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A Remark on Koszul Complexes

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Let R be a commutative ring and M an R-module. A linear form $\varphi: M \to R$ induces the Koszul complex

$$\mathcal{K}(\varphi)\colon \cdots \longrightarrow \textstyle\bigwedge^i M \xrightarrow{\partial_\varphi} \bigwedge^{i-1} M \longrightarrow \cdots \longrightarrow \bigwedge^2 M \longrightarrow M \longrightarrow R \longrightarrow 0$$

where $\partial_\varphi(m_1 \wedge \ldots \wedge m_i) = \sum_j (-1)^{j-1}\varphi(m_j) m_1 \wedge \cdots \widehat{m}_j \cdots \wedge m_i$ for all $m_1, \ldots, m_i \in M$. When M is a finite free R-module, then the vanishing of the homology of $\mathcal{K}(\varphi)$ is determined by the grade g_φ of the ideal $\operatorname{Im}\varphi$: one has $H_j(\mathcal{K}(\varphi)) = 0$ for $j = n-g_\varphi+1, \ldots, n$ and $H_{n-g_\varphi}(\mathcal{K}(\varphi)) \neq 0$. Giving a linear form on a finite free R-module of rank n is equivalent to specifying a sequence $a_1, \ldots, a_n$ of elements of R.

If M is not a free R-module, then the picture changes drastically. In this note we investigate the simplest case of a non-free module, namely $M = F/Rx$ where x is a non-zero element of the finite free R-module F. We define the linear map $\psi: R \to F$ by the assignment $1 \mapsto x$, so that $M = F/\operatorname{Im}\psi$. In plain terms, we now consider two sequences of elements of R, namely the sequence (x_i) of the components of x, and the sequence $(y_i) = (\varphi(\bar{e}_i))$ where $\bar{e}_i$ is the residue class of the ith element in a basis of F. These two sequences are connected by the equation $\sum_i x_i y_i = 0$.

It is of course not surprising that one can expect completely satisfactory results only when the ideals $\sum Ry_i = \operatorname{Im}\varphi$ and $\sum Rx_i = \operatorname{Im}\psi^*$ have "almost" their maximal grade (the superscript * denotes the R-dual). The grade g_{ψ^*} of $\operatorname{Im}\psi^*$ determines the homological properties of the exterior powers of M.

This note was motivated by a paper of Boffi [1], and its main application certainly is a simple and characteristic free approach to Boffi's results.

For simplicity we assume in the following that R is noetherian. Using the general theory of grade (for example, see Bruns and Herzog [3], Chapter 9) one can often replace R by an arbitrary commutative ring.

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Let F be a free R-module of rank n, and $\psi\colon R \to F$ an R-homomorphism with cokernel M, as above. Furthermore we have a linear form φ on F such that $\varphi\psi = 0$; the induced linear form on M is denoted by $\bar\varphi$ (it was φ above). For every non-negative integer i the map $\bigwedge^i F \to \bigwedge^{i+1} F$, $x \mapsto x \wedge \psi(1)$, induces an exact sequence

$$\textstyle\bigwedge^i M \xrightarrow{\psi_{i+1}} \bigwedge^{i+1} F \xrightarrow{\bigwedge^{i+1}\pi} \bigwedge^{i+1} M \to 0,$$

where $\pi\colon F \to M$ denotes the natural projection.

It is crucial in the following to compare the Koszul complexes $\mathcal{K}(\varphi)$ of φ and $\mathcal{K}(\bar\varphi)$ of $\bar\varphi$. Since obviously

$$\textstyle\partial_{\bar\varphi} \circ (\bigwedge^i \pi) = (\bigwedge^{i-1}\pi) \circ \partial_\varphi \quad \text{and} \quad \partial_\varphi \circ \psi_i = \psi_{i-1} \circ \partial_{\bar\varphi},$$

we obtain an exact sequence of complexes

$$\mathcal{K}(\bar\varphi)[-1] \xrightarrow{(\psi_i)} \mathcal{K}(\varphi) \xrightarrow{(\bigwedge^i \pi)} \mathcal{K}(\bar\varphi) \to 0.$$

Proposition 1. *For all i with $n - g_\varphi + 1 \le i < g_{\psi^*}$ one has*

$$H_{i+1}(\mathcal{K}(\bar\varphi)) \cong H_{i-1}(\mathcal{K}(\bar\varphi)).$$

Proof. For all j the module $\bigwedge^j M$ has rank $\binom{n-1}{j}$, so $\operatorname{Ker}\psi_{j+1} = t(\bigwedge^j M)$ (where $t(N)$ denotes the torsion submodule of the R-module N).

Since $i < g_{\psi^*}$, the modules $\bigwedge^j M$ are torsionfree for $j = 0, \dots, i$; see Lebelt [4], Satz 0. Since this special case of Lebelt's result is very easy, we include the argument: the sequence

$$0 \to R \xrightarrow{\psi(1)\wedge} F \xrightarrow{\psi(1)\wedge} \cdots \xrightarrow{\psi(1)\wedge} \textstyle\bigwedge^j F \to \bigwedge^j M \to 0$$

is exact because of $g_{\psi^*} > j$; it is in fact an augmented "tail" of the dual of the Koszul complex $\mathcal{K}(\psi^*)$. An easy depth count yields $\operatorname{depth} \bigwedge^j M_{\mathfrak p} \ge \min(1, \operatorname{depth} R_{\mathfrak p})$ for all prime ideals of R, and therefore $\bigwedge^j M$ is torsionfree.

Thus the diagram

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\bigwedge^{i+2} M & \longrightarrow & \bigwedge^{i+1} M & \xrightarrow{\partial_{\bar\varphi}} & \bigwedge^{i} M & \longrightarrow & \bigwedge^{i-1} M \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle \bigwedge^i \pi} & & \uparrow \\
\bigwedge^{i+2} F & \longrightarrow & \bigwedge^{i+1} F & \xrightarrow{\partial_{\varphi}} & \bigwedge^{i} F & \longrightarrow & \bigwedge^{i-1} F \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle \psi_i} & & \uparrow \\
\bigwedge^{i+1} M / t(\bigwedge^{i+1} M) & \longrightarrow & \bigwedge^{i} M & \xrightarrow{\partial_{\bar\varphi}} & \bigwedge^{i-1} M & \longrightarrow & \bigwedge^{i-2} M \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & 0 & & 0
\end{array}$$

has exact columns (the maps in the lower left corner are the induced ones). Since $n - g_\varphi + 1 \le i$, we have $H_{i+1}(\mathcal{K}(\varphi)) = H_i(\mathcal{K}(\varphi)) = 0$. The assertion now follows from the long exact sequence in homology. □

Now we discuss the case in which g_φ has its maximal value n, or equivalently, $\mathcal{K}(\varphi)$ is acyclic.

Proposition 2. *With the notation from above, suppose that $g_{\psi^*} = n$. Then $\operatorname{Im}\varphi \subset \operatorname{Im}\psi^*$, and for all i with $n - g_\varphi \le i \le n$ one has*

$$H_i(\mathcal{K}(\bar\varphi)) = \begin{cases} R/\operatorname{Im}\psi^* & i \equiv n \bmod 2, \\ 0 & i \not\equiv n \bmod 2. \end{cases}$$

In particular, if $g_{\psi^*} = n = g_\varphi$, then n is even and $\operatorname{Im}\varphi = \operatorname{Im}\psi^*$.

Proof. That $H_n(\mathcal{K}(\bar\varphi)) = R/\operatorname{Im}\psi^*$ is a consequence of $\partial_{\bar\varphi}(\bigwedge^n M) \subset t(\bigwedge^{n-1} M) = 0$ and $\bigwedge^n M = R/\operatorname{Im}\psi^*$. The last equality implies $\operatorname{Im}\varphi \subset \operatorname{Im}\psi^*$. Furthermore one has $g_\varphi \ge 1$ if and only if $H_{n-1}(\mathcal{K}(\bar\varphi)) = 0$: if $g_\varphi \ge 1$, then $H_n(\mathcal{K}(\varphi)) = 0$, so $\partial_{\bar\varphi}\colon \bigwedge^{n-1} M \to \bigwedge^{n-2} M$ is injective. Conversely, if $g_\varphi = 0$, then there is an $a \in R$, $a \ne 0$, with $a \operatorname{Im}\varphi = 0$. Since $\bigwedge^{n-1} M$ has rank 1, one has $a\bigwedge^{n-1} M \ne 0$; but $\partial_{\bar\varphi}(a\bigwedge^{n-1} M) = 0$, so $H_{n-1}(\mathcal{K}(\bar\varphi)) \ne 0$. The remaining assertions follow from Proposition 1. □

We add two

Remarks.

(1) By the Koszul complex, associated to the map $\bar\varphi\colon M \to R$, one often understands the complex

$$\mathcal{K}^*(\bar\varphi)\colon 0 \to R^* \xrightarrow{d_{\bar\varphi}} M^* \xrightarrow{d_{\bar\varphi}} (\textstyle\bigwedge^2 M)^* \to \dots \xrightarrow{d_{\bar\varphi}} (\textstyle\bigwedge^{n-1} M)^*$$

where $d_{\bar\varphi}$ sends the element $\alpha \in (\bigwedge^i M)^*$ to $\bar\varphi \wedge \alpha$. (Here we use that $\bigoplus_i (\bigwedge^i M)^*$ is equipped with a multiplication $\wedge$ which makes it an exterior algebra; for example, see Bruns [2], Theorem 1.4.) Choose an orientation χ on F and set

$$\mu(\bigwedge^{n-1} \pi(x)) = \chi(x \wedge \psi(1)),$$

for all $x \in \bigwedge^{n-1} F$, and

$$(\mu^i(u))(v) = \mu(u \wedge v)$$

for all $u \in \bigwedge^i M$, $v \in \bigwedge^{n-1-i} M$, $0 \le i \le n-1$, $\mu^n = 0$; clearly $\mu^i\colon \bigwedge^i M \to (\bigwedge^{n-1-i} M)^*$ is a homomorphism, and $d_{\bar\varphi} \circ \mu^i = (-1)^n \mu^{i-1} \circ \partial_{\bar\varphi}$. If $g_{\psi^*} = n$, then μ^i is an isomorphism for $i = 0, \dots, n-2$ and μ^{n-1} is injective (see Bruns [2], Theorem 2.4). Thus, in this case, the homology of $\mathcal{K}^*(\bar\varphi)$ is completely determined by the homology of $\mathcal{K}(\bar\varphi)$ (and vice versa).

(2) Suppose that $x_1, \dots, x_n$ is a regular sequence in R, and consider the first syzygy N of the ideal in R generated by $x_1, \dots, x_n$. We claim that there is a $y \in N$, $y \ne 0$, such that $(N/Ry)_\mathfrak{p}$ is free for every prime ideal $\mathfrak{p} \not\supseteq \sum Rx_i$ if and only if n is even. To prove this set $F = R^n$ and $\varphi(e_j) = x_j$ where $e_1, \dots, e_n$ are the elements of the canonical basis of F. For

the 'only if' part consider the map $\psi\colon R \to F = R^n$ given by $\psi(1) = y$. The condition for y means that $\operatorname{Im}\psi^*$ has grade $n = \operatorname{grade}\varphi$. Thus, by Proposition 2, n must be even. If n is even, then $y = \sum_{j=0}^{n-1}(-1)^j x_{n-j}e_{j+1} \in \operatorname{Ker}\varphi = N$ satisfies the condition in the claim. (This construction is closely related to the construction, given by Trautmann in [5], 4.2.4, of a reflexive coherent analytic sheaf on $\mathbb{C}^n$ with rank $n-2$ which is locally free on $\mathbb{C}^n \setminus \{0\}$ but not on $\mathbb{C}^n$.)

The subsequent example has been treated by Boffi in [1]. It was the impulse to our considerations.

Example. Let k be a field, and $x_1, \dots, x_m, y_1, \dots, y_p$ indeterminates over k. Set $\underline{x} = x_1, \dots, x_m$, $\underline{y} = y_1, \dots, y_p$, $R = k[\underline{x}, \underline{y}]$, and

$$A = \sum_{\substack{1\le i\le m\\ 1\le j\le p}} a_{ij}x_iy_j, \quad a_{ij} \in k.$$

Put $F = R^m \oplus R^p$, and define the R-homomorphism $\psi\colon R \to F$ by $\psi(1) = (\underline{x}, \underline{y})$. Denote by $d_1, \dots, d_m$ and $e_1, \dots, e_p$ the canonical bases of R^m and R^p. The linear form φ on F is given by $\varphi(d_i) = \frac{\partial A}{\partial x_i}$, $\varphi(e_j) = -\frac{\partial A}{\partial y_j}$. Clearly $\varphi\psi = 0$. As above let $\bar{\varphi}$ be the induced linear form on the cokernel of ψ.

Observe that

$$(\varphi(d_1), \dots, \varphi(e_p)) = \psi(1)\begin{pmatrix} 0 & -(a_{ij}) \\ (a_{ij})^t & 0 \end{pmatrix}$$

where the upper t means 'transpose'.

From Proposition 2 we draw:

Proposition 3. *Set $r = \operatorname{rank}(a_{ij})$. Then $H_{m+p-\lambda}(\mathcal{K}(\bar{\varphi})) \neq 0$ for $\lambda = 2r, \dots, m+p$. Furthermore, for $0 \le \lambda \le 2r$ one has*

$$H_{m+p-\lambda}(\mathcal{K}(\bar{\varphi})) = \begin{cases} k & \text{for } \lambda \equiv 0 \bmod 2 \\ 0 & \text{for } \lambda \not\equiv 0 \bmod 2. \end{cases}$$

Proof. Since $\operatorname{grade}\operatorname{Im}\varphi = 2r$, the second statement is an immediate consequence of Proposition 2. To prove the first one, observe that $H_\lambda(\mathcal{K}(\bar{\varphi})) = 0$ implies $\operatorname{Ker}\partial_\varphi \subset \operatorname{Im}\partial_\varphi + \operatorname{Im}\psi_\lambda$. Consider the systems of linear equations

$$\sum_{i=1}^m a_{ij}a_i = 0, \quad j = 1, \dots, p, \qquad \text{and} \qquad \sum_{j=1}^p a_{ij}b_j = 0, \quad i = 1, \dots, m,$$

over k. There are linearly independent elements $a^{(1)}, \dots, a^{(m-r)}$ in k^m solving the first system and linearly independent elements $b^{(1)}, \dots, b^{(p-r)}$ in k^p solving the second one. Set $v_i = (a^{(i)}, 0) \in k^{m+p}$, $i = 1, \dots, m-r$, and $w_j = (0, b^{(j)}) \in k^{m+p}$, $j = 1, \dots, p-r$. Obviously $\varphi(v_i) = \varphi(w_j) = 0$. So if $0 \le \rho \le m-r$, $0 \le \sigma \le p-r$, and $v_{i_1}, \dots, v_{i_\rho}$, $w_{j_1}, \dots, w_{j_\sigma}$

are distinct, then $v_{i_1} \wedge \ldots \wedge v_{i_\rho} \wedge w_{j_1} \wedge \ldots \wedge w_{j_\sigma}$ lies in $\operatorname{Ker} \partial_\varphi$ but not in $\operatorname{Im} \partial_\varphi + \operatorname{Im} \psi_{\rho+\sigma}$. Consequently $H_{\rho+\sigma}(\mathcal{K}(\bar\varphi)) \neq 0$. □

In particular Proposition 3 implies the main results of Boffi ([1], Theorem 7 and Propositions 8–10) and answers some questions asked there:

Proposition 4. *The following conditions are equivalent:*

(1) $m = p$ *and the matrix* (a_{ij}) *is invertible.*

(2) $$H_i(\mathcal{K}(\bar\varphi)) = \begin{cases} k & \text{if } i \leq 2m \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

(3) $H_1(\mathcal{K}(\bar\varphi)) = 0$.

Also in the case in which g_{ψ^*} and g_φ are "submaximal" one gets a satisfactory result on the behaviour of $\mathcal{K}(\bar\varphi)$.

Proposition 5. *With the notation introduced above Proposition* 1 *suppose that* $g_{\psi^*} = g_\varphi = \dim R = n - 1$. *Then all homology modules* $H_i(\mathcal{K}(\bar\varphi))$, $1 \leq i \leq n$, *have the same (finite) length. In particular, the following conditions are equivalent:*

(1) $H_i(\mathcal{K}(\bar\varphi)) = 0$ *for* $i = 1, \ldots, n$;

(2) $H_j(\mathcal{K}\bar\varphi)) = 0$ *for some* j, $1 \leq j \leq n$.

Proof. First observe that $\bigwedge^i M$ is torsionfree for $i = 0, \ldots, n-2$. Corresponding to the diagram in the proof of Proposition 1 we obtain an exact sequence of complexes

$$0 \to \mathcal{K}(\bar\varphi)[-1]' \xrightarrow{(\psi_i')} \mathcal{K}(\varphi) \xrightarrow{(\bigwedge^i \pi)} \mathcal{K}(\bar\varphi) \to 0$$

where $\mathcal{K}(\bar\varphi)[-1]'$ is the complex $\mathcal{K}(\bar\varphi)[-1]$ except that the map $\partial_{\bar\varphi}\colon \bigwedge^{n-1} M \to \bigwedge^{n-2} M$ is replaced by the induced map $\bigwedge^{n-1} M / t(\bigwedge^{n-1} M) \to \bigwedge^{n-2} M$; correspondingly one chooses $\psi_i' = \psi_i$ for $i < n$ and replaces ψ_n by the induced map ψ_n'. From the long exact sequence in homology we then cut the exact piece

$$0 \longrightarrow H_2(\mathcal{K}(\bar\varphi)) \longrightarrow R/\operatorname{Im}\varphi \longrightarrow H_1(\mathcal{K}(\varphi)) \longrightarrow H_1(\mathcal{K}(\bar\varphi)) \longrightarrow 0.$$

Since $\ell(H_1(\mathcal{K}(\varphi)) = \ell(H_0(\mathcal{K}(\varphi)) = \ell(R/\operatorname{Im}\varphi) < \infty$ (ℓ means 'length'), it follows that $\ell(H_2(\mathcal{K}(\bar\varphi)) = \ell(H_1(\mathcal{K}(\bar\varphi))$. The remaining assertions result from Proposition 1. □

Example. Let k be a field of characteristic 0, P the polynomial ring over k in the indeterminates $X_1, \ldots, X_n$, graded by $\deg X_i = m_i > 0$, $i = 1, \ldots, n$. Choose a non-constant homogeneous polynomial $f \in P$, and set $R = P/(f)$. Let $F = R^n$ and define the R-homomorphism $\psi\colon R \to F$ via $\psi(1) = (\partial f/\partial X_1, \ldots, \partial f/\partial X_n) \bmod f \cdot F$. Then the cokernel of ψ is the R-module $\mathrm{D}_k(R)$ of Kähler differentials of R over k. The linear form φ on F maps the ith unit vector to $m_i x_i$ where x_i denotes the residue of X_i in R; so $\bar\varphi$

is the linear form on $\mathrm{D}_k(R)$ induced by the Euler derivation. Clearly $\varphi\psi = 0$. A direct computation yields $H_1(\mathcal{K}(\bar{\varphi})) = 0$.

Suppose now that $g_{\psi^*} = n - 1$; this means that (the affine variety with coordinate ring) R has an isolated singularity at 0. By Proposition 5, it results the well-known fact that the Koszul complex of $\bar{\varphi}$ is exact.

The same is true for $\mathcal{K}(\bar{\psi}^*)$; by $\bar{\psi}^*$ we of course mean the linear form induced by ψ^* on the cokernel of $\varphi^*: R \to F^*$. In fact, there are only two possibilities for $H_n(\mathcal{K}(\bar{\psi}^*))$, namely $H_n(\mathcal{K}(\bar{\psi}^*)) = 0$ and $H_n(\mathcal{K}(\bar{\psi}^*)) = k$; but the second case is actually impossible since it would imply that $H_1(\mathcal{K}(\varphi)) = 0$.

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