard monomials consisting entirely of factors from Ω. Therefore $B[\Omega]$ is an ASL in a natural way, and $\dim B[\Omega] = \operatorname{rk} \Omega$ (cf. (1.3)). Furthermore $B[\Omega]$ is a direct $B[\Omega]$-summand, and now one applies the cohomological argument detailed in (2.3),(b).

If hypothesis (ii) is satisfied, one passes to $\overline{A} = A/C(\Omega)$ which is an ASL on Ω in a natural way. Let $\overline{I} = \overline{A}\Omega$. Then

$$\operatorname{ara} I \geq \operatorname{ara} \overline{I} \geq \operatorname{ht} \overline{I} = \operatorname{rk} \Omega. \text{ —}$$

(3.4) COROLLARY. *Let K be an algebraically closed field. The minimal number of equations defining the Schubert variety $\Omega(a_1, \ldots, a_m)$ as a subvariety of $\mathrm{G}_m(V)$, $\dim V = n$, is given by*

$$\max_{a_k - k < n - m} m(n-m) - (m-k+1)(a_k - k) + 1.$$

PROOF: Using the information presented in (1.4),(c) we let $b_i = n - a_{m-i+1} + 1$, $i = 1, \ldots, m$, and $\gamma = [b_1, \ldots, b_m]$. The maximal elements of $\Omega = \{\delta \in \Gamma(X) \colon \delta \not\geq \gamma\}$ are given by

$$\tau_i = [b_i - i, \ldots, b_i - 1, n - (m-i) + 1, \ldots, n], \qquad b_i > i.$$

An easy computation yields

$$\operatorname{rk} \tau_i = m(n-m) + 1 - i(n - m - b_i + i + 1).$$

Replacing i by $m - k + 1$ and b_i by $n - a_k + 1$, one obtains the desired result. —

(3.5) REMARK. Let B be an integral domain, $A = B[X]$, $I = \mathrm{I}_t(X)$ as in (2.2). The "symbolic graded ring"

$$\widetilde{A} = \bigoplus_{i=0}^{\infty} I^{(i)}/I^{(i+1)}$$

is a graded ASL over B on the poset Δ^* given by the leading forms of the minors of X. The ideal $\Omega^* \subset \Delta^*$ consisting of the leading forms of the ideal $\Omega \subset \Delta$ generating I satisfies both of the hypotheses (i) and (ii) of (3.2) though $\widetilde{A}$ is not a symmetric ASL, cf. [**BV.1**], Section 10. It follows that $\operatorname{ara} \widetilde{A}\Omega^* = mn - t^2 + 1$, a result we cannot prove by ASL methods for $I = A\Omega$. —

4. Straightening Laws on Modules.

It occurs frequently that a module M over an ASL A has a structure closely related to that of A: the generators of M are partially ordered, a distinguished set of "standard elements" forms a B-basis of M, and the multiplication $A \times M \to A$ satisfies a straightening law similar to the straightening law in A itself. In this section we introduce the notion of a module with straightening law whereas the next section contains a strengthening of this notion.

DEFINITION. Let A be an ASL over B on Π. An A-module M is called a *module with straightening law* (MSL) on the finite poset $\mathcal{X} \subset M$ if the following conditions are satisfied:
(MSL-1) For every $x \in \mathcal{X}$ there exists an ideal $\mathcal{I}(x) \subset \Pi$ such that the elements

$$\xi_1 \cdots \xi_n x, \qquad x \in \mathcal{X}, \quad \xi_1 \notin \mathcal{I}(x), \quad \xi_1 \leq \cdots \leq \xi_n, \quad n \geq 0,$$

constitute a B-basis of M. These elements are called *standard elements.*
(MSL-2) For every $x \in \mathcal{X}$ and $\xi \in \mathcal{I}(x)$ one has

$$\xi x \in \sum_{y<x} Ay.$$

It follows immediately by induction on the rank of x that the element ξx as in (MSL-2) has a standard representation

$$\xi x = \sum_{y<x} (\sum b_{\xi x \mu y} \mu) y, \qquad b_{\xi x \mu y} \in B, \ b_{\xi x \mu y} \neq 0,$$

in which each μy is a standard element.

(4.1) REMARKS. (a) Suppose M is an MSL, and $\mathcal{T} \subset \mathcal{X}$ an ideal. Then the submodule of M generated by $\mathcal{T}$ is an MSL, too. This fact allows one to prove theorems on MSLs by noetherian induction on the set of ideals of $\mathcal{X}$.

(b) It would have been enough to require that the standard elements are linearly independent. If just (MSL-2) is satisfied then the induction principle in (a) proves that M is generated as a B-module by the standard elements. —

(4.2) EXAMPLES. (a) A itself is an MSL if one takes $\mathcal{X} = \{1\}$, $\mathcal{I}(1) = \emptyset$. Another choice is $\mathcal{X} = \Pi \cup \{1\}$, $\mathcal{I}(\xi) = \{\pi \in \Pi \colon \pi \not\geq \xi\}$, $\mathcal{I}(1) = \Pi$, $1 > \pi$

for each $\pi \in \Pi$. The relations necessary for (MSL-2) are then given by the identities $\pi 1 = \pi$, the straightening relations

$$\xi\upsilon = \sum b_\mu \mu, \qquad \xi, \upsilon \quad \text{incomparable},$$

and the Koszul relations

$$\xi\upsilon = \upsilon\xi, \qquad \xi < \upsilon.$$

By (4.1),(a) for every poset ideal $\Psi \subset \Pi$ the ideal $A\Psi$ is an MSL, too.

(b) Suppose that Ψ as in (a) additionally satisfies the following condition: Whenever $\phi, \psi \in \Psi$ are incomparable, then every standard monomial μ in the standard representation $\phi\psi = \sum a_\mu \mu$, $a_\mu \neq 0$, contains at least two factors from Ψ. This condition appears in **[Hu]**, **[EH]**, **[BST]**, and in **[BV.1]**, Section 9 where the ideal $I = A\Psi$ is called *straightening-closed*. As a consequence of (d) below the powers I^n of $I = A\Psi$ are MSLs. Observe in particular that the condition above is satisfied if every μ a priori contains at most two factors and Ψ consists of the elements in Π of highest degree.

(c) In order to prove and to generalize the statements in (b) let us consider an MSL M on $\mathcal{X}$ and an ideal $\Psi \subset \Pi$ such that $I = A\Psi$ is straightening-closed and the following condition holds:

(∗) The standard monomials in the standard representation of a product ψx, $\psi \in \Psi$, $x \in \mathcal{X}$, all contain a factor from Ψ.

Then it is easy to see that IM is again an MSL on the set $\{\psi x \colon x \in \mathcal{X},\ \psi \in \Psi \setminus \mathcal{I}(x)\}$ partially ordered by

$$\psi x \leq \phi y \qquad \Longleftrightarrow \qquad x < y \quad \text{or} \quad x = y,\ \psi \leq \phi,$$

if one takes

$$\mathcal{I}(\psi x) = \{\pi \in \Pi \colon \pi \not\geq \psi\}.$$

Furthermore (∗) holds again. Thus $I^n M$ is an MSL for all $n \geq 1$, and in particular one obtains (b) from the special case $M = A$.

The residue class module M/IM also carries the structure of an MSL on the set $\overline{\mathcal{X}}$ of residues of $\mathcal{X}$ if we let

$$\mathcal{I}(\overline{x}) = \mathcal{I}(x) \cup \Psi.$$

Combining the previous arguments we get that $I^n M / I^{n+1} M$ is an MSL for all $n \geq 0$.

In the situation just considered the associated graded ring $\mathrm{Gr}_I A$ is an ASL on the set Π^* of leading forms (ordered in the same way as Π), cf. [**BST**] or [**BV.1**], (9.8), and obviously $\mathrm{Gr}_I M$ is an MSL on $\mathcal{X}^*$.

(d) Let $A = B[X]/\mathrm{I}_{r+1}(X)$ as in (1.4), (b), $0 \le r \le \min(m,n)$ (so $A = B[X]$ is included). The matrix $\overline{X}$ over A whose entries are the residue classes of the indeterminates defines a map $A^m \to A^n$, also denoted by $\overline{X}$. The modules $\mathrm{Im}\,\overline{X}$ and $\mathrm{Coker}\,\overline{X}$ have been investigated in [**Br.1**]. A simplified treatment has been given in [**BV.1**], Section 13, from where we draw some of the arguments below. Let $d_1,\dots,d_m$ and $e_1,\dots,e_n$ denote the canonical bases of A^m and A^n. Then we order the system $\overline{e}_1,\dots,\overline{e}_n$ of generators of $\mathrm{Coker}\,\overline{X}$ linearly by

$$\overline{e}_1 > \cdots > \overline{e}_n.$$

Furthermore we put

$$\mathcal{I}(\overline{e}_i) = \begin{cases} \{\, \delta \in \Delta_r(X) : \delta \not\geq [1,\dots,r|1,\dots,\widehat{i},\dots,r+1] \,\} & \text{for } i \le r, \\ \emptyset & \text{else,} \end{cases}$$

if $r < n$, and in the case in which $r = n$

$$\mathcal{I}(\overline{e}_i) = \{\, \delta \in \Delta_r(X) : \delta \not\geq [1,\dots,r-1|1,\dots,\widehat{i},\dots,r] \,\}.$$

(where $\widehat{i}$ denotes that i is to be omitted). We claim: $\mathrm{Coker}\,\overline{X}$ is an MSL with respect to these data.

Suppose that $\delta \in \mathcal{I}(\overline{e}_i)$. Then

$$\delta = [a_1,\dots,a_s|1,\dots,i,b_{i+1},\dots,b_s], \qquad s \le r.$$

The element

$$\sum_{j=1}^{s}(-1)^{j+i}[a_1,\dots,\widehat{a_j},\dots,a_s|1,\dots,i-1,b_{i+1},\dots,b_s]\overline{X}(d_{a_j})$$

of $\mathrm{Im}\,\overline{X}$ is a suitable relation for (MSL-2):

(1) $$\delta\overline{e}_i = \sum_{k=i+1}^{n} \pm[a_1,\dots,a_s|1,\dots,i-1,k,b_{i+1},\dots,b_s]\overline{e}_k.$$

Rearranging the column indices $1,\dots,i-1,k,b_{i+1},\dots,b_s$ in ascending order one makes (1) the standard representation of $\delta\overline{e}_i$, and observes the following fact recorded for later purpose:

(2) $\delta \notin \mathcal{I}(\overline{e}_k)$ for all $k \geq i+1$ such that

$$[a_1,\dots,a_s|1,\dots,i-1,k,b_{i+1},\dots,b_s] \neq 0.$$

In order to prove the linear independence of the standard elements one may assume that $r < n$ since $\mathrm{I}_n(X)$ annihilates M. Let

$$\widetilde{M} = \sum_{i=r+1}^{n} A\overline{e}_i, \quad \Psi = \{\, \delta \in \Delta_r(X) \colon \delta \not\geq [1,\dots,r|1,\dots,r-1,r+1] \,\}$$

and $I = A\Psi$.

We claim:
(i) $\widetilde{M}$ is a free A-module.
(ii) $M/\widetilde{M}$ is (over A/I) isomorphic to the ideal generated by the minors $[1,\dots,r|1,\dots,\widehat{i},\dots,r+1]$, $1 \leq i \leq r$, in A/I.
In fact, the minors just specified form a linearly ordered ideal in the poset $\Delta_r(X) \setminus \Psi$ underlying the ASL A/I, and the linear independence of the standard elements follows immediately from (i) and (ii).

Statement (i) simply holds since $\operatorname{rank} \overline{X} = r$, and the $r \times r$-minor in the left upper corner of $\overline{X}$, being the minimal element of $\Delta_r(X)$, is not a zero-divisor in A. For (ii) one applies (4.5) below to show that $M/\widetilde{M}$ and the ideal in (ii) have the same representation given by the matrix

$$\begin{pmatrix} \overline{X}_{11} & \dots & \overline{X}_{1r} \\ \vdots & & \vdots \\ \overline{X}_{m1} & \dots & \overline{X}_{mr} \end{pmatrix},$$

the entries taken in A/I: The assignment $\overline{e}_i \to (-1)^{i+1}[1,\dots,r|1,\dots,\widehat{i},\dots,r+1]$ induces the isomorphism. The computations needed for the application of (4.5) are covered by (1).

By similar arguments one can show that $\operatorname{Im} \overline{X}$ is also an MSL, see [**BV.1**], proof of (13.6) where a filtration argument is given which shows the linear independence of the standard elements. Such a filtration argument could also have been applied to prove (MSL-1) for M.

(e) Another example is furnished by the modules defined by generic alternating maps. Recalling the notations of (1.4), (d) we let $A = B[X]/\operatorname{Pf}_{r+2}(X)$ and M be the cokernel of the linear map

$$\overline{X} \colon F \longrightarrow F^*, \qquad F = A^n.$$

In complete analogy with the preceding example M is an MSL on $\{\overline{e}_1, \dots, \overline{e}_n\}$, the canonical basis of F^*, $\overline{e}_1 > \dots > \overline{e}_n$, if one puts

$$\mathcal{I}(\overline{e}_i) = \begin{cases} \{\, \pi \in \Phi_r(X) : \pi \not\geq [1, \dots, \widehat{i}, \dots, r+1] \,\} & \text{for} \quad i \leq r, \\ \emptyset & \text{else,} \end{cases}$$

if $r < n$, and in the case in which $r = n$

$$\mathcal{I}(\overline{e}_i) = \begin{cases} \{\, \pi \in \Phi(X) : \pi \not\geq [1, \dots, \widehat{i}, \dots, r-1] \,\} & \text{for} \quad i \leq n-1, \\ \{\, [1, \dots, n] \,\} & \text{for} \quad i = n. \end{cases}$$

The straightening law (1) is replaced by the equation

$$(1') \qquad \pi \overline{e}_i = \sum_{k=i+1}^{n} \pm [1, \dots, i-1, k, b_{i+1}, \dots, b_s]\, \overline{e}_k,$$

obtained from Laplace type expansion of pfaffians as (1) has been derived from Laplace expansion of minors. Observe that the analogue (2′) of (2) is satisfied. The linear independence of the standard elements is proved in entire analogy with (d).

A notable special case is n odd, $r = n-1$. In this case $\operatorname{Coker} X \cong \operatorname{Pf}_r(X)$ is an ideal of projective dimension 2 [**BE**] and generated by a linearly ordered poset ideal in $\Phi(X)$. —

The following proposition helps to detect further MSLs:

(4.3) Proposition. *Let M, M_1, M_2 be modules over an ASL A, connected by an exact sequence*

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0.$$

Let M_1 and M_2 be MSLs on $\mathcal{X}_1$ and $\mathcal{X}_2$, and choose a splitting f of the epimorphism $M \to M_2$ over B. Then M is an MSL on $\mathcal{X} = \mathcal{X}_1 \cup f(\mathcal{X}_2)$ ordered by $x_1 < f(x_2)$ for all $x_1 \in \mathcal{X}_1$, $x_2 \in \mathcal{X}_2$, and the given partial orders on $\mathcal{X}_1$ and the copy $f(\mathcal{X}_2)$ of $\mathcal{X}_2$. Moreover one chooses $\mathcal{I}(x)$, $x \in \mathcal{X}_1$, as in M_1 and $\mathcal{I}(f(x)) = \mathcal{I}(x)$ for all $x \in \mathcal{X}_2$.

The proof is straightforward and can be left to the reader.

(4.4) Example. The preceding proposition helps to supplement (4.2),(c). Under the hypotheses there one has that $M/I^n M$ is an MSL for all $n \geq 1$.

It has been stated in (4.2),(c) that all the quotients $I^nM/I^{n+1}M$ are MSLs, and therefore we may argue inductively by the exact sequence

$$0 \longrightarrow I^nM/I^{n+1}M \longrightarrow M/I^{n+1}M \longrightarrow M/I^nM \longrightarrow 0.$$

In particular one has that A/I^n is an MSL over A (though it is not an ASL for $n \geq 2$). —

In terms of generators and relations an ASL is defined by its generating poset and its straightening relations, cf. (1.1). This holds similarly for MSLs:

(4.5) PROPOSITION. *Let A be an ASL on Π over B, and M an MSL on $\mathcal{X}$ over A. Let e_x, $x \in \mathcal{X}$, denote the elements of the canonical basis of the free module $A^{\mathcal{X}}$. Then the kernel $K_{\mathcal{X}}$ of the natural epimorphism*

$$A^{\mathcal{X}} \longrightarrow M, \qquad e_x \longrightarrow x,$$

is generated by the relations required for (MSL-2)*:*

$$\rho_{\xi x} = \xi e_x - \sum_{y<x} a_{\xi x y} e_y, \qquad x \in \mathcal{X},\ \xi \in \mathcal{I}(x).$$

PROOF: We use the induction principle indicated in (4.1), (a). Let $\tilde{x} \in \mathcal{X}$ be a maximal element. Then $\mathcal{T} = \mathcal{X} \setminus \{\tilde{x}\}$ is an ideal. By induction $A\mathcal{T}$ is defined by the relations $\rho_{\xi x}$, $x \in \mathcal{T}$, $\xi \in \mathcal{I}(x)$. Furthermore (MSL-1) and (MSL-2) imply

$$M/A\mathcal{T} \cong A/A\mathcal{I}(\tilde{x}) \tag{3}$$

If $a_{\tilde{x}}\tilde{x} - \sum_{y\in\mathcal{T}} a_y y = 0$, one has $a_{\tilde{x}} \in A\mathcal{I}(\tilde{x})$ and subtracting a linear combination of the elements $\rho_{\xi\tilde{x}}$ from $a_{\tilde{x}}e_{\tilde{x}} - \sum_{y\in\mathcal{T}} a_y e_y$ one obtains a relation of the elements $y \in \mathcal{T}$ as desired. —

The kernel of the epimorphism $A^{\mathcal{X}} \to M$ is again an MSL:

(4.6) PROPOSITION. *With the notations and hypotheses of (4.5) the kernel $K_{\mathcal{X}}$ of the epimorphism $A^{\mathcal{X}} \to M$ is an MSL if we let*

$$\mathcal{I}(\rho_{\xi x}) = \{\pi \in \Pi \colon \pi \not\geq \xi\}$$

and

$$\rho_{\xi x} \leq \rho_{\upsilon y} \qquad \Longleftrightarrow \qquad x < y \quad \text{or} \quad x = y,\ \xi \leq \upsilon.$$

PROOF: Choose $\tilde{x}$ and $\mathcal{T}$ as in the proof of (4.5). By virtue of (4.5) the projection $A^{\mathcal{X}} \to Ae_{\tilde{x}}$ with kernel $A^{\mathcal{T}}$ induces an exact sequence

$$0 \longrightarrow K_{\mathcal{T}} \longrightarrow K_{\mathcal{X}} \longrightarrow A\mathcal{I}(\tilde{x}) \longrightarrow 0.$$

Now (4.3) and induction finish the argument. —

If a module M is given in terms of generators and relations, it is in general more difficult to establish (MSL-1) than (MSL-2). For (MSL-2) one "only" has to show that elements $\rho_{\xi x}$ as in the proof of (4.5) can be obtained as linear combinations of the given relations. In this connection the following proposition may be useful: it is enough that the module generated by the $\rho_{\xi x}$ satisfies (MSL-2) again.

(4.7) PROPOSITION. *Let the data $M, \mathcal{X}, \mathcal{I}(x), x \in \mathcal{X}$, be given as in the definition, and suppose that* (MSL-2) *is satisfied. Suppose that the kernel $K_{\mathcal{X}}$ of the natural epimorphism $A^{\mathcal{X}} \to M$ is generated by the elements $\rho_{\xi x} \in A^{\mathcal{X}}$ representing the relations in* (MSL-2). *Order the $\rho_{\xi x}$ and choose $\mathcal{I}(\rho_{\xi x})$ as in* (4.6). *If $K_{\mathcal{X}}$ satisfies* (MSL-2) *again, M is an MSL.*

PROOF: Let $\tilde{x} \in \mathcal{X}$ be a maximal element, $\mathcal{T} = \mathcal{X} \setminus \{\tilde{x}\}$. We consider the induced epimorphism

$$A^{\mathcal{T}} \longrightarrow A\mathcal{T}$$

with kernel $K_{\mathcal{T}}$. One has $K_{\mathcal{T}} = K_{\mathcal{X}} \cap A^{\mathcal{T}}$. Since the $\rho_{\xi x}$ satisfy (MSL-2), every element in $K_{\mathcal{X}}$ can be written as a B-linear combination of standard elements, and only the $\rho_{\xi\tilde{x}}$ have a nonzero coefficient with respect to $e_{\tilde{x}}$. The projection onto the component $Ae_{\tilde{x}}$ with kernel $A^{\mathcal{T}}$ shows that $K_{\mathcal{T}}$ is generated by the $\rho_{\xi x}$, $x \in \mathcal{T}$. Now one can argue inductively, and the split-exact sequence

$$0 \longrightarrow A\mathcal{T} \longrightarrow M \longrightarrow M/A\mathcal{T} \cong A/A\mathcal{I}(\tilde{x}) \longrightarrow 0$$

of B-modules finishes the proof. —

Modules with a straightening law have a distinguished filtration with cyclic quotients; by the usual induction this follows immediately from the isomorphism (3) above:

(4.8) PROPOSITION. *Let M be an MSL on $\mathcal{X}$ over A. Then M has a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that each quotient M_{i+1}/M_i*

is isomorphic with one of the residue class rings $A/A\mathcal{I}(x)$, $x \in \mathcal{X}$, and conversely each such residue class ring appears as a quotient in the filtration.

As a consequence one can bound the depth of an MSL (cf. Section 1 for the definition of depth in this context).

(4.9) COROLLARY. *Let M be an MSL on $\mathcal{X}$ over A. Then*

$$\operatorname{depth} M \geq \min\{\operatorname{depth} A/A\mathcal{I}(x) \colon x \in \mathcal{X}\}.$$

We specialize to ASLs over wonderful posets (cf. [Ei], [**DEP.2**], or [**BV.1**] for this notion and the properties of ASLs over wonderful posets).

(4.10) COROLLARY. *Let A be an ASL on the wonderful poset Π.*
(a) *If M is an MSL on $\mathcal{X}$ over A, then*

$$\operatorname{depth} M \geq \min\{\operatorname{rk}\Pi - \operatorname{rk}\mathcal{I}(x) \colon x \in \mathcal{X}\}.$$

(b) *Let $\Psi \subset \Pi$ be an ideal. Then*

$$\operatorname{depth} A/A\Psi \geq \operatorname{rk}\Pi - \operatorname{rk}\Psi.$$

(c) *Suppose furthermore that $I = A\Psi$ is straightening-closed. Then*

$$\operatorname{depth} A/I^n \geq \operatorname{rk}\Pi - \operatorname{rk}\Psi \qquad \textit{for all} \quad n \geq 1.$$

PROOF: In (b) and (c) $A\Psi$ and I^n resp. are MSLs on a certain poset $\mathcal{X}$, cf. (4.2),(b) and (c) above. In both cases one has

$$\mathcal{I}(x) = \{\pi \in \Pi \colon \pi \not\geq \psi\} \qquad \text{for some} \quad \psi \in \Psi$$

for all $x \in \mathcal{X}$. $\Pi \setminus \mathcal{I}(x)$ is wonderful again (cf. [**DEP.2**], 8.2 or [**BV.1**], (5.13)) and therefore

$$\operatorname{depth} A/A\mathcal{I}(x) \geq \operatorname{rk}\Pi - \operatorname{rk}\psi + 1 \geq \operatorname{rk}\Pi - \operatorname{rk}\Psi + 1$$

by virtue of [**DEP.2**], 8.1. Now one applies (4.9) and switches from $A\Psi$ and I^n to the residue class rings. Part (a) finally follows from (4.9) and (b). —

Of course the inequalities (4.9) and (4.10) can be improved in many cases. For example, $A/A\Psi$ may be a Cohen-Macaulay ring. On the other hand there is a class of ideals I such that one has equality in (4.10),(c) for $n \gg 0$, cf. [**BV.1**], (9.22). The depth of the generic modules (4.2),(d) and (e) has been determined in [**BV.1**], Section 13 and [**BV.2**] resp. using the fact that, with the notations of (4.2),(d), the depth of $M/\widetilde{M}$ can be computed exactly.

Further consequences concern the annihilator, the localizations with respect to prime ideals $P \in \operatorname{Ass} A$, and the rank of an MSL.

(4.11) PROPOSITION. *Let M be an MSL on $\mathcal{X}$ over A, and*

$$J = A(\bigcap_{x\in\mathcal{X}} \mathcal{I}(x)).$$

Then

$$J \supset \operatorname{Ann} M \supset J^n, \qquad n = \operatorname{rk} \mathcal{X}.$$

PROOF: Note that $A(\bigcap \mathcal{I}(x)) = \bigcap A\mathcal{I}(x)$ (as a consequence of (1.2)). Since $\operatorname{Ann} M$ annihilates every subquotient of M, the inclusion $\operatorname{Ann} M \subset J$ follows from (4.8). Furthermore (MSL-2) implies inductively that

$$J^i M \subset \sum_{\operatorname{rk} x \leq \operatorname{rk} \Pi - i} Ax$$

for all i, in particular $J^n M = 0$. —

(4.12) PROPOSITION. *Let M be an MSL on $\mathcal{X}$ over A, and $P \in \operatorname{Ass} A$.*
(a) *Then $\{\pi \in \Pi : \pi \notin P\}$ has a single minimal element σ, and σ is also a minimal element of* Π.
(b) *Let $\mathcal{Y} = \{x \in \mathcal{X} : \sigma \notin \mathcal{I}(x)\}$. Then $\mathcal{Y}$ is a basis of the free A_P-module M_P. Furthermore $(K_{\mathcal{X}})_P$ is generated by the elements $\varrho_{\sigma x}$, $x \notin \mathcal{Y}$.*

PROOF: (a) If π_1, π_2, $\pi_1 \neq \pi_2$, are minimal elements of $\{\pi \in \Pi : \pi \notin P\}$, then, by (ASL-2), $\pi_1, \pi_2 \in P$. So there is a single minimal element σ. It has to be a single minimal element of Π, too, since otherwise P would contain all the minimal elements of Π whose sum, however, is not zero-divisor in A ([**BV.1**], (5.11)).

(b) Consider the exact sequence

$$0 \longrightarrow A\mathcal{T} \longrightarrow M \longrightarrow A/A\mathcal{I}(\tilde{x}) \longrightarrow 0$$

introduced in the proof of (4.5). If $\tilde{x} \notin \mathcal{Y}$, then $\tilde{x} \in A_P\mathcal{T}$ by the relation $\varrho_{\sigma\tilde{x}}$, and we are through by induction. If $\tilde{x} \in \mathcal{Y}$, then σ and all the elements of $\mathcal{I}(\tilde{x})$ are incomparable, so they are annihilated by σ (because of (ASL-2)). Consequently $(A/A\mathcal{I}(\tilde{x}))_P \cong A_P$, $\tilde{x}$ generates a free summand of M_P, and induction finishes the argument again. —

We say that a module M over A has rank r if $M \otimes L$ is free of rank r as an L-module, L denoting the total ring of fractions of A. Cf. [**BV.1**], 16.A for the properties of this notion.

(4.13) COROLLARY. *Let M be an MSL on $\mathcal{X}$ over the ASL A on Π. Suppose that Π has a single minimal element π, a condition satisfied if A is a domain. Then*

$$\operatorname{rank} M = |\{x \in \mathcal{X} : \mathcal{I}(x) = \emptyset\}|.$$

5. Modules with a Strict Straightening Law.

Some MSLs satisfy further natural axioms which strengthen (MSL-1) and (MSL-2). Let M be an MSL on $\mathcal{X}$ over A. The first additional axiom:

(MSL-3) For all $x, y \in \mathcal{X}$: $\quad x < y \;\Rightarrow\; \mathcal{I}(x) \subset \mathcal{I}(y)$.

The property (MSL-3) implies that $\Pi \cup \mathcal{X}$ is a partially ordered set if we order its subsets Π and $\mathcal{X}$ as given and all other relations are given by

$$x < \xi \qquad \Longleftrightarrow \qquad \xi \notin \mathcal{I}(x).$$

(MSL-3) simply guarantees transitivity. If it is satisfied, one can consider the following strengthening of (MSL-2):

(MSL-4) $\xi x = \sum_{y<x,\xi} a_{\xi x y} y \quad$ for all $x \in \mathcal{X}$, $\xi \in \mathcal{I}(x)$.

DEFINITION. We say that M has a *strict straightening law* if it is an MSL satisfying (MSL-3) and (MSL-4).

An ideal $I \subset A$ generated by an ideal $\Psi \subset \Pi$ is a trivial example of a module with a strict straightening law, and the generic modules (4.2),(d) and (e) may be considered significant examples. On the other hand not every MSL has a strict straightening law. The following proposition which strengthens (4.11) excludes all the modules $M/I^n M$, $n \geq 2$, as in (4.4), in particular the residue class rings $A/I^n A$, $n \geq 2$, $I = A\Psi$ straightening-closed.

(5.1) PROPOSITION. *Let M be a module with a strict straightening law on $\mathcal{X}$ over A. Then*

$$\operatorname{Ann} M = A\Big(\bigcap_{x\in\mathcal{X}} \mathcal{I}(x)\Big).$$

PROOF: In fact, if $\xi \in \bigcap \mathcal{I}(x)$, then $\xi x = 0$ for all $x \in \mathcal{X}$, since there is no element $y \in \mathcal{X}$, $y < \xi$. —

Suppose that $\mathcal{X}$ is linearly ordered. Then the straightening laws (MSL-4) and (ASL-2) constitute a set of straightening relations on $\Pi \cup \mathcal{X}$, and the following question suggests itself: Is the symmetric algebra $\mathrm{S}(M)$ an ASL over B? In general the answer is "no", as the following example demonstrates: $A = B[X_1, X_2, X_3]$, $X_1 < X_2 < X_3$,

$$M = A^3/(A(X_1,0,0) + A(X_2,0,0) + A(0,X_1,X_3)),$$

the residue classes of the canonical basis ordered by $\overline{e}_1 > \overline{e}_2 > \overline{e}_3$. On the other hand $\mathrm{S}(I)$ is an ASL if I is generated by a linearly ordered poset ideal, cf. [**BV.1**], (9.13) or [**BST**]; one uses that the Rees algebra $\mathcal{R}(I)$ of A with respect to I is an ASL, and concludes easily that the natural epimorphism $\mathrm{S}(I) \to \mathcal{R}(I)$ is an isomorphism. We will give a new proof of this fact below.

The following proposition may not be considered ultima ratio, but it covers the case just discussed and also the generic modules.

(5.2) PROPOSITION. *Let M be a graded module with strict straightening law on the linearly ordered set $\mathcal{X} = \{x_1, \ldots, x_n\}$, $x_1 < \cdots < x_n$. Put $\mathcal{X}_i = \{x_1, \ldots, x_i\}$, $M_i = A\mathcal{X}_i$, $\overline{M}_{i+1} = M/M_i$, $i = 0, \ldots, n$. Suppose that for all $j > i$ and all prime ideals $P \in \mathrm{Ass}(A/A\mathcal{I}(x_j))$ the localization $(\overline{M}_i)_P$ is a free $(A/A\mathcal{I}(x_i))_P$-module, $i = 1, \ldots, n$.*
(a) *Then $\mathrm{S}(M)$ is an ASL on $\Pi \cup \mathcal{X}$.*
(b) *If $\mathcal{I}(x_1) = \emptyset$, then $\mathrm{S}(M)$ is a torsionfree A-module.*

PROOF: Since $\Pi \cup \mathcal{X}$ generates $\mathrm{S}(M)$ as a B-algebra (and $\mathrm{S}(M)$ is a graded B-algebra in a natural way) and (ASL-2) is obviously satisfied, it remains to show that the standard monomials containing k factors from $\mathcal{X}$ are linearly independent for all $k \geq 0$. Since $\mathrm{S}^0(M) = A$ this is obviously true for $k = 0$, and it remains true if $\mathrm{Ann}\, M = A\mathcal{I}(x_1)$ is factored out; since this does not affect the symmetric powers $\mathrm{S}^k(M)$, $k > 0$, we may assume that $\mathrm{Ann}\, M = 0$. If $n = 1$, then M is now a free A-module and the contention holds for trivial reasons.

The hypotheses indicate that an inductive argument is in order. Independent of the special assumptions on M_i and $\mathcal{I}(x_i)$ there is an exact sequence

$$(5) \qquad \mathrm{S}^k(M) \xrightarrow{g} \mathrm{S}^{k+1}(M) \xrightarrow{f} \mathrm{S}^{k+1}(M/Ax_1) \longrightarrow 0$$

in which f is the natural epimorphism and g is the multiplication by x_1. Let $P \in \mathrm{Ass}\, A$. By (4.12) x_1 generates a free direct summand of M_P. Therefore (5) splits over A_P, and $g \otimes A_P$ is injective. It is now enough to show that $\mathrm{S}^k(M)$ is torsionfree; then g is injective itself and (5) splits as a sequence of B-modules as desired: By induction the standard elements in $\mathrm{S}^k(M)$ as well as in $\mathrm{S}^{k+1}(M/Ax_1)$ are linearly independent.

The linear independence of the standard elements in $\mathrm{S}^k(M)$ implies that $\mathrm{S}^k(M)$ is an MSL over A on the set of monomials of length k in $\mathcal{X}$ with respect to a suitable partial order and the choice

$$\mathcal{I}(x_{i_1}\cdots x_{i_k})=\mathcal{I}(x_{i_k}),\qquad i_1\le\cdots\le i_k.$$

Let $P\in\operatorname{Spec}A$, $P\notin\operatorname{Ass}A$. Then $P\notin\operatorname{Ass}(A/A\mathcal{I}(x_1))$, since $\mathcal{I}(x_1)=\emptyset$ by assumption. If $P\notin\operatorname{Ass}(A/A\mathcal{I}(x_j))$ for all $j=2,\dots,n$, then $P\notin\operatorname{Ass}\mathrm{S}^k(M)$ by virtue of (4.8); otherwise $\mathrm{S}^k(M)_P$ is a free A_P-module by hypothesis. Altogether: $\operatorname{Ass}\mathrm{S}^k(M)=\operatorname{Ass}A$, and $\mathrm{S}^k(M)$ is torsionfree. —

(5.3) COROLLARY. *With the notations and hypotheses of (5.2), the symmetric algebra* $\mathrm{S}(M_i)$ *is an ASL on* $\Pi\cup\mathcal{X}_i$ *for all* $i=1,\dots,n$. $\mathrm{S}(M_i)$ *is a sub-ASL of* $\mathrm{S}(M)$ *in a natural way.*

PROOF: There is a natural homomorphism $\mathrm{S}(M_i)\to\mathrm{S}(M)$ induced by the inclusion $M_i\to M$. Since $\mathrm{S}(M_i)$ satisfies (ASL-2), it is generated as a B-module by the standard monomials in $\Pi\cup\mathcal{X}_i$. Since these standard monomials are linearly independent in $\mathrm{S}(M)$, they are linearly independent in $\mathrm{S}(M_i)$, too, and $\mathrm{S}(M_i)$ is a subalgebra of $\mathrm{S}(M)$. —

The following corollary has already been mentioned:

(5.4) COROLLARY. *Let* A *be an ASL on* Π, *and* $\Psi\subset\Pi$ *a linearly ordered ideal. Then* $\mathrm{S}(A\Psi)$ *is an ASL on the disjoint union of* Π *and* Ψ.

PROOF: For each $\psi\in\Psi$ the poset $\Pi\setminus\mathcal{I}(\psi)$ has ψ as its single minimal element. Let $\Psi=\{\psi_1,\dots,\psi_n\}$, $\psi_1<\cdots<\psi_n$. If $P\in\operatorname{Ass}(A/A\mathcal{I}(\psi_j))$, then $\psi_j\notin P$ since ψ_j is not a zero-divisor of the ASL $A/A\mathcal{I}(\psi_j)$. Consequently $(A\Psi/(\sum_{k=1}^{i}A\psi_k))_P$ is isomorphic to $(A/\mathcal{I}(\psi_i))_P$ for all $i<j$. —

We want to apply (5.2) to the generic modules discussed in (4.2), (d), and recall the notations introduced there: $A=B[X]/\mathrm{I}_{r+1}(X)$ is an ASL on $\Delta_r(X)$, the set of all i-minors, $i\le r$, of X. M is the cokernel of the map $A^m\to A^n$ defined by the matrix $\overline{X}$, $\overline{e}_1,\dots,\overline{e}_n$ are the residue classes of the canonical basis $e_1,\dots,e_n$ of A^n. (Thus M_k is the submodule of M generated by $\overline{e}_{n-k+1},\dots,\overline{e}_n$.)

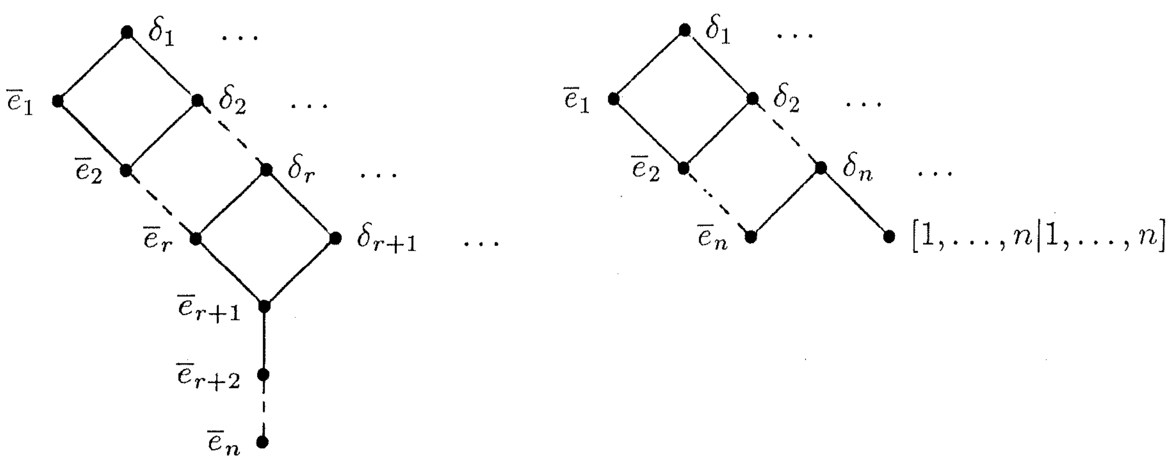
(5.5) COROLLARY. (a) *With the notations just recalled, the symmetric algebra of a generic module* M *is an ASL. If* $r+1\le n$, $\mathrm{S}(M)$ *is torsionfree over* A.
(b) *Let* B *be a Cohen-Macaulay ring.* $\mathrm{S}(M)$ *is Cohen-Macaulay if and only if* $r+1\le n$ *or* $r=m=n$.

PROOF: (a) Factoring out the ideal generated by $\mathcal{I}(\overline{e}_n)$ we may suppose that $r<n$. Note that with the notations introduced in (4.2),(d) one has

$\overline{e}_n < \cdots < \overline{e}_1$. Because of statement (ii) in (4.2),(d) the validity of the hypothesis of (5.2) for $i \geq n-r+1$ follows from the proof of (5.4).

Let $i \leq n-r$, $j > i$, $k = n-j+1$, $\delta = [1,\ldots,r|1,\ldots,r]$ for $k \geq r+1$ and $\delta = [1,\ldots,r|1,\ldots,\widehat{k},\ldots,r+1]$ for $k \leq r$. Then δ is the minimal element of the poset underlying $A/\mathcal{I}(\overline{x}_j) = A/\mathcal{I}(\overline{e}_k)$, thus not contained in an associated prime ideal of the latter. On the other hand $(\overline{M}_i)_P$ is free for every prime P not containing δ.

(b) in order to form the poset $\Pi \cup \{\overline{e}_1,\ldots,\overline{e}_n\}$ one attaches $\{\overline{e}_1,\ldots,\overline{e}_n\}$ to Π as indicated by the following diagrams for the cases $r+1 \leq n$ and $r = m = n$ resp. In the first case we let $\delta_i = [1,\ldots,r|1,\ldots,\widehat{i},\ldots,r+1]$, in the second $\delta_i = [1,\ldots,r-1|1,\ldots,\widehat{i},\ldots,r]$.



It is an easy exercise to show that $\Pi \cup \{\overline{e}_1,\ldots,\overline{e}_n\}$ and $\Pi \cup \{\overline{e}_{n-k+1},\ldots,\overline{e}_n\}$ are wonderful, implying the Cohen-Macaulay property for ASL's defined on the poset ([**BV.1**], Section 5 or [**DEP.2**]).

In the case in which $m > n = r$, the ideal $\mathrm{I}_n(X)\,\mathrm{S}(M)$ annihilates $\bigoplus_{i>0} \mathrm{S}^i(M)$, and $\dim \mathrm{S}(M)/\mathrm{I}_n(X) < \dim \mathrm{S}(M)$ by (1.3), excluding the Cohen-Macaulay property. —

Admittedly the preceding corollary is not a new result. In fact, let Y be an $n \times 1$ matrix of new indeterminates. Then

$$\mathrm{S}(M) \cong B[X,Y]/(\mathrm{I}_{r+1}(X) + \mathrm{I}_1(XY))$$

can be regarded as the coordinate ring of a variety of complexes, which has been shown to be a Hodge algebra in [**DS**]. The results of [**DS**] include part (b) of (5.5) as well as the fact that $\mathrm{S}(M)$ is a (normal) domain if $r+1 \leq n$ and B is a (normal) domain. The divisor classgroup of $\mathrm{S}(M)$ in case $r+1 \leq b$, B normal, has been computed in [**Br.2**]: $\mathrm{Cl}(\mathrm{S}(M)) = \mathrm{Cl}(B)$ if

$m = r < n-1$, $\mathrm{Cl}(\mathrm{S}(M)) = \mathrm{Cl}(B) \oplus \mathbb{Z}$ else. The algebras $\mathrm{S}(M)$, in particular for the cases $r+1 > \min(m,n)$, i.e. $A = B[X]$, and $r+1 = \min(m,n)$, have received much attention in the literature, cf. [**Av.2**], [**BE**], [**BKM**], and the references given there. Note that (5.5) also applies to the subalgebras $\mathrm{S}(M_k)$. In the case $A = B[X]$, $m \le n$, these rings have been analyzed in [**BS**].

The analogue (5.6) of (5.5) seems to be new however. We recall the notations of (4.2), (e): X is an alternating $n \times n$-matrix of indeterminates, $A = B[X]/\mathrm{Pf}_{r+2}(X)$, $F = A^n$, $\overline{X}\colon F \to F^*$ given by the residue class of X, and $M = \operatorname{Coker} \overline{X}$.

(5.6) COROLLARY. (a) *With the notations just recalled, the symmetric algebra of an "alternating" generic module M is an ASL. If $r < n$, $\mathrm{S}(M)$ is a torsionfree A-module.*
(b) *Let B be a Cohen-Macaulay ring. Then $\mathrm{S}(M)$ is Cohen-Macaulay if and only if $r < n$.*
(c) *Let B be a (normal) domain. Then $\mathrm{S}(M)$ is a (normal) domain if and only if $r < n$.*
(d) *Let B be normal and $r < n$. Then $\mathrm{Cl}(\mathrm{S}(M)) \cong \mathrm{Cl}(B) \oplus \mathbb{Z}$ if $r = n-1$, and $\mathrm{Cl}(\mathrm{S}(M)) \cong \mathrm{Cl}(B)$ if $r < n-1$. In particular $\mathrm{S}(M)$ is factorial if $r < n-1$ and B is factorial.*

PROOF: (a) and (b) are proved in the same way as (5.5).

Standard arguments involving flatness reduce (c) to the case in which B is a field (cf. [**BV.1**], Section 3 for example). Thus we may certainly suppose that B is a normal domain.

In the case in which $r = n-1$ the module M is just $I = \mathrm{Pf}_{n-1}(X)$ as remarked above, an ideal generated by a linearly ordered poset ideal. Then (i) $\mathrm{Gr}_I A$ is an ASL, in particular reduced, and (ii) $\mathrm{S}(M)$ is the Rees algebra of A with respect to I (cf. [**BST**] for example). Thus we can apply the main result of [**HV**] to conclude (c) and (d).

Let $r \le n-2$ now. In the spirit of this paper a "linear" argument seems to be most appropriate: By [**Fo**], Theorem 10.11 and [**Av.1**] it is sufficient that all the symmetric powers of M are reflexive. Since M_P, hence $\mathrm{S}^k(M_P)$ is free for prime ideals $P \not\supset \mathrm{Pf}_r(\overline{X})$ it is enough to show that $\mathrm{Pf}_r(\overline{X})$ contains an $\mathrm{S}^k(M)$-sequence of length 2 for every k. Each $\mathrm{S}^k(M)$ is an MSL whose data $\mathcal{I}(\dots)$ coincide with those of M itself. Therefore (4.8) can be applied and we can replace the $\mathrm{S}^k(M)$ by the residue class rings A/I_i, $I_i = A\{\pi \in \Phi_r(x)\colon \pi \not\geq [1,\dots,\widehat{i},\dots,r+1]\}$, $i = 1,\dots,r$. One has

$\mathrm{Pf}_r(X) \supset I_i$.

The poset Π underlying A/I_i is wonderful (cf. [**DEP.2**], Lemma 8.2 or [**BV.1**], (5.13)). Therefore the elements

$$[1,\ldots,\widehat{i},\ldots,r+1] = \sum_{\substack{\pi\in\Pi \\ rk\pi=1}} \pi \qquad \text{and} \qquad \sum_{\substack{\pi\in\Pi \\ rk\pi=2}} \pi$$

form an A/I_i-sequence by [**DEP.2**], Theorem 8.1. Both these elements are contained in $\mathrm{Pf}_r(\overline{X})$.

References

[**Av.1**] L. L. Avramov, *A class of factorial domains*, Serdica **5** (1979), 378–379.

[**Av.2**] L. L. Avramov, *Complete intersections and symmetric algebras*, J. Algebra **73** (1981), 248–263.

[**Br.1**] W. Bruns, *Generic maps and modules*, Compos. Math. **47** (1982), 171–193.

[**Br.2**] W. Bruns, *Divisors on varieties of complexes*, Math. Ann. **264** (1983), 53–71.

[**BKM**] W. Bruns, A. Kustin and M. Miller, *The resolution of the generic residual intersection of a complete intersection*, (to appear).

[**BS**] W. Bruns and A. Simis, *Symmetric algebras of modules arising from a fixed submatrix of a generic matrix.*, J. Pure Appl. Algebra **49** (1987), 227-245.

[**BST**] W. Bruns, A. Simis and Ngô Việt Trung, *Blow-up of straightening closed ideals in ordinal Hodge algebras*, (to appear).

[**BV.1**] W. Bruns and U. Vetter, "Determinantal rings," Springer Lect. Notes Math. **1327**, 1988.

[**BV.2**] W. Bruns and U. Vetter, *Modules defined by generic symmetric and alternating maps*, (to appear).

[**BE**] D. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3*, Amer. J. Math. **99** (1977), 447-485.

[**CN**] R. Cowsik and M. V. Nori, *On the fibers of blowing up*, J. Indian Math. Soc. **40** (1976), 217–222.

[**DEP.1**] C. De Concini, D. Eisenbud and C. Procesi, *Young diagrams and determinantal varieties*, Invent. Math. **56** (1980), 129–165.

[**DEP.2**] C. De Concini, D. Eisenbud and C. Procesi, "Hodge algebras," Astérisque **91**, 1982.

[**DS**] C. De Concini and E. Strickland, *On the variety of complexes*, Adv. Math. **41** (1981), 57–77.

[**Ei**] D. Eisenbud, *Introduction to algebras with straightening laws*, in "Ring Theory and Algebra III," M. Dekker, New York and Basel, 1980, pp. 243–267.

[**EH**] D. Eisenbud and C. Huneke, *Cohen-Macaulay Rees algebras and their specializations*, J. Algebra **81** (1983), 202–224.

[**Gr**] H.-G. Gräbe, *Über den arithmetischen Rang quadratfreier Potenzproduktideale*, Math. Nachr. **120** (1985), 217–227.

[**Ha**] R. Hartshorne, *Cohomological dimension of algebraic varieties*, Ann. Math. **88** (1968), 403–450.

[**HV**] J. Herzog and V. Vasconcelos, *On the divisor class group of Rees algebras*, J. Algebra **93** (1985), 182–188.

[**Ho**] M. Hochster, *Grassmannians and their Schubert subvarieties are arithmetically Cohen-Macaulay*, J. Algebra **25** (1973), 40–57.

[**HE**] M. Hochster and J. A. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020–1058.

[**Hi**] T. Hibi, *Distributive lattices, affine semigroup rings and algebras with straightening laws*, in "Commutative algebra and combinatorics," Advanced Studies in Pure Mathematics **11**, 1987, pp. 93–109.

[**Hu**] C. Huneke, *Powers of ideals generated by weak d-sequences*, J. Algebra **68** (1981), 471–509.

[**KL**] H. Kleppe and D. Laksov, *The algebraic structure and deformation of Pfaffian schemes*, J. Algebra **64** (1980), 167–189.

[**Ma**] H. Matsumura, "Commutative Algebra," Second Ed., Benjamin/Cummings, Reading, 1980.

[Ne] P. E. Newstead, *Some subvarieties of Grassmannians of codimension 3*, Bull. London Math. Soc. **12** (1980), 176–182.

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