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Cohomology of partially ordered sets and local cohomology of section rings

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Abstract

We study local cohomology of rings of global sections of sheafs on the Alexandrov space of a partially ordered set. We give a criterion for a splitting of the local cohomology groups into summands determined by the cohomology of the poset and the local cohomology of the stalks. The face ring of a rational pointed fan can be considered as the ring of global sections of a flasque sheaf on the face poset of the fan. Thus we obtain a decomposition of the local cohomology of such face rings. Since the Stanley–Reisner ring of a simplicial complex is the face ring of a rational pointed fan, our main result can be interpreted as a generalization of Hochster's decomposition of local cohomology of Stanley–Reisner rings.

Keywords: Local cohomology; Partially ordered set; Face rings; Sections rings

1. Introduction

In this paper we study local cohomology of rings of global sections of sheafs on the Alexandrov space of a partially ordered set. Before we introduce the concepts needed to state our main results, we describe some consequences.

Recall that a *simplicial complex* Δ on a finite vertex set V is a collection Δ of subsets of V closed under inclusion, that is, if $F \subseteq G$ and $G \in \Delta$, then $F \in \Delta$. A rational pointed fan Σ in $\mathbb{R}^d$

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is a finite collection of rational pointed cones in $\mathbb{R}^d$ such that for $C' \subseteq C$ with $C \in \Sigma$ we have that C' is a face of C if and only if $C' \in \Sigma$, and such that if $C, C' \in \Sigma$, then $C \cap C'$ is a common face of C and C'. The face poset $P(\Sigma)$ of Σ is the partially ordered set of faces of Σ ordered by inclusion. The *face ring* $K[\Sigma]$ of Σ over a field K is defined as follows: As a K-vector space $K[\Sigma]$ has one basis element x^a for each a in the intersection of $\mathbb{Z}^d$ and the union of the faces of Σ. Multiplication in $K[\Sigma]$ is defined by

$$x^a x^b = \begin{cases} x^{a+b} & \text{if } a \text{ and } b \text{ are elements of a common face of } \Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

A simplicial complex Δ on the vertex set $V = \{1, \ldots, d-1\}$ has an associated rational fan $\Sigma(\Delta)$ with one cone $C(F)$ for each face F of Δ. The cone $C(F)$ is equal to the set of vectors in $\mathbb{R}^d_{\geqslant 0} = \{(x_1, \ldots, x_d)\colon x_1, \ldots, x_d \geqslant 0\}$ with $x_d = \sum_{i=1}^{d-1} x_i$ and $x_i = 0$ for $i \in V \setminus F$. The face ring $K[\Sigma(\Delta)]$ is called the *Stanley–Reisner ring* $K[\Delta]$ of Δ.

Stanley observed in [13, Lemma 4.6] that the face ring $K[\Sigma]$ of a rational pointed fan Σ is a Cohen–Macaulay ring if the Stanley–Reisner ring $K[\Delta(P(\Sigma))]$ of the order complex of the face poset of Σ is a Cohen–Macaulay ring. Stanley reduces the proof of this observation to a theorem of Yuzvinsky [18, Theorem 6.4]. Theorem 1.2 below generalizes Yuzvinsky's theorem, and Stanley's observation is a direct consequence of it.

We consider every poset P as a topological space with the *Alexandrov topology*, that is, the topology where the open sets are the lower subsets (also called order ideals) of P. If $\mathscr{T}$ is a sheaf of K-algebras on P, then for every $x \in P$ there is a restriction homomorphism $H^0(P, \mathscr{T}) \to \mathscr{T}_x$ from the zeroth cohomology group of P with coefficients in $\mathscr{T}$ to the stalk $\mathscr{T}_x$ of $\mathscr{T}$ at x. Given an ideal I in a commutative ring R and an R-module M we denote the local cohomology groups of M by $H^i_I(M)$ for $i \geqslant 0$. The following is our decomposition of local cohomology.

Theorem 1.1. *Let K be a field, let $\mathscr{T}$ be a sheaf of K-algebras on a finite poset P and let I be an ideal of the zeroth cohomology group $H^0(P, \mathscr{T})$ of P with coefficients in $\mathscr{T}$. For $x \in P$ we let d_x denote the Krull dimension of the stalk $\mathscr{T}_x$ of $\mathscr{T}$ at x and we assume that*:

(i) *$H^0(P, \mathscr{T})$ is a Noetherian ring and $H^i(P, \mathscr{T}) = 0$ for every $i > 0$,*
(ii) *$H^i_I(\mathscr{T}_x) = 0$ for every $x \in P$ and every $i \neq d_x$,*
(iii) *if $x < y$ in P then $d_x < d_y$.*

Then there is an isomorphism

$$H^i_I\big(H^0(P, \mathscr{T})\big) \cong \bigoplus_{x \in P} \widetilde{H}^{i-d_x-1}\big((x, 1_{\widehat{P}}); K\big) \otimes_K H^{d_x}_I(\mathscr{T}_x)$$

of K-modules, where $\widetilde{H}^{i-d_x-1}((x, 1_{\widehat{P}}); K)$ denotes the reduced cohomology of the partially ordered set $(x, 1_{\widehat{P}}) = \{y \in P\colon x < y\}$ with coefficients in K. If $\mathscr{T}$ is a sheaf of $\mathbb{Z}^d$-graded K-algebras, the above isomorphism is an isomorphism of $\mathbb{Z}^d$-graded K-modules.

The zeroth cohomology ring $H^0(P, \mathscr{T})$ is naturally identified with the ring of global sections of $\mathscr{T}$. Under this name it was studied by Yuzvinsky [18,19] and Caijun [8]. The following immediate corollary of Theorem 1.1 generalizes the results [8, Theorem 2.4] and [18, Theorem 6.4] of Caijun and Yuzvinsky.

Theorem 1.2. *Suppose in the situation of Theorem* 1.1 *that there exists a unique graded maximal ideal* $\mathfrak{m}$ *in* $H^0(P,\mathscr{T})$ *which is maximal considered as an ideal of* $H^0(P,\mathscr{T})$*. If the assumptions of Theorem* 1.1 *are satisfied by the ideal* $\mathfrak{m}$*, then the ring* $H^0(P,\mathscr{T})$ *is Cohen–Macaulay if and only if there exists a number* n *such that the reduced cohomology* $\widetilde{H}^*(\Delta((x,1_{\widehat{P}})),K)$ *of the simplicial complex* $\Delta((x,1_{\widehat{P}}))$ *associated to the poset* $(x,1_{\widehat{P}})$ *is concentrated in degree* $(n-d_x-1)$ *for every* $x\in P$ *with* $\mathscr{T}_x\neq 0$.

Let us return to the situation where $P=P(\Sigma)$ is the face poset of a rational pointed fan Σ in $\mathbb{R}^d$. There is a unique flasque sheaf $\mathscr{T}$ of $\mathbb{Z}^d$-graded K-algebras associated to Σ with stalks $\mathscr{T}_C=K[C]$ given by the monoid algebras on the cones of Σ, with a natural isomorphism $K[\Sigma]\cong H^0(P,\mathscr{T})$ between the face ring of Σ and the ring of global sections of $\mathscr{T}$ and with restriction homomorphisms $H^0(P,\mathscr{T})\cong K[\Sigma]\to K[C]\cong\mathscr{T}_C$ acting by the identity on x^a if $a\in C$ and taking x^a to zero otherwise (see [3, Theorem 4.7]). Note that the face ring $K[\Sigma]$ is Noetherian, $\mathbb{Z}^d$-graded and has a unique maximal $\mathbb{Z}^d$-graded ideal $\mathfrak{m}$. The K-algebras $K[C]$ are normal and thus Cohen–Macaulay of Krull dimension $d_C=\dim(C)$. Hence the following result is a direct consequence of Theorem 1.1 since every flasque sheaf of rings on a poset satisfies assumption (i).

Theorem 1.3. *Let* Σ *be a rational pointed fan in* $\mathbb{R}^d$ *with face poset* P*,* K *be a field and* $\mathfrak{m}$ *be the graded maximal ideal of the face ring* $K[\Sigma]$*. Then there is an isomorphism*

$$H^i_{\mathfrak{m}}\big(K[\Sigma]\big)\cong\bigoplus_{C\in P}\widetilde{H}^{i-\dim(C)-1}\big((C,1_{\widehat{P}});K\big)\otimes_K H^{\dim(C)}_{\mathfrak{m}}\big(K[C]\big)$$

of $\mathbb{Z}^d$*-graded* K*-modules.*

If $\Sigma=\Sigma(\Delta)$ for a simplicial complex Δ, then the posets $(C,1_{\widehat{P}})=(C(F),1_{\widehat{P}})$ in the above formula are isomorphic to the face posets $P(\operatorname{lk}_\Delta F)\setminus\emptyset$ of the links in Δ. Thus the above theorem generalizes Hochster's decomposition of local cohomology of Stanley–Reisner rings (see [7]). Just like Reisner's topological characterization of the Cohen–Macaulay property of Stanley–Reisner rings is a consequence of Hochster's decomposition of local cohomology of Stanley–Reisner rings, the observation [13, Lemma 4.6] of Stanley mentioned above is a consequence of Theorem 1.2.

Sheaves on a poset P can be described in a different way. More precisely, let R be a commutative ring, then a sheaf $\mathscr{T}$ of R-algebras on P is described by a unique collection $(T_x)_{x\in P}$ of R-algebras and homomorphisms $T_{xy}:T_y\to T_x$ for $x\leqslant y$ in P with the property that T_{xx} is the identity on T_x and that $T_{xy}\circ T_{yz}=T_{xz}$ for every $x\leqslant y\leqslant z$ in P. Moreover, every such collection describes a unique sheaf on P, and we have that $H^0(P,\mathscr{T})$ is the (inverse) limit $\lim T_x$. In the body of this paper we call $T=(T_x,T_{xy})$ an RP*-algebra*, and we work with RP-algebras instead of with sheaves. One reason for this change of perspective is that homological algebra of RP-algebras is more accessible than sheaf cohomology. In fact, Theorem 1.1 is a consequence of general homological arguments.

Algebras of type $\lim T$ for an RP-algebra T appear at many places in commutative algebra and combinatorics. For example, Bruns and Gubeladze studied such algebras in a series of papers [5,6]. Brun and Römer considered the relationship between initial ideals of the defining ideal of the face ring of a rational fan and subdivisions of that fan in [3].

This paper is organized as follows. In Section 2 we recall definitions and notations related to posets and abstract simplicial complexes. In Section 3 we introduce RP-algebras and give examples from commutative algebra and combinatorics. In Section 4 we study the local cohomology of limits of RP-algebras. In particular, we prove Theorem 1.1. In Section 5 we present further applications.

The proof of Theorem 1.1 uses homological algebra over a poset P. We recollect in Sections 6 and 7 some notation and basic facts needed in the proof. Perhaps apart from Proposition 6.6 the results in Sections 6 and 7 are well known. We have included these sections as a bridge between the questions considered in this paper and the literature on homological algebra of functor categories. They are also intended as a soft introduction to the language of functor categories for the reader who is not very familiar with category theory. More detailed accounts of the concepts introduced here can be found in the papers of Baues–Wirsching [2] and of Mitchell [11]. Sections 8 is the technical heart of the paper. Here we use the theory collected in Sections 6 and 7 to study the local cohomology of RP-modules.

2. Prerequisites

In this paper $P = (P, \leqslant)$ always denotes a *partially ordered set* (poset for short). Given P, the opposite poset $P^{\mathrm{op}} = (P, \preccurlyeq)$ has the same underlying set as P and the opposite partial order $\preccurlyeq$, that is, $y \preccurlyeq x$ if and only if $x \leqslant y$. We write $x < y$ if $x \neq y$ and $x \leqslant y$. We also consider a poset as a category "with morphisms pointing down," that is, for $x, y \in P$ there is a unique arrow $y \to x$ if and only if $x \leqslant y$. If P contains a unique maximal element, then this element is called the *initial element* of P, and it will be denoted 1_P. Analogously a unique minimal element of P is called a *terminal element* of P and it is denoted 0_P. The poset $\widehat{P} = (\widehat{P}, \leqslant)$ associated with P has underlying set $\widehat{P} = P \cup \{0_{\widehat{P}}, 1_{\widehat{P}}\}$ obtained by adding a terminal element $0_{\widehat{P}}$ and an initial element $1_{\widehat{P}}$ to P (in spite of a terminal or an initial element which may already exist in P). The closed interval $[x, y]$ of elements between x and y in P is the set $[x, y] = \{z \in P\colon\ x \leqslant z \leqslant y\}$ considered as a sub-poset of P. The half-open interval $[x, y)$, the half-open interval $(x, y]$ and the open interval (x, y) are described similarly. Note that if $x \in \widehat{P}$ and $y \in P$, then $(x, y]$ is a sub-poset of P. A finite poset P is called a *graded* poset, if all maximal chains (i.e. totally ordered subsets) of P have the same length $\operatorname{rank}(P)$. In this situation it is possible to define a unique rank function on P such that for $x \in P$ we have that $\operatorname{rank}(x)$ is the common length of maximal chains in P ending at x.

The *Alexandrov topology* [1] on a poset P is the topology where the open subsets are the lower subsets (also called order ideals), that is, the subsets U such that $y \in U$ and $x \leqslant y$ implies $x \in U$. The subsets of the form $(0_{\widehat{P}}, x]$ form a basis for this topology. (Sometimes, e.g. in [18], this is called the *order topology* on P^{op}.)

The poset P is *locally finite* if every closed interval of the form $[x, y]$ for $x, y \in P$ is finite and it is *topologically finite* if every interval of the form $(0_{\widehat{P}}, x]$ for $x \in P$ is finite. Note that locally finite does not imply topologically finite but the converse is true and that P is topologically finite if and only if every element of P has a finite neighborhood in the Alexandrov topology.

If a simplicial complex Δ on a finite vertex set V is non-empty, then the empty set is a terminal element in the *face poset*, that is, in the partially ordered set $P(\Delta) = (\Delta, \subseteq)$ of elements in Δ ordered by inclusion. The elements F of Δ are called *faces*. If F contains $d + 1$ vertices, that is, $d + 1$ elements of V, then F is called a *d-dimensional face*, and we write $\dim F = d$. The empty set is a face of dimension -1. The *dimension* $\dim \Delta$ is the supremum of the dimensions of the faces of Δ.

Above we constructed a poset $P(\Delta)$ associated to every simplicial complex Δ. Conversely, the *order complex* $\Delta(P)$ of P is the simplicial complex on the vertex set P consisting of the chains in P ordered by inclusion.

Recall that if F is a face of a simplicial complex Δ then the *link* $\operatorname{lk}_\Delta F$ of F in Δ is the simplicial complex $\operatorname{lk}_\Delta F = \{G \setminus F\colon G \in \Delta \text{ and } F \subseteq G\}$. It is easy to see that the correspondence $G \setminus F \mapsto G = (G \setminus F) \cup F$ defines an order-preserving bijection

$$P(\operatorname{lk}_\Delta F) \xrightarrow{\cong} [F, 1_{\widehat{P(\Delta)}}).$$

Thus the barycentric subdivision $\Delta(P(\operatorname{lk}_\Delta F) \setminus \emptyset)$ of $\operatorname{lk}_\Delta F$ has the same reduced simplicial (co-)homology as $\Delta((F, 1_{\widehat{P(\Delta)}}))$. This fact will be used several times in this paper.

For more details on simplicial complexes and posets see, for example, the corresponding chapters in the books of Bruns–Herzog [7] and Stanley [14,15].

3. Examples of RP-algebras

Fix a commutative ring R and a poset $(P, \leqslant)$. An *RP-algebra* T is a system $(T_x)_{x\in P}$ of R-algebras and homomorphisms $T_{xy}: T_y \to T_x$ for $x \leqslant y$ in P with the property that T_{xx} is the identity on T_x and that $T_{xy} \circ T_{yz} = T_{xz}$ for every $x \leqslant y \leqslant z$ in P. The (inverse) *limit* of T is the subring $\lim T$ of $\prod_{x\in P} T_x$ consisting of sequences $r = (r_x)_{x\in P}$ with the property that $T_{xy}(r_y) = r_x$ for every $x \leqslant y$ in P. In particular, an RP-algebra is an RP-module in the sense explained in Section 6 and we can apply the theory developed in that section. The RP-algebra T is called *cyclic* if the homomorphism $R \to T_x$ is surjective for every $x \in P$. We call T an RP-algebra of $\mathbb{Z}^d$-graded RP-algebras if the R-algebras T_x are $\mathbb{Z}^d$-graded, and the homomorphisms $T_{xy}: T_y \to T_x$ and $R \to T_x$ are homogeneous of degree zero for $x \leqslant y$. In this case $\lim T$ is a $\mathbb{Z}^d$-graded R-algebra. Similarly, we call T an RP-algebra of Cohen–Macaulay rings, if all T_x are Cohen–Macaulay rings.

The following examples motivate the study of RP-algebras in commutative algebra and algebraic combinatorics. For a field K and a set F we let $K[F] = K[x_i\colon i \in F]$ be the polynomial ring with one indeterminate for each $i \in F$.

Example 3.1 *(Stanley–Reisner ring).* Let K be a field, Δ be a simplicial complex on the vertex set $V = \{1, \ldots, d\}$ and $P = P(\Delta)$. For $F \in P$ define $T_F = K[F]$. For $G \subseteq F$, we define $T_{GF}: K[F] \to K[G]$ to be the natural projection. If we let $R = K[\bigcup_{F\in P} F]$, then T is a cyclic RP-algebra and

$$\lim T \cong R/I_\Delta,$$

where I_Δ is generated by all squarefree monomials $\prod_{i\in G} x_i$ for $G \subseteq V$, $G \notin \Delta$. Hence $\lim T$ is the usual *Stanley–Reisner ring* in this case.

Note that the polynomial algebra $R = K[x_1, \ldots, x_d]$ is $\mathbb{Z}^d$-graded and T is a cyclic RP-algebra of $\mathbb{Z}^d$-graded R-algebras.

Example 3.2 *(Toric face rings).* We consider a rational pointed fan Σ in $\mathbb{R}^d$, that is, a collection of rational pointed cones in $\mathbb{R}^d$ such that for $C' \subseteq C$ with $C \in \Sigma$ we have that C' is a face of C if and only if $C' \in \Sigma$, and if $C, C' \in \Sigma$, then $C \cap C'$ is a common face of C and C'. Let P be the face poset of Σ ordered by inclusion. For a rational pointed cone $C \in P$ we let T_C be the monoid ring $K[C \cap \mathbb{Z}^d]$ over a field K. The homomorphisms $T_{C'C} : T_C \to T_{C'}$ are induced by the natural face projection. Then T is a $\mathbb{Z}^d$-graded KP-algebra. For a suitable polynomial ring R over K the homomorphisms $R \to T_C$ are all surjective and $\mathbb{Z}^d$-graded. Then T is a cyclic RP-algebra of $\mathbb{Z}^d$-graded R-algebras. In any case, $\lim T$ is the *toric face ring* of Σ. This examples goes back to a construction of Stanley in [13]. It was generalized, and these rings were studied by Brun–Römer in [3]. The limits $\lim T$ of such algebras were intensively studied by Bruns–Gubeladze (see [5,6]).

Note that Stanley–Reisner rings of simplicial complexes are covered by this example: Let Δ be a simplicial complex on the vertex set $V = \{1, \dots, d-1\}$. To a subset F of V we associate the pointed cone C_F in $\mathbb{R}^d$ generated by the set of elements of the form $e_i + e_d$ for $i \in F$. Here e_i denotes the standard basis vector $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ of $\mathbb{R}^d$. If Σ denotes the fan in $\mathbb{R}^d$ consisting of the cones C_F for $F \in \Delta$, then the face posets of Δ and of Σ are isomorphic and the RP-algebras of Examples 3.1 and 3.2 correspond to each other via this isomorphism.

Borrowing notation from the theory of sheaves we call an RP-algebra T *flasque* if $\lim T|_U \to \lim T|_V$ is surjective for all open sets $V \subseteq U$ of the poset P. In the third example we present a general construction to produce flasque RP-algebras. (See also [18].)

Example 3.3. Let R be a commutative ring and $\mathscr{D}$ a distributive lattice of ideals in R (with respect to sum and intersection). Moreover, let P be a finite subset of $\mathscr{D}$ such that $I + J \in P$ for all $I, J \in P$, and consider P as a poset with $I \geqslant J$ if $I \subseteq J$. Let T be the RP-algebra given by $T_I = R/I$ for all $I \in P$ and T_{JI} the natural epimorphism $R/I \to R/J$. For $I_1, \dots, I_n$ in P let U be the smallest open subset of P containing $I_1, \dots, I_n$. We claim that

$$\lim T|_U = R/(I_1 \cap \cdots \cap I_n).$$

In fact, clearly $\lim T|_U$ is the kernel of the map

$$\Phi : R/I_1 \times \cdots \times R/I_n \to \prod_{i<j} R/(I_i + I_j),$$

where $\Phi(\bar{a}_1, \dots, \bar{a}_n) = (\bar{a}_i - \bar{a}_j \colon\ i < j)$ and $\bar{\ }$ denotes the residue class with respect to the appropriate ideal. For $n = 2$ the claim is proved by the classical exact sequence

$$0 \to R/(I_1 \cap I_2) \to R/I_1 \times R/I_2 \to R/(I_1 + I_2) \to 0.$$

By induction we can assume that an element in the kernel of Φ has the form $(\bar{a}, \dots, \bar{a}, \bar{b})$, and it remains to show that the sequence

$$0 \to R/\big((I_1 \cap \cdots \cap I_{n-1}) \cap I_n\big) \to (R/I_1 \cap \cdots \cap I_{n-1}) \times R/I_n \xrightarrow{\Phi'} \prod_{i=1}^{n-1} R/(I_i + I_n)$$

is exact. By the case $n = 2$, it is enough that the target of Φ' can be replaced by $R/(I_1 \cap \cdots \cap I_{n-1}) + I_n$, and this follows immediately from distributivity. Finally, it is now immediate that T is flasque.

Distributive lattices of ideals in rings (with respect to sum and intersection) appear naturally in commutative algebra. For example, let $S = K[x_1, \dots, x_n]$ be the polynomial ring. Assume that we have fixed a standard K-basis $\mathscr{S}$ of S and $\mathscr{D}$ is the set of ideals which have a subset of $\mathscr{D}$ as a K-basis. Then $\mathscr{D}$ is a distributive lattice. If $\mathscr{S}$ is the set of usual monomials in S, then $\mathscr{D}$ is the set of monomial ideals. Choosing a finite set $\mathscr{P}$ of monomial prime ideals closed under summation of ideals gives back the example of Stanley–Reisner rings.

Example 3.3 is in fact quite general. First let us note that if T is a flasque RP-algebra on a finite poset P with the property that the homomorphism $R \to \lim T$ is surjective, then the kernel of this homomorphism is given by the intersection of the kernels I_x of the compositions $R \to \lim T \to T_x$ for $x \in P$. Indeed, the homomorphism $R/\bigcap_{x\in P} I_x \to \lim T$ is surjective, and it is injective since the composition $R/\bigcap_{x\in P} I_x \to \lim T \to \prod_{x\in P} T_x$ is injective. Next note that if Q denotes the poset consisting of nonzero sums of ideals of the form I_x for $x \in P$, with order given by reverse inclusion, there is an order-preserving map $P \to Q$ taking $x \in P$ to $I_x \in Q$. Since $T_x \cong R/I_x$, there is a canonical homomorphism $\lim_{I\in Q} R/I \to \lim T$. Since the kernel of $R \to \lim_{I\in Q} R/I$ is equal to the kernel of $R \to \lim T$, we have an isomorphism $\lim_{I\in Q} R/I \cong \lim T$. Finally, let us note that the map $P \to Q$ is injective if and only if $x < y$ in P implies that the homomorphism $T_y \to T_x$ has a nonzero kernel. For further examples of RP-algebras see [4].

4. Decomposition results

This section contains our main results on the local cohomology of the limit of an RP-algebra T. The following theorem is our most general decomposition of local cohomology groups. The proof will be given in Section 8.

Theorem 4.1. *Let R be a Noetherian commutative algebra over a field K and $I \subset R$ an ideal. Let P be a finite poset and let T be an RP-algebra. For $x \in P$ we let d_x denote the Krull dimension of T_x and we assume that*:

(i) $H_I^i(T_x) = 0$ *for every $x \in P$ and every $i \neq d_x$,*
(ii) $\operatorname{Ext}^i_{RP}(R, T) = 0$ *for every $i > 0$,*
(iii) *if $x < y$ in P then $d_x < d_y$.*

Then there is an isomorphism

$$H_I^i(\lim T) \cong \bigoplus_{x\in P} \widetilde{H}^{i-d_x-1}\big((x, 1_{\widehat{P}}); K\big) \otimes_K H_I^{d_x}(T_x)$$

of K-modules.

If R is a Noetherian commutative $\mathbb{Z}^d$-graded algebra over a field K, $I \subset R$ is a graded ideal and T is an RP-algebra of $\mathbb{Z}^d$-graded R-algebras, then the above isomorphism is an isomorphism of $\mathbb{Z}^d$-graded K-modules.

Recall that a graded maximal ideal in a $\mathbb{Z}^d$-graded ring R is a graded ideal $\mathfrak{m}$ that is maximal among the proper graded ideals in R, and that R is graded local if it contains a unique graded maximal ideal. In the next section we give applications of the following (graded) version of Theorem 4.1.

Corollary 4.2. *Let R be a Noetherian $\mathbb{Z}^d$-graded local commutative algebra over a field K with graded maximal ideal $\mathfrak{m}$ which is maximal considered as an ideal of R. Let P be a finite poset and let T be a cyclic RP-algebra of $\mathbb{Z}^d$-graded R-algebras. For $x \in P$ we let d_x denote the Krull dimension of T_x and we assume that*:

(i) *T_x is a Cohen–Macaulay ring for every $x \in P$,*
(ii) $\mathrm{Ext}^i_{RP}(R, T) = 0$ *for every $i > 0$,*
(iii) *if $x < y$ in P then $d_x < d_y$.*

Then there is an isomorphism

$$H^i_{\mathfrak{m}}(\lim T) \cong \bigoplus_{x \in P} \widetilde{H}^{i-d_x-1}\big((x, 1_{\widehat{P}}); K\big) \otimes_K H^{d_x}_{\mathfrak{m}}(T_x)$$

of $\mathbb{Z}^d$-graded K-modules.

Proof. We first note that under the given assumptions T_x is Cohen–Macaulay if and only if $(T_x)_{\mathfrak{m}}$ is Cohen–Macaulay. (See [7, Exercise 2.1.27].) By Grothendieck's Vanishing Theorem on local cohomology modules (see [7, Theorem 3.5.7]) this is the case if and only if condition (i) of Theorem 4.1 is satisfied. The other conditions of Theorem 4.1 are part of our assumptions. □

5. Applications

In this section we explain how the results of Section 4 generalize the Hochster formulas for local cohomology. Let P be a poset and let R be a commutative ring. The following lemma is a consequence of Lemma 7.3.

Lemma 5.1. *Let R be a commutative ring and let P be a poset. If T is a flasque RP-algebra, then* $\mathrm{Ext}^i_{RP}(R, T) = 0$ *for all $i > 0$.*

Using Corollary 4.2 it is possible to study several properties of the ring $\lim T$. The next result generalizes results of Yuzvinsky [18, Theorem 6.4] and Caijun [8, Theorem 2.4] in the graded situation.

Corollary 5.2. *Let R be a $\mathbb{Z}^d$-graded local Noetherian algebra over a field K with unique graded maximal ideal $\mathfrak{m}$ which is maximal considered as an ideal of R. Assume that T is a flasque cyclic RP-algebra of $\mathbb{Z}^d$-graded Cohen–Macaulay R-algebras such that $x < y$ in P implies $d_x < d_y$. The following statements are equivalent*:

(i) *$\lim T$ is a Cohen–Macaulay ring,*
(ii) *$\widetilde{H}^p((x, 1_{\widehat{P}}); K) = 0$ for $x \in P$ and $p \neq \dim(\lim T) - d_x - 1$.*

Proof. Recall that by standard arguments $\lim T$ is a Cohen–Macaulay ring if and only if $H^i_{\mathfrak{m}}(\lim T) = 0$ for $i \neq \dim(\lim T)$. The equivalence of (i) and (ii) is a direct consequence of Corollary 4.2 and Lemma 5.1. □

We need the following result.

Lemma 5.3. *Let T be a KP-algebra. Assume that there exists $x \in P$ with the following properties*:

(i) $T_y = T_x$ *for every* $y \in [x, 1_{\widehat{P}})$,
(ii) T_{xy} *is the identity homomorphism on* T_x *for every* $y \in [x, 1_{\widehat{P}})$,
(iii) $T_y = 0$ *if* $y \notin [x, 1_{\widehat{P}})$.

Then T is a flasque KP-algebra.

Proof. If U is an open subset of P then $T|_U = 0$ if $x \notin U$, and otherwise $\lim T|_U \cong T_x$. If $V \subseteq U$ is an inclusion of open subsets of P then the natural projection $\lim T|_U \to \lim T|_V$ is isomorphic to the identity on T_x if $x \in V$, and otherwise $\lim T|_V = 0$. Thus T is a flasque KP-algebra. □

The following result generalizes an observation of Yuzvinsky [18, Proposition 7.6].

Proposition 5.4. *Let Σ be a rational pointed fan in $\mathbb{R}^d$ with face lattice P. The $\mathbb{Z}^d$-graded KP-algebra T of Example* 3.2 *is flasque and if $D \subset C$ in P, then the Krull dimension of T_D is strictly less than the Krull dimension of T_C. In particular, for every simplicial complex Δ, the $\mathbb{Z}^d$-graded $KP(\Delta)$-algebra of Example* 3.1 *is flasque.*

Proof. T has the decomposition $T = \bigoplus_{a \in \mathbb{Z}^d} T(a)$, where $T(a)_C = K$ if $a \in C$ and $T(a)_C = 0$ otherwise.

Let $D = \bigcap_{C \in P\colon a \in C} C$. Then $T(a)_C = 0$ unless $D \subseteq C$, and in this case $T(a)_C = K$ and the map $T(a)_{DC}$ is the identity map on K. It follows from 5.3 that T is a direct sum of flasque KP-algebras, and this implies that T is a flasque KP-algebra.

The statement about Krull dimensions holds since the Krull dimension of the monoid ring $K[C \cap \mathbb{Z}^d]$ is equal to the dimension of the cone C. □

Let Δ be a simplicial complex with face lattice P. We have observed that for every $F \in P$ the posets $P(\operatorname{lk}_\Delta F)$ and $[F, 1_{\widehat{P}})$ are isomorphic. Considering the rational pointed fan induced by a simplicial complex as in Example 3.2, and using the barycentric subdivision homeomorphism, the following theorem recovers Hochster's decomposition of the local cohomology of Stanley–Reisner rings.

Theorem 5.5. *Let Σ be a rational pointed fan in $\mathbb{R}^d$ with face lattice P. If $K[\Sigma]$ denotes the toric face ring of Σ over a field K and if $\mathfrak{m}$ denotes the graded maximal ideal of $K[\Sigma]$, then there is an isomorphism*

$$H^i_{\mathfrak{m}}\bigl(K[\Sigma]\bigr) \cong \bigoplus_{C \in P} \widetilde{H}^{i-\dim C-1}\bigl((C, 1_{\widehat{P}}); K\bigr) \otimes_K H^{\dim C}_{\mathfrak{m}}\bigl(K\bigl[C \cap \mathbb{Z}^d\bigr]\bigr)$$

of $\mathbb{Z}^d$-graded K-modules. In particular, $K[\Sigma]$ is a Cohen–Macaulay ring if and only if $\widetilde{H}^p((C, 1_{\widehat{P}}); K) = 0$ for $C \in P$ and $p \neq \dim(K[\Sigma]) - d_C - 1$.

Proof. Let T be the KP-algebra of Example 3.2. We have that

$$H_{\mathfrak{m}}^i\bigl(K[\Sigma]\bigr) \cong H_{\mathfrak{m}}^i(\lim T).$$

For $C \in P$ the ring $T_C = K[C \cap \mathbb{Z}^d]$ is a normal monoid ring of Krull dimension $d_C = \dim C$. Thus it is Cohen–Macaulay (see [7, Theorem 6.3.5]). By Proposition 5.4 the KP-algebra T is flasque and $d_D < d_C$ for $D \subset C$ in P. Since T_C is a Cohen–Macaulay ring for every $C \in P$ it follows from 4.2 that there is an isomorphism

$$H_{\mathfrak{m}}^i\bigl(K[\Sigma]\bigr) \cong \bigoplus_{C \in P} \widetilde{H}^{i-\dim C-1}\bigl((C, 1_{\widehat{P}}); K\bigr) \otimes H_{\mathfrak{m}}^{\dim C}\bigl(K\bigl[C \cap \mathbb{Z}^d\bigr]\bigr)$$

of $\mathbb{Z}^d$-graded K-modules for every $i \geqslant 0$. $\square$

Remark 5.6. Reisner's Cohen–Macaulay criterion for a simplicial complex Δ states that $K[\Delta]$ is a Cohen–Macaulay ring if and only if

$$\widetilde{H}^i(\mathrm{lk}_\Delta F; K) \cong \widetilde{H}_i(\mathrm{lk}_\Delta F; K) = 0 \quad \text{for all } F \in \Delta \text{ and all } i < \dim \mathrm{lk}_\Delta F.$$

This criterion is a direct consequence of Hochster's decomposition of the local cohomology of Stanley–Reisner rings. In particular, since $\widetilde{H}^p(\mathrm{lk}_\Delta F; K) \cong \widetilde{H}^p((F, 1_{\widehat{P(\Delta)}}); K)$ for $F \in \Delta$ and all p, it is a consequence of Theorem 5.5.

The following corollary of Theorem 5.5 is an observation of Stanley in [13].

Corollary 5.7. *Let $\Sigma \subset \mathbb{R}^d$ be a rational pointed fan, let P be the face poset of Σ and let T be the KP-algebra considered in Example* 3.2*. If $\Delta(P)$ is K-Cohen–Macaulay, then* $\lim T$ *is Cohen–Macaulay.*

Proof. If $\Delta(P)$ is Cohen–Macaulay, then

$$\widetilde{H}^p\bigl((C, 1_{\widehat{P}}); K\bigr) = 0 \quad \text{for } C \in P \text{ and } p \neq \dim \Delta(C, 1_{\widehat{P}}),$$

where the intervals $(C, 1_{\widehat{P}})$ are taken as sub-posets of P. It is known that P is a graded poset, i.e. all maximal chains of P have the same length. We prove by induction on $\operatorname{rank} P - \operatorname{rank} C$ that $\dim \Delta(C, 1_{\widehat{P}}) = \dim(\lim T) - d_C - 1$ for $C \in P$. This will conclude the proof by the remark of 5.5.

If $\operatorname{rank} C = \operatorname{rank} P$, then C is a maximal face and it is easy to see that $\operatorname{Ker}(\lim T \to T_C)$ is a minimal prime ideal of $\lim T$. Hence

$$\dim \Delta(C, 1_{\widehat{P}}) - 1 = \dim(\lim T) - \dim T_C - 1 = \dim(\lim T) - \dim C - 1$$

and we are done in this case. If $\operatorname{rank} C < \operatorname{rank} P$, then it is well known that there exists a face $C' \in P$ such that $\dim C' = \dim C + 1$ and $C \subseteq C'$. Thus by the induction hypothesis

$$\dim \Delta(C, 1_{\widehat{P}}) = \dim \Delta(C', 1_{\widehat{P}}) + 1 = \dim(\lim T) - \dim C' = \dim(\lim T) - \dim C - 1. \qquad \square$$

The next goal will be to find a result similar to 5.2 for Buchsbaum rings, at least in the $\mathbb{Z}$-graded case. Assume that K is a field and let $R = K[x_1, \dots, x_d]$ be an $\mathbb{N}$-graded polynomial ring (i.e. R is $\mathbb{Z}$-graded such that $R_0 = K$ and $R_i = 0$ for $i < 0$) with graded maximal ideal $\mathfrak{m} = (x_1, \dots, x_d)$. Recall that a finitely generated graded R-module N is called a *Buchsbaum module* if the $S_{\mathfrak{m}}$-module $N_{\mathfrak{m}}$ is Buchsbaum as defined in [16]. If K is an infinite field and R is generated in degree 1, then N is Buchsbaum if and only if $\dim N = 0$ or $\dim N > 0$ and every homogeneous system of parameters $y_1, \dots, y_{\dim N}$ of N is a *weak N-sequence*, i.e. $(y_1, \dots, y_{i-1})N : y_i = (y_1, \dots, y_{i-1})N : \mathfrak{m}$ for $i = 1, \dots, \dim N$. If N is a Buchsbaum module, then $\dim_K H^i_{\mathfrak{m}}(N) < \infty$ for $i = 1, \dots, \dim N - 1$. In general the converse is not true. But if there exists an $r \in \mathbb{Z}$ such that $H^i_{\mathfrak{m}}(N)_j = 0$ for $j \neq r$ and $i = 1, \dots, \dim N - 1$, then N is Buchsbaum (see [12, Satz 4.3.1]). For more details on the theory of Buchsbaum modules we refer to Schenzel [12] and Stückrad and Vogel [16].

Corollary 5.8. *Let $R = K[x_1, \dots, x_d]$ be the $\mathbb{N}$-graded polynomial ring over a field K. Let P be a finite poset with a terminal element 0_P and let T be a cyclic flasque RP-algebra of $\mathbb{Z}$-graded Cohen–Macaulay R-algebras such that $d_x = \dim T_x < d_y = \dim T_y$ for $x < y$ in P. Assume that $T_{0_P} = K$. Then the following statements are equivalent*:

(i) $\lim T$ *is a Buchsbaum ring,*
(ii) *for all $x \in P \setminus \{0_P\}$ we have $\widetilde{H}^p((x, 1_{\widehat{P}}); K) = 0$ for $p \neq \dim(\lim T) - d_x - 1$.*

Proof. Note that $d_x > 0$ for $x > 0_P$. If $\lim T$ is Buchsbaum, then it follows from Corollary 4.2 that for all $x \in P$ such that $x \neq 0_P$ we have

$$\widetilde{H}^p\big((x, 1_{\widehat{P}}); K\big) = 0 \quad \text{for } p \neq \dim(\lim T) - d_x - 1,$$

because otherwise $H^i_{\mathfrak{m}}(\lim T)$ is not a finitely generated K-vector space since we have $\dim_K H^{d_x}_{\mathfrak{m}}(T_x) = \infty$ for $x \neq 0_p$.

If condition (ii) holds, then we have that $H^i_{\mathfrak{m}}(\lim T)_j = 0$ for $i < \dim(\lim T)$ and $j \neq 0$. Thus $\lim T$ is a Buchsbaum ring. □

6. AP-modules

In the following sections we give proofs of the results in Section 4. Let A be an associative and unital ring and let P be a poset. A *left AP-module* M is a system $(M_x)_{x \in P}$ of left A-modules and homomorphisms $M_{xy} : M_y \to M_x$ for $x \leqslant y$ in P with the property that M_{xx} is the identity on M_x and that $M_{xy} \circ M_{yz} = M_{xz}$ for every $x \leqslant y \leqslant z$ in P. A *homomorphism* $f : M \to N$ of left AP-modules consists of homomorphisms $f_x : M_x \to N_x$ of left A-modules for $x \in P$ with the property that $f_x \circ M_{xy} = N_{xy} \circ f_y$ for every $x \leqslant y$ in P. We denote the abelian group of homomorphisms from M to N by $\mathrm{Hom}_{AP}(M, N)$. The category-minded reader recognizes that a left AP-module is a functor from P to the category of left A-modules, and that a homomorphism of left AP-modules is a natural transformation of such functors. We denote the category of left AP-modules by AP-Mod.

More generally, for every small category $\mathscr{C}$ enriched in the category of abelian groups we could consider the category $\mathscr{C}$-Mod of enriched functors from $\mathscr{C}$ to the category of abelian groups. Everything we do in this section and in Section 7 in the category AP-Mod can also be done in the category $\mathscr{C}$-Mod. Since we are interested in certain particular properties of the categories AP-Mod we focus on these.

A homomorphism $f:M\to N$ of left AP-modules is a *monomorphism* if $f_x:M_x\to N_x$ is injective for every $x\in P$, and it is an *epimorphism* if f_x is surjective for every $x\in P$. A left AP-module L is *projective* if for every epimorphism $f:M\to N$ and every homomorphism $g:L\to N$ of left AP-modules there exists a homomorphism $\bar g:L\to M$ with $f\circ\bar g=g$. Dually a left AP-module I is *injective* if for every monomorphism $f:M\to N$ and every homomorphism $g:M\to I$ of left AP-modules there exists a homomorphism $\bar g:N\to I$ with $\bar g\circ f=g$. The category AP-Mod of left AP-modules is an abelian category.

Example 6.1.

(i) To an order-preserving map $f:P\to Q$ of posets and a left AQ-module M there is an associated left AP-module f^*M with $(f^*M)_x=M_{f(x)}$ and $(f^*M)_{xy}=M_{f(x)f(y)}$. We shall write M instead of f^*M when it is clear from the context that we are working with left AP-modules.

(ii) If Q is the one element poset, then the category of left AQ-modules is isomorphic to the category of left A-modules.

(iii) Given a poset P there is a unique order-preserving map $f:P\to Q$ from P to the one element poset Q. Thus we can consider a left A-module E firstly as an AQ-module and secondly as an AP-module f^*E. Again, when it is clear from the context we write E instead of f^*E. Note that f^*E is the constant left AP-module with constant value E, that is, $f^*E_x=E$ for $x\in P$ and $f^*E_{xy}=\mathrm{id}_E$ for $x\leqslant y$ in P.

(iv) For $z\in P$ there is a *left AP-module AP^z represented by z*. The left AP-module AP^z takes x to $AP^z_x=A$ if $x\leqslant z$ and to 0 otherwise. The homomorphism AP^z_{xy} is the identity on A if $x\leqslant y\leqslant z$ and otherwise it is the zero homomorphism. If M is another left AP-module, then the abelian group of homomorphisms from AP^z to M is isomorphic to the underlying abelian group of M_z. In particular, AP^z is a projective left AP-module.

(v) To a family $(M_i)_{i\in I}$ of left AP-modules we can associate left AP-modules $\bigoplus_{i\in I}M_i$ and $\prod_{i\in I}M_i$ with

$$\Big(\bigoplus_{i\in I}M_i\Big)_x=\bigoplus_{i\in I}(M_i)_x\quad\text{and}\quad\Big(\prod_{i\in I}M_i\Big)_x=\prod_{i\in I}(M_i)_x\quad\text{for } x\in P.$$

Definition 6.2. The *limit* of a left AP-module M is the left A-submodule $\lim M$ of $\prod_{x\in P}M_x$ consisting of sequences $m=(m_x)_{x\in P}$ with the property that $M_{xy}(m_y)=m_x$ for every $x\leqslant y$ in P.

We call a left AP-module M *finitely generated* is there exists an epimorphism of the form $\bigoplus_{i\in I}AP^{z_i}\to M$ for some finite index set I. Note that a finitely generated left AP-module is projective if and only if it is a direct summand of a finitely generated left AP-module $\bigoplus_{i\in I}AP^{z_i}$ for some finite set I. A left AP-module M is called *Noetherian* if every increasing sequence $M_1\subseteq M_2\subseteq\cdots$ of left sub-AP-modules of M stabilizes. Recall that a poset P is topologically finite if $(0_{\widehat P},x]$ is finite for every $x\in P$.

Lemma 6.3. *Suppose that A is a left Noetherian ring and that P is a topologically finite poset. Then AP^z is a Noetherian left AP-module for every $z\in P$.*

Proof. A left sub-AP-module of AP^z is uniquely determined by a family of left ideals $I(x)$ of A for $x \leqslant z$ in P with the property that $I(y) \subseteq I(x)$ if $x \leqslant y \leqslant z$. By our assumptions this is a finite family of left ideals of a left Noetherian ring. □

It is well known that if A is left Noetherian, then every left submodule of a finitely generated free A-module is finitely generated. This also holds in the category of left AP-modules in the following way:

Proposition 6.4. *If AP^z is a Noetherian left AP-module for every $z \in P$, then every submodule of a left AP-module $\bigoplus_{i \in I} AP^{z_i}$ for some finite set I is finitely generated.*

Remark 6.5. If P is finite and A is a commutative ring, then we can consider the *incidence algebra $I(P, A)$* of P over A, that is, the A-algebra with underlying A-module

$$I(P, A) = \bigoplus_{x \leqslant y} A \cdot e_{x \leqslant y}$$

and with multiplication defined via $(e_{x \leqslant y})(e_{y \leqslant z}) = e_{x \leqslant z}$ and with $(e_{x \leqslant y})(e_{x' \leqslant y'}) = 0$ if $y \neq x'$. (See [15, Definition 3.6.1] or [11, p. 33].) Note that $\sum_{x \in P} e_{x \leqslant x}$ is the multiplicative unit in $I(A, P)$ and that the elements $e_{x \leqslant x}$ are idempotent in $I(A, P)$.

Given a left module M over the ring $I(A, P)$, we obtain A-modules $M_x := e_{x \leqslant x} M$ for every $x \in P$, and A-linear homomorphisms $M_{xy} : M_y \to M_x$ given by

$$(e_{y \leqslant y})m \mapsto (e_{x \leqslant y})(e_{y \leqslant y})m = (e_{x \leqslant y})m = (e_{x \leqslant x})(e_{x \leqslant y})m$$

for every $x \leqslant y$ in P. In this situation the A-modules $(M_x)_{x \in P}$ and the A-homomorphisms M_{xy} form a left AP-module. Conversely, if M is a left AP-module, then the direct sum $\bigoplus_{x \in P} M_x$ of the A-modules can be given the structure of a left module over the ring $I(A, P)$ by defining $e_{x \leqslant y} m_y = M_{xy}(m_y)$ for $m_y \in M_y$ and $e_{x \leqslant y} m = 0$ if $m \notin M_y$. This correspondence shows that the category of left modules over the ring $I(A, P)$ is equivalent to the category of left AP-modules. This justifies the above terminology and it shows that in the case where P is finite the concept of AP-modules is really nothing new. However, as we shall see, many left $I(A, P)$-modules become more transparent when considered as left AP-modules.

Observe that the left AP-modules AP^z correspond to the ideals $I(P, A)e_{z \leqslant z}$. The module $I(P, A)$ corresponds to the left AP-module $\bigoplus_{z \in P} AP^z$.

The next result is well known to specialists. Since we did not find a proof in the literature we include it for the sake of completeness.

Proposition 6.6. *Let P be a poset and let A be an associative and unital ring. The category of left AP-modules is equivalent to the category of sheaves of left A-modules on P with the Alexandrov topology.*

Proof. Let $\mathscr{F}$ be a sheaf of left A-modules on P. We let $\Phi(P)$ denote the left AP-module with $\Phi(P)_x = \mathscr{F}((0_{\widehat{P}}, x])$ for $x \in P$ and with $\Phi(P)_{xy}$ equal to the restriction homomorphism associated to the inclusion $(0_{\widehat{P}}, x] \subseteq (0_{\widehat{P}}, y]$ for $x \leqslant y$. This defines a functor Φ from the category of sheaves of left A-modules on P to the category of left AP-modules.

Let M be a left AP-module. For $U \subseteq P$ open define

$$\Psi(M)(U) = \left\{(s_x) \in \prod_{x \in U} M_x\colon\ s_x = M_{xy}(s_y) \text{ for } x \leqslant y\right\} = \lim M|_U. \tag{1}$$

Given an inclusion $\iota\colon V \subseteq U$ of open subsets of P, the restriction map $\Psi(M)(U) \to \Psi(M)(V)$ is the natural restriction map of limits. Using that $(0_{\widehat{P}}, x]$ is contained in every neighborhood of x it is straightforward to check that $\Psi(M)$ is isomorphic to its associated sheaf of left A-modules on P. (Compare [9, Proposition-Definition II.1.2].) Thus $\Psi(M)$ is a sheaf of left A-modules on P. We have defined a functor Ψ from the category of left AP-modules to the category of sheaves of left A-modules on P.

If M is a left AP-module, then the homomorphism

$$M_y \to \Phi\big(\Psi(M)\big)_y = \lim M|_{(0_{\widehat{P}}, y]}$$

induced by the structure maps $M_{xy}: M_y \to M_x$ is an isomorphism.

Conversely, if $\mathscr{F}$ is a sheaf of left A-modules on P and U is an open subset of P, then the intervals $(0_{\widehat{P}}, x]$ for $x \in U$ form an open cover of U. Since the homomorphism

$$\mathscr{F}\big((0_{\widehat{P}}, y]\big) \to \Psi\big(\Phi(M)\big)\big((0_{\widehat{P}}, y]\big) = \lim_{x \in (0_{\widehat{P}}, y]} \mathscr{F}\big((0_{\widehat{P}}, x]\big)$$

is an isomorphism for every $y \in P$, the sheaf condition on $\mathscr{F}$ ensures that the homomorphism

$$\mathscr{F}(U) \to \Psi\big(\Phi(\mathscr{F})\big)(U) = \lim_{x \in U} \mathscr{F}\big((0_{\widehat{P}}, x]\big)$$

is an isomorphism for every open subset U of P. This concludes the proof. $\square$

We will also need to consider right AP-modules. A *right AP-module* is a system $(M^x)_{x \in P}$ of right A-modules and homomorphisms $M^{xy}: M^x \to M^y$ for $x \leqslant y$ in P with the property that M^{xx} is the identity on M^x and that $M^{yz} \circ M^{xy} = M^{xz}$ for every $x \leqslant y \leqslant z$ in P. A homomorphism $f: M \to N$ of right AP-modules consists of homomorphisms $f^x: M^x \to N^x$ of right A-modules for $x \in P$ with the property that $f^y \circ M^{xy} = N^{xy} \circ f^x$ for every $x \leqslant y$ in P. (In other words, a right AP-module is a left $A^{\mathrm{op}}P^{\mathrm{op}}$-module.) The category Mod-AP of right AP-modules is also an abelian category.

Example 6.7. For $x \in P$ there is a *right AP-module AP_x represented by x*. The right AP-module AP_x takes z to $AP_x^z = A$ if $x \leqslant z$ and to 0 otherwise. The homomorphism AP_x^{yz} is the identity on A if $x \leqslant y \leqslant z$ and otherwise it is the zero homomorphism. If M is another left AP-module, then the abelian group of homomorphisms of right AP-modules from M to AP_x is isomorphic to the underlying abelian group of M_x. In particular, AP_x is a projective right AP-module.

If $P = (P, \leqslant)$ is a poset we let $(P, =)$ denote the poset with the same elements as P and with the partial order where no distinct elements are comparable. Given a right AP-module M and a left AP-module N the *tensor product* $M \otimes_{A(P,=)} N$ *of* M *and* N *over* $A(P,=)$ is the abelian group $M \otimes_{A(P,=)} N = \bigoplus_{x\in P} M^x \otimes_A N_x$. For every $x \in P$ there is a homomorphism

$$N_* : AP_x \otimes_{A(P,=)} N = \bigoplus_{y\in P} AP_x^y \otimes_A N_y \to N_x$$

induced by the unique homomorphisms $AP_x^y \otimes_A N_y \to N_x$ taking $1 \otimes n$ to $N_{xy}(n)$ for $n \in N_y$ and $x \leqslant y$. Similarly, there is a homomorphism

$$M^* : M \otimes_{A(P,=)} AP^y = \bigoplus_{x\in P} M^x \otimes_A AP_x^y \to M^y.$$

The *tensor product* $M \otimes_{AP} N$ *of* M *and* N *over* AP is the abelian group given by the cokernel of the homomorphism

$$\bigoplus_{x,y\in P} M^x \otimes_A AP_x^y \otimes_A N_y \xrightarrow{M^*\otimes_A 1 - 1\otimes_A N_*} \bigoplus_{x\in P} M^x \otimes_A N_x.$$

Given posets P and Q and associative and unital rings A and B, an AP-BQ-bimodule is a system $(M_x^u)_{x\in P, u\in Q}$ of A-B-bimodules together with a left AP-module structure on $(M_x^u)_{x\in P}$ for every $u \in Q$ and a right BQ-module structures on $(M_x^u)_{u\in Q}$ for every $x \in P$ subject to the condition that $M_{xy}^v \circ M_y^{uv} = M_x^{uv} \circ M_{xy}^u$ for $x \leqslant y$ in P and $u \leqslant v$ in Q. A homomorphism $f : M \to N$ of AP-BQ-bimodules consists of homomorphisms $f_x^u : M_x^u \to N_x^u$ for $u \in Q$ and $x \in P$ such that for every $x \in P$ the homomorphisms $(f_x^u)_{u\in Q}$ form a homomorphism of right BQ-modules and for every $u \in Q$ the homomorphisms $(f_x^u)_{x\in P}$ form a homomorphism of left AP-modules. We denote by $\mathrm{Hom}_{AP\text{-}BQ}(M,N)$ the abelian group of homomorphisms of AP-BQ-bimodules from M to N. If P is a one-point poset, then we say that M is an A-$A'Q$-bimodule instead of saying that it is an AP-$A'Q$-bimodule. Similarly, if Q is a one-point poset, then we say that M is an AP-A'-bimodule.

Example 6.8. For every poset P and every associative unital ring A we can consider the AP-AP-bimodule AP with $AP_x^y = A$ if $x \leqslant y$ and $AP_x^y = 0$ otherwise.

Let P, Q and R be posets and let A, A' and A'' be associative and unital rings. If M is an $A'Q$-AP-bimodule and N is an AP-$A''R$-bimodule, then the tensor product $M \otimes_{AP} N$ inherits the structure of an $A'Q$-$A''R$-bimodule. Observe that if P is a one-point poset, then this is the $A'Q$-$A''R$-bimodule $M \otimes_A N$ induced by $(M \otimes_A N)_x^y = M_x \otimes_A N^y$.

If further L is an AP-$A'Q$-bimodule, then we let $\mathrm{Hom}_{AP}(L,N)$ denote the $A'Q$-$A''R$-bimodule with $\mathrm{Hom}_{AP}(L,N)_u^a$ given by the set of AP-homomorphisms from $L^u = (L_x^u)_{x\in P}$ to $N^a = (N_x^a)_{x\in P}$, and with structure homomorphisms induced from those of L and N. Observe that if P is a one-point poset, then this is the $A'Q$-$A''R$-bimodule $\mathrm{Hom}_A(L,N)$ induced by $\mathrm{Hom}_A(L,N)_u^a = \mathrm{Hom}_A(L^u, N^a)$. Note also that $\mathrm{Hom}_{AP}(L,N)$ is a left A'-module in the particular case where Q and R are one-point posets and $A'' = \mathbb{Z}$.

Suppose that A and A' are algebras over a commutative ring K, that is, there are given ring-homomorphism from K to the centers of A and A'. If M is a right AP-module and M' is a right $A'P'$-module, then the tensor product $M \otimes_K M'$ of M and M' over K is the right $(A \otimes_K A')(P \times P')$-module with $(M \otimes_K M')^{(x,x')} = M^x \otimes_K M'^{x'}$. Similarly, if N is a left AP-module and N' is a left $A'P'$-module, then $N \otimes_K N'$ is the left $(A \otimes_K A')(P \times P')$-module with $(N \otimes_K N')_{(x,x')} = N_x \otimes_K N'_{x'}$.

Proposition 6.9. *Suppose that A and A' are associative and unital algebras over a commutative ring K. If M is a finitely generated projective left AP-module and M' is a finitely generated projective left $A'P'$-module, then for every left AP-module N and every left $A'P'$-module N' the* Hom-$\otimes$ *interchange homomorphism*

$$\begin{aligned} \mathrm{Hom}_{AP}(M, N) \otimes_K \mathrm{Hom}_{A'P'}(M', N') &\to \mathrm{Hom}_{(A\otimes_K A')(P\times P')}(M \otimes_K M', N \otimes_K N'), \\ (f \otimes f') &\mapsto \big(m \otimes m' \mapsto f(m) \otimes f'(m')\big) \end{aligned}$$

is an isomorphism.

Proof. Since every retract of an isomorphism is an isomorphism we can without loss of generality assume that M and M' are left AP-modules of the form $\bigoplus_{i\in I} AP^{z_i}$ for some finite set I. By direct sums, the statement is now reduced to the case $M = AP^y$ and $M' = A'P'^{y'}$. In this case

$$M \otimes_K M' \cong (A \otimes_K A')(P \times P')^{(y,y')}.$$

We have that

$$\begin{aligned} &\mathrm{Hom}_{AP}(M, N) \otimes_K \mathrm{Hom}_{A'P'}(M', N') \\ &\quad = \mathrm{Hom}_{AP}\big(AP^y, N\big) \otimes_K \mathrm{Hom}_{A'P'}\big(A'P'^{y'}, N'\big) \\ &\quad \cong N_y \otimes_K N'_{y'} = (N \otimes_K N')_{(y,y')} \\ &\quad \cong \mathrm{Hom}_{(A\otimes_K A')(P\times P')}\big((A \otimes_K A')(P \times P')^{(y,y')}, N \otimes_K N'\big) \\ &\quad \cong \mathrm{Hom}_{(A\otimes_K A')(P\times P')}(M \otimes_K M', N \otimes_K N'), \end{aligned}$$

where we used the fact (the Yoneda lemma) that the homomorphisms

$$\begin{aligned} \mathrm{Hom}_{AP}\big(AP^y, N\big) &\to N_y, \\ \varphi &\mapsto \varphi_y(1_A) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}_{(A\otimes_K A')(P\times P')}\big((A \otimes_K A')(P \times P')^{(y,y')}, N \otimes_K N'\big) &\to (N \otimes_K N')_{(y,y')}, \\ \varphi &\mapsto \varphi_{(y,y')}(1_{A\otimes_K A'}) \end{aligned}$$

are isomorphisms. $\square$

We leave the proof of the following lemma to the reader.

Lemma 6.10. *Let A, A' and A'' be commutative rings and let P, P' and P'' be posets. Given an AP-$A'P'$-bimodule X, an $A'P'$-$A''P''$-bimodule Y and an AP-$A''P''$-bimodule Z the homomorphism*

$$\begin{aligned}\mathrm{Hom}_{AP\text{-}A''P''}(X \otimes_{A'P'} Y, Z) \to \mathrm{Hom}_{A'P'\text{-}A''P''}\big(Y, \mathrm{Hom}_{AP}(X, Z)\big),\\ f \mapsto \big(y \mapsto \big(x \mapsto f(x \otimes y)\big)\big)\end{aligned}$$

is an isomorphism.

7. Homological algebra

A *chain complex C* of left AP-modules is a collection $(C_n)_{n\in\mathbb{Z}}$ of left AP-module together with homomorphisms $d = d_n : C_n \to C_{n-1}$ with the property that $d \circ d = 0$. We call a chain complex C *positive* if $C_n = 0$ for $n < 0$ and we call it *negative* if $C_n = 0$ for $n > 0$. The *homology $H_*(C)$* of a chain complex C is the collection $(H_n(C))_{n\in\mathbb{Z}}$ of the left AP-modules $H_n(C) = \ker(d : C_n \to C_{n-1})/\operatorname{Im}(d : C_{n+1} \to C_n)$. A *homomorphism $f : C \to D$ of chain complexes of left AP-modules* is a collection of homomorphisms $f_n : C_n \to D_n$ of left AP-modules with the property that $df_n = f_n d$. We denote the category of chain complexes of left AP-modules $\mathrm{Ch}(AP\text{-Mod})$.

A *positive resolution* of a left AP-module M is a positive chain complex C with the properties that $H_0(C) \cong M$ and that $H_n(C) = 0$ for $n \neq 0$. A *projective resolution* of M is a positive resolution C of M with the property that C_n is a projective left AP-module for every n.

Proposition 7.1. *If AP^z is a Noetherian left AP-module for every $z \in P$, then there exists a* (*homological*) *degreewise finitely generated projective resolution of every finitely generated left AP-module M.*

Proof. There exists a short exact sequence of the form

$$0 \to G_0 \to F_0 \to M \to 0,$$

where F_0 is a finitely generated left AP-module of the form $\bigoplus_{i\in I} AP^{z_i}$ for some finite set I. By Proposition 6.4 G_0 is finitely generated, and proceeding by induction we obtain a resolution $F \to M$, with F_i a finitely generated and projective left AP-module for every $i \geqslant 0$. □

A *negative resolution* of a left AP-module M is a negative chain complex C with the properties that $H_0(C) \cong M$ and that $H_n(C) = 0$ for $n \neq 0$. An *injective resolution* of M is a negative resolution of M consisting of injective left AP-modules. Every left AP-module has a projective resolution, so the category AP-Mod is an abelian category with enough projectives. The usual argument showing that module categories are categories with enough injectives (see [17]) also shows that every left AP-module has an injective resolution. This fact can also be deduced from Proposition 6.6 and the fact that the category of sheaves of left A-modules on P has enough injective objects (see [9, Proposition 2.2]). The above observations also apply to right AP-modules and to AP-LQ-bimodules.

Suppose that A is an algebra over a commutative ring K. The functor

$$\otimes_{AP} : AP\text{-Mod} \times \text{Mod-}AP \to K\text{-Mod}, \quad (M, N) \mapsto M \otimes_{AP} N$$

is additive and right exact in both M and N and it can be extended to a functor

$$\otimes_{AP} : \mathrm{Ch}(AP\text{-Mod}) \times \mathrm{Ch}(\text{Mod-}AP) \to \mathrm{Ch}(K\text{-Mod}), \quad (C, D) \mapsto C \otimes_{AP} D.$$

More precisely, $(C \otimes_{AP} D)_n = \bigoplus_{r+s=n} C_r \otimes_{AP} D_s$ and if $m \in (C_r)^x$ and $n \in (D_s)_x$, then the differential takes the element in $C_r \otimes_{AP} D_s$ represented by $m \otimes n \in (C_r)^x \otimes_A (D_s)_x$ to the sum of the elements in $(C \otimes_{AP} D)_{n-1}$ represented by $dm \otimes n$ and $(-1)^r m \otimes dn$.

If C_M is a projective resolution of M and C_N is a projective resolution of N, then the $\mathrm{Tor}_n^{AP}(M, N)$ denotes the isomorphism class of

$$H_n(C_M \otimes_{AP} N) \cong H_n(M \otimes_{AP} C_N).$$

It is a well-known fact of homological algebra that the above homology groups are universal δ-functors and that they are naturally isomorphic by a unique isomorphism. See, for example, Weibel [17]. Similarly, the functor

$$\mathrm{Hom}_{AP} : AP\text{-Mod} \times AP\text{-Mod} \to K\text{-Mod}, \quad (M, N) \mapsto \mathrm{Hom}_{AP}(M, N)$$

is additive in both its entries. It is left exact in M and in N and it can be extended to a functor

$$\mathrm{Hom}_{AP} : \mathrm{Ch}(AP\text{-Mod}) \times \mathrm{Ch}(AP\text{-Mod}) \to \mathrm{Ch}(K\text{-Mod}), \quad (C, D) \mapsto \mathrm{Hom}_{AP}(C, D)$$

with $\mathrm{Hom}_{AP}(C, D)_n = \prod_{-r+s=n} \mathrm{Hom}_{AP}(C_r, D_s)$ and with df given by $(df)(c) = f(dc) - d(f(c))$ for $f \in \mathrm{Hom}_{AP}(C, D)_n$. If D_M is a projective resolution of M and C_N is an injective resolution of N, then the functors $\mathrm{Ext}^n_{AP}(M, N)$ denotes the isomorphism class of

$$\mathrm{Ext}^n_{AP}(M, N) = H_{-n}\big(\mathrm{Hom}_{AP}(D_M, N)\big) = H_{-n}\big(\mathrm{Hom}_{AP}(M, C_N)\big).$$

Again it is a well-known fact of homological algebra that the above homology groups are universal δ-functors and that they are naturally isomorphic by a unique isomorphism.

Remark 7.2. Given a sheaf $\mathscr{F}$ of left A-modules let $H^n(P, \mathscr{F})$ denote the sheaf cohomology. For a left AP-module M we have that

$$\mathrm{Ext}^0_{AP}(A, M) \cong \mathrm{Hom}^0_{AP}(A, M) \cong H^0\big(P, \Psi(M)\big),$$

where $\Psi(M)$ is the sheaf associated to M as constructed in (1) in the proof of 6.6. Using Proposition 6.6 we see that if M is injective then $\Psi(M)$ is an injective sheaf on P. It follows that there is a natural isomorphism

$$\mathrm{Ext}^n_{AP}(A, M) \cong H^n\big(P, \Psi(M)\big)$$

for every n. The above groups are isomorphic to the Hochschild–Mitchell cohomology groups $H^n(P, M)$ of P with coefficients in M. More precisely, the Hochschild–Mitchell complex is the chain complex $\mathrm{Hom}_{AP}(B^K(AP, AP, A), M)$, where $B^K(AP, AP, A)$ is a particular projective resolution of A over AP called the bar-construction (see [2,11]). If M is a left A-module considered as a constant AP-module, that is, M_{xy} is the identity on M for every $x \leqslant y$ in P, then we can consider the cohomology $H^n(\Delta(P), M)$ of the simplicial complex $\Delta(P)$ with coefficients in M. In this case the chain complex $\mathrm{Hom}_{AP}(B^K(AP, AP, A), M)$ is isomorphic to the chain complex computing $H^n(\Delta(P), M)$. Thus there are natural isomorphisms

$$H^n\big(\Delta(P), M\big) \cong H^n(P, M) \cong \mathrm{Ext}^n_{AP}(A, M) \cong H^n\big(P, \Psi(M)\big). \tag{2}$$

Furthermore McCord has shown that the cohomology groups $H^n(P, M)$ are isomorphic to the singular cohomology groups of P with the Alexandrov topology and with coefficients in M [10].

Borrowing notation from the theory of sheaves we call a left AP-module M *flasque* if $\lim M|_U \to \lim M|_V$ is surjective for all open sets $V \subseteq U$ of the poset P. Note that $\mathrm{Hom}_{AP}(A, M) \cong \lim M$ is isomorphic to the group of global sections of the sheaf associated to M under the equivalence of categories between left AP-modules and sheaves of A-modules on P of Proposition 6.6. Since the higher cohomology groups of a flasque sheaf vanish (see [9, Proposition III.2.5]) we have:

Lemma 7.3. *Let A be an associative and unital ring and let P be a poset. If M is a flasque left AP-module, then $\mathrm{Ext}^i_{AP}(A, M) = 0$ for all $i > 0$.*

Proposition 7.4. *Suppose that A and A' are associative and unital algebras over a commutative ring K. If C is a positive chain complex of finitely generated projective left AP-modules and C' is a positive chain complex of finitely generated projective left $A'P'$-modules, then for every negative chain complex D of left AP-modules and every negative chain complex D' of left $A'P'$-modules the* Hom-$\otimes$ *interchange homomorphism*

$$\mathrm{Hom}_{AP}(C, D) \otimes_K \mathrm{Hom}_{A'P'}(C', D') \to \mathrm{Hom}_{(A\otimes_K A')(P\times P')}(C \otimes_K C', D \otimes_K D'),$$
$$(f \otimes f') \mapsto \big(m \otimes m' \mapsto (-1)^{|f'||m|} f(m) \otimes f'(m')\big)$$

is an isomorphism of chain complexes.

Proof. This is a direct consequence of Proposition 6.9. □

Note that Lemma 6.10 also holds for chain complexes.

8. Local cohomology

Let R be a commutative ring, let I be a finitely generated ideal in R and let N be an R-module. The natural projections $R/I^{n+1} \to R/I^n$ induce maps $\mathrm{Ext}^q_R(R/I^n, N) \to \mathrm{Ext}^q_R(R/I^{n+1}, N)$. Our model for the *$q$th local cohomology group of N with respect to I* is the colimit

$$H^q_I(N) = \operatorname{colim} \mathrm{Ext}^q_R\big(R/I^n, N\big).$$

Proposition 8.1. *Let I be a finitely generated ideal of a commutative ring R, let P be a poset and let M be a left RP-module. Suppose that there exist a degreewise finitely generated projective* *resolution of R over RP and a degreewise finitely generated free resolution of R/I^n over R. If $\operatorname{Ext}^q_{RP}(R, M) = 0$ for $q > 0$, then there is a natural isomorphism*

$$H^q_I(\lim M) \cong \operatorname{colim} \operatorname{Ext}^q_{RP}\big(R/I^n, M\big) \quad \textit{of } R\textit{-modules for every } q \geqslant 0.$$

Proof. Let E be a degreewise finitely generated projective resolution of R over RP and let $F_n \to R/I^n$ be a degreewise finitely generated free resolution of R/I^n over R. We may consider $F_n \otimes_R E$ as a projective resolution of R/I^n over RP. The vanishing of $\operatorname{Ext}^q_{RP}(R, M)$ implies that the homomorphism $E \to R$ induces a quasi-isomorphism $\operatorname{Hom}_{RP}(R, M) \to \operatorname{Hom}_{RP}(E, M)$. Applying this quasi-isomorphism, $\otimes$-Hom-interchange and basic isomorphisms of the form $\lim M \cong \operatorname{Hom}_R(R, \lim M)$ and $R \otimes_R N \cong N$ and noting that $\operatorname{Hom}_R(F_n, R)$ is a degreewise free R-module, we obtain the following chain of isomorphisms and quasi-isomorphisms:

$$\begin{aligned}
\operatorname{Hom}_{RP}(F_n \otimes_R E, M) &\cong \operatorname{Hom}_{R\otimes_R R(*\times P)}(F_n \otimes_R E, R \otimes_R M) \\
&\cong \operatorname{Hom}_R(F_n, R) \otimes_R \operatorname{Hom}_{RP}(E, M) \\
&\simeq \operatorname{Hom}_R(F_n, R) \otimes_R \operatorname{Hom}_{RP}(R, M) \\
&\cong \operatorname{Hom}_R(F_n, R) \otimes_R \lim M \\
&\cong \operatorname{Hom}_R(F_n, R) \otimes_R \operatorname{Hom}_R(R, \lim M) \\
&\cong \operatorname{Hom}_R(F_n \otimes_R R, R \otimes_R \lim M) \\
&\cong \operatorname{Hom}_R(F_n, \lim M).
\end{aligned}$$

Taking cohomology we get the natural isomorphism

$$\operatorname{Ext}^q_{RP}\big(R/I^n, M\big) \cong \operatorname{Ext}^q_R\big(R/I^n, \lim M\big)$$

of R-modules. Forming the colimit of these isomorphism we obtain the isomorphism

$$\operatorname{colim} \operatorname{Ext}^q_{RP}\big(R/I^n, M\big) \cong \operatorname{colim} \operatorname{Ext}^q_R\big(R/I^n, \lim M\big) = H^q_I(\lim M). \qquad \square$$

In our applications we need a graded version of the above result. If R is a $\mathbb{Z}^d$-graded commutative ring and N and N' are $\mathbb{Z}^d$-graded R-modules, then the group $\operatorname{Hom}^{\mathrm{gr}}_R(N, N')$ of homogeneous homomorphisms from N to N' is a $\mathbb{Z}^d$-graded R-module. Choosing a projective resolution E of N in the category of $\mathbb{Z}^d$-graded R-modules and homogeneous homomorphisms of degree zero we obtain a chain complex $\operatorname{Hom}^{\mathrm{gr}}_R(E, N')$ of $\mathbb{Z}^d$-graded R-modules.

If F is a finitely generated free $\mathbb{Z}^d$-graded R-module, then $\operatorname{Hom}_R(F, N')$ is isomorphic to $\operatorname{Hom}^{\mathrm{gr}}_R(F, N')$ for every $\mathbb{Z}^d$-graded R-module N'. Since both $\operatorname{Hom}_R(-, N')$ and $\operatorname{Hom}^{\mathrm{gr}}_R(-, N')$ are left exact functors it follows for every finitely presented $\mathbb{Z}^d$-graded R-module N that $\operatorname{Hom}^{\mathrm{gr}}_R(N, N')$ and $\operatorname{Hom}_R(N, N')$ are isomorphic for every $\mathbb{Z}^d$-graded R-module N'. In particular, a finitely generated projective $\mathbb{Z}^d$-graded R-module is also projective considered as a nongraded R-module. If R is Noetherian and N is a finitely generated $\mathbb{Z}^d$-graded R-module, we obtain a $\mathbb{Z}^d$-grading of

$$\operatorname{Ext}^q_R(N, N') \cong H_{-q}\big(\operatorname{Hom}(E, N')\big)$$

for every $q \geqslant 0$. Here E is a degreewise finitely generated projective resolution of the $\mathbb{Z}^d$-graded R-module N. In this case we obtain a grading of $H_I^q(N)$ if I is a finitely generated graded ideal in R.

Recall that for a poset P a $\mathbb{Z}^d$-graded left RP-module is a left RP-module M together with gradings of the R-modules M_x such that the homomorphisms $M_{xy}: M_y \to M_x$ are homogeneous of degree zero for every $x \leqslant y$ in P. The proof of Proposition 8.1 can easily be modified to a proof of the following result.

Proposition 8.2. *Let I be a finitely generated graded ideal of a commutative $\mathbb{Z}^d$-graded ring R, let P be a poset and let M be a $\mathbb{Z}^d$-graded left RP-module. Suppose that there exist a degreewise finitely generated $\mathbb{Z}^d$-graded projective resolution of R over RP and a degreewise finitely generated $\mathbb{Z}^d$-graded free resolution of R/I^n over R. If $\operatorname{Ext}^q_{RP}(R, M) = 0$ for $q > 0$, then there is a natural isomorphism*

$$H_I^q(\lim M) \cong \operatorname{colim} \operatorname{Ext}^q_{RP}\big(R/I^n, M\big)$$

of $\mathbb{Z}^d$-graded R-modules for every $q \geqslant 0$.

Definition 8.3. Let P be a poset and R a commutative ring. For $x \in P$ and $q \in \mathbb{Z}$, the *left RP-skyscraper chain complex $R(x, q)$* is the left RP-module with $R(x, q)_y = 0$ for $y \neq x$ and with $R(x, q)_x = R[q]$ equal to the chain complex consisting of a copy of R in homological degree $-q$.

In the following a zig-zag chain of quasi-isomorphisms between complexes C and D consists of chain complexes $E_0, \ldots, E_{2n}$ with $E_0 = C$, $E_{2n} = D$, of quasi-isomorphisms $E_{2k-1} \to E_{2k-2}$ and of quasi-isomorphisms $E_{2k-1} \to E_{2k}$ for $k = 1, \ldots, n$.

Recall the discussion about bimodules after Example 6.8. For example we need the following. Assume that $K \to R$ is a homomorphism of commutative rings, F an R-module, P a poset and M a left RP-module. Then M is also an R-KP^{op}-bimodule. Hence $\operatorname{Hom}_R(F, M)$ has a R-KP^{op}-bimodule structure and thus a left RP-module structure induced by $\operatorname{Hom}_R(N, M)_x = \operatorname{Hom}_R(N, M_x)$ for $x \in P$. This gives also an RP-module structure on $\operatorname{Ext}_R(N, M)$.

Theorem 8.4. *Let $K \to R$ be a homomorphism of commutative rings, let I be a finitely generated ideal of R, let P be a poset and let M be a left RP-module. Suppose that E is a positive chain complex of finitely generated projective left KP-modules and that we have degreewise finitely generated free resolutions $F_n \to R/I^n$ of R/I^n over R together with homomorphisms $F_{n+1} \to F_n$ inducing the natural projections $R/I^{n+1} \to R/I^n$ in homology. Assume that the following conditions are satisfied*:

- (i) *there exists a zig-zag chain of quasi-isomorphisms between the chain complex of left KP-modules $\operatorname{colim} \operatorname{Hom}_R(F_n, M)$ and the chain complex of left KP-modules $H_I^{-*}(M) := \operatorname{colim} \operatorname{Ext}_R^{-*}(R/I^n, M)$ with $H_I^{-*}(M)_x = H_I^{-*}(M_x)$ for $x \in P$,*
- (ii) *for every $x < y$ in P the homomorphism $H_I^{-*}(M_y) \to H_I^{-*}(M_x)$ is the zero-homomorphism.*

Then there is a natural zig-zag chain of quasi-isomorphisms of chain complexes of K-modules of the form

$$\operatorname{colim} \operatorname{Hom}_{RP}(F_n \otimes_K E, M) \simeq \bigoplus_{x \in P} \bigoplus_{q \geqslant 0} \operatorname{Hom}_{KP}\big(E, K(x, q)\big) \otimes_K H_I^q(M_x).$$

Proof. The asserted weak equivalence is the composition of the following homomorphisms:

$$\begin{aligned}
\operatorname{colim}\operatorname{Hom}_{RP}(F_n \otimes_K E, M) &\cong \operatorname{colim}\operatorname{Hom}_{R\text{-}KP^{\mathrm{op}}}(F_n \otimes_K E, M)\\
&\cong \operatorname{colim}\operatorname{Hom}_{K\text{-}KP^{\mathrm{op}}}\big(E, \operatorname{Hom}_R(F_n, M)\big)\\
&\cong \operatorname{Hom}_{KP}\big(E, \operatorname{colim}\operatorname{Hom}_R(F_n, M)\big)\\
&\simeq \operatorname{Hom}_{KP}\big(E, H_I^{-*}(M)\big)\\
&\cong \operatorname{Hom}_{KP}\Big(E, \bigoplus_{x\in P}\bigoplus_{q\geqslant 0} K(x,q) \otimes_K H_I^q(M)\Big)\\
&\cong \bigoplus_{x\in P}\bigoplus_{q\geqslant 0}\operatorname{Hom}_{KP}\big(E, K(x,q) \otimes_K H_I^q(M)\big)\\
&\cong \bigoplus_{x\in P}\bigoplus_{q\geqslant 0}\operatorname{Hom}_{KP}\big(E, K(x,q)\big) \otimes_K H_I^q(M).
\end{aligned}$$

Here the first isomorphism is clear, the second isomorphism is given by Lemma 6.10 and the third isomorphism is due to the facts that E is positive and degreewise finitely generated and that $\operatorname{Hom}_R(F_n, M)$ is negative. The zig-zag of quasi-isomorphism on the fourth line exists by condition (i) and the fact that $\operatorname{Hom}_{KP}(E, -)$ preserves quasi-isomorphisms between negative chain complexes. The isomorphism on the fifth line is a direct consequence of condition (ii). The isomorphism on the sixth line is again due to the fact that E is positive and degreewise finitely generated and that $K(x,q)$ is concentrated in one homological degree. The last isomorphism is a direct application of Proposition 7.4. □

For reference we state a $\mathbb{Z}^d$-graded version of the above result. It is proved in exactly the same way.

Theorem 8.5. *Let $K \to R$ be a homomorphism of $\mathbb{Z}^d$-graded commutative rings. Let I be a finitely generated graded ideal in R, let P be a poset and let M be a $\mathbb{Z}^d$-graded left RP-module. Suppose that E is a positive chain complex of finitely generated $\mathbb{Z}^d$-graded projective left KP-modules and that we have degreewise finitely generated $\mathbb{Z}^d$-graded free resolutions $F_n \to R/I^n$ of R/I^n over R together with homomorphisms $F_{n+1} \to F_n$ inducing the natural projections $R/I^{n+1} \to R/I^n$ in homology. Assume that the following conditions are satisfied*:

- (i) *there exists a zig-zag chain of homogeneous quasi-isomorphisms of degree zero between the chain complex of left KP-modules* $\operatorname{colim}\operatorname{Hom}_R(F_n, M)$ *and the chain complex of left KP-modules $H_I^{-*}(M)$ with $H_I^{-*}(M)_x = H_I^{-*}(M_x)$ for $x \in P$,*
- (ii) *for every $x < y$ in P the homomorphism $H_I^{-*}(M_y) \to H_I^{-*}(M_x)$ is the zero-homomorphism.*

Then there is a natural zig-zag chain of homogeneous quasi-isomorphisms of chain complexes of $\mathbb{Z}^d$-graded K-modules of the form

$$\operatorname{colim}\operatorname{Hom}_{RP}(F_n \otimes_R E, M) \simeq \bigoplus_{x\in P}\bigoplus_{q\geqslant 0}\operatorname{Hom}_{KP}\big(E, K(x,q)\big) \otimes_K H_I^q(M_x).$$

Proposition 8.6. *Let P be a finite poset, let K be a field and let C be a chain complex of left KP-modules. Suppose that for every $x \in P$ there exists $n_x \in \mathbb{Z}$ such that $H_*(C_x)$ is concentrated in degree n_x and that $x < y$ implies $n_x > n_y$. Then there exists a zig-zag chain of quasi-isomorphisms between the chain complexes of left KP-modules C and $H_*(C)$.*

Proof. Let ∂ be the differential of C and

$$B_k(C) = \mathrm{Im}(\partial_{k+1}) \subseteq C_k, \qquad Z_k(C) = \mathrm{Ker}(\partial_k) \subseteq C_k$$

for all k. We define a chain complex C' of left KP-modules with

$$\left(C'_x\right)_k = \begin{cases} (C_x)_k, & k > n_x, \\ B_k(C_x) \oplus H_k(C_x), & k = n_x, \\ 0, & k < n_x. \end{cases}$$

For $k > n_x + 1$ the boundary $d : (C'_x)_k \to (C'_x)_{k-1}$ is the boundary map of C_x, for $k = n_x + 1$ it is the map $(d, 0) : (C_x)_{n_x+1} \to B_{n_x}(C_x) \oplus H_{n_x}(C_x)$, and for $k \leqslant n_x$ it is the zero map. We give C' the structure of a chain complex of left KP-modules, where the map $(C'_{xy})_k : (C'_y)_k \to (C'_x)_k$ is the map $(C_{xy})_k : (C_y)_k \to (C_x)_k$ for $k > n_x$, the zero map for $k < n_x$ and the map

$$(C_y)_k \xrightarrow{(C_{xy})_k} (C_x)_k \to B_k(C_x) \oplus H_k(C_x)$$

for $k = n_x$. Here $(C_x)_{n_x} \to B_{n_x}(C_x)$ is a (chosen) retract of the inclusion $B_{n_x}(C_x) \to (C_x)_{n_x}$. Similarly, the map $(C_x)_{n_x} \to H_{n_x}(C_x)$ is the composition $(C_x)_{n_x} \to Z_{n_x}(C_x) \to H_{n_x}(C_x)$, where $(C_x)_{n_x} \to Z_{n_x}(C_x)$ is a (chosen) retract of the inclusion $Z_{n_x}(C_x) \to (C_x)_{n_x}$. Note that these retracts exist since K is a field.

There is a quasi-isomorphism $f : C \to C'$ with $(f_x)_k$ equal to the identity on $(C_x)_k$ if $k > n_x$, the zero map if $k < n_x$ and the map $(C_x)_k \to B_k(C_x) \oplus H_k(C_x)$ defined above for $k = n_x$.

On the other hand, the inclusion $H_*(C) \to C'$ is a quasi-isomorphism of chain complexes of left KP-modules. Thus there are quasi-isomorphisms $C \to C' \leftarrow H_*(C)$. □

Lemma 8.7. *Let P be a poset, let $x \in P$ and let $q \geqslant 0$. For every n there is a natural isomorphism*

$$\widetilde{H}^{n-q-1}\big((x, 1_{\widehat{P}}); R\big) \cong \mathrm{Ext}^n_{RP}\big(R, R(x, q)\big).$$

Proof. Let $F = RP^y$ for $y \in P$, that is, $F_x = R$ if $x \leqslant y$ and $F_x = 0$ otherwise. Given a left RP-module M, the so-called Yoneda lemma provides an isomorphism

$$\mathrm{Hom}_{R(x,1_{\widehat{P}})}(F|_{R(x,1_{\widehat{P}})}, M|_{R(x,1_{\widehat{P}})}) \cong \begin{cases} M_y & \text{if } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

If $x < y$ then the above isomorphism takes $\varphi : F|_{R(x,1_{\widehat{P}})} \to M|_{R(x,1_{\widehat{P}})}$ to $\varphi_y(1)$, where 1 is the unit of $R = RP^y_y$. Similarly, there are isomorphisms

$$\mathrm{Hom}_{R[x,1_{\widehat{P}})}(F|_{R[x,1_{\widehat{P}})}, M|_{R[x,1_{\widehat{P}})}) \cong \begin{cases} M_y & \text{if } x \leqslant y, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\operatorname{Hom}_{RP}\bigl(F, R(x,q)\bigr) \cong \begin{cases} R[q] & \text{if } x = y, \\ 0 & \text{if } y \neq x. \end{cases}$$

Let E be a projective resolution of R as a left RP-module. Since the above isomorphisms are natural in the projective left RP-module F, there is a short exact sequence of left RP-modules of the form

$$\begin{aligned} 0 &\to \operatorname{Hom}_{R(x,1_{\widehat{P}})}\bigl(E|_{R(x,1_{\widehat{P}})}, R[q]\bigr) \to \operatorname{Hom}_{R[x,1_{\widehat{P}})}\bigl(E|_{R[x,1_{\widehat{P}})}, R[q]\bigr) \\ &\to \operatorname{Hom}_{RP}\bigl(E, R(x,q)\bigr) \to 0. \end{aligned}$$

Note that $E|_{R(x,1_{\widehat{P}})}$ is a projective resolution of R over $R(x,1_{\widehat{P}})$ and that $E|_{R[x,1_{\widehat{P}})}$ is a projective resolution of R over $R[x,1_{\widehat{P}})$. Thus the long exact sequence associated to the above short exact sequence of chain complexes is of the form

$$\operatorname{Ext}^{n-1}_{RP}\bigl(R, R(x,q)\bigr) \to \operatorname{Ext}^{n-q}_{R(x,1_{\widehat{P}})}(R,R) \to \operatorname{Ext}^{n-q}_{R[x,1_{\widehat{P}})}(R,R) \to \operatorname{Ext}^{n}_{RP}\bigl(R, R(x,q)\bigr) \to \cdots.$$

The result now follows from the fact that $[x,1_{\widehat{P}})$ is contractible since by Eq. (2) in Remark 7.2 we have $\operatorname{Ext}^j_{RQ}(R,R) \cong H^j(Q,R)$ for any poset Q. $\square$

Now we are able to present the proof of Theorem 4.1.

Proof of Theorem 4.1. Since P is finite, K is a finitely generated KP-module and it follows from Lemma 6.3 and Proposition 7.1 that K has a degreewise finitely generated projective resolution E over KP. All the rings R/I^k have free resolution F_k over R which are degreewise finitely generated. By Proposition 8.1 there is an isomorphism

$$H^i_I(\lim T) \cong \operatorname{colim} \operatorname{Ext}^i_{RP}\bigl(R/I^k, T\bigr).$$

By the assumption (i) on T_x we have $H^i_I(T_x) = 0$ for $i \neq d_x$. By Proposition 8.6 there exists a zig-zag chain of quasi-isomorphisms between the left KP-modules $\operatorname{colim} \operatorname{Hom}_R(F_k, T)$ and

$$H^\bullet_I(T) := \operatorname{colim} \operatorname{Ext}^{-\bullet}_R\bigl(R/I^k, T\bigr) \cong H_\bullet\bigl(\operatorname{colim} \operatorname{Hom}_R(F_k, T)\bigr).$$

(Here we used the fact that homology and filtered colimits commute.) Since $x < y$ implies $d_x < d_y$ the homonorphism $H^i_I(T_y) \to H^i_I(T_x)$ is the zero-homomorphism. Hence Theorem 8.4 implies that there is a natural zig-zag chain of quasi-isomorphisms of the form

$$\operatorname{colim} \operatorname{Hom}_{RP}(F_n \otimes_K E, T) \simeq \bigoplus_{x\in P} \bigoplus_{q \geqslant 0} \operatorname{Hom}_{KP}\bigl(E, K(x,q)\bigr) \otimes_K H^q_I(T_x).$$

Since K is a field we obtain the isomorphism

$$H^i_I(\lim T) \cong \operatorname{colim} \operatorname{Ext}^i_{RP}\bigl(R/I^k, T\bigr) \cong \bigoplus_{x\in P} \bigoplus_{0 \leqslant q \leqslant i} \operatorname{Ext}^i_{KP}\bigl(K, K(x,q)\bigr) \otimes_K H^q_I(T_x)$$

of K-modules. Lemma 8.7 implies that

$$H_I^i(\lim T) \cong \bigoplus_{x \in P} \bigoplus_{0 \leqslant q \leqslant i} \widetilde{H}^{i-q-1}\big((x, 1_{\widehat{P}}); K\big) \otimes_K H_I^q(T_x).$$

By the assumption (i) we obtain

$$H_I^i(\lim T) \cong \bigoplus_{x \in P} \widetilde{H}^{i-d_x-1}\big((x, 1_{\widehat{P}}); K\big) \otimes_K H_I^{d_x}(T_x).$$

The $\mathbb{Z}^d$-graded version can be proved in exactly the same way since Proposition 8.1 and Theorem 8.4 have $\mathbb{Z}^d$-graded versions. This concludes the proof. $\square$

Remark 8.8. Some of the results in this section can be interpreted in terms of Grothendieck spectral sequences. Given abelian categories $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ with enough injectives and left exact functors $G : \mathscr{A} \to \mathscr{B}$ and $F : \mathscr{B} \to \mathscr{C}$ the Grothendieck spectral sequence with E_2-term $E_2^{p,q}(A) = (R^p F)(R^q G(A))$ given by composed right derived functors converges to the right derived functor $(R^{p+q}(FG))(A)$ of FG for every object A of $\mathscr{A}$.

The qth local cohomology $H_I^q(N)$ of an R-module N is the qth right derived of the zeroth local cohomology $H_I^0(N) : R\text{-Mod} \to R\text{-Mod}$ and the group $\operatorname{Ext}^q_{RP}(R, M)$ is the qth right derived functor of the functor $\lim_{x \in P} = \operatorname{Hom}_{RP}(R, -) : RP\text{-Mod} \to R\text{-Mod}$. Writing out the definition of H_I^0 we see that the composed functor $H_I^0 \circ \operatorname{Hom}_{RP}(R, -)$ is isomorphic to the functor $\operatorname{colim}_n \operatorname{Hom}_{RP}(R/I^n, -)$. Proposition 8.1 also follows from this isomorphism and the Grothendieck spectral sequence. On the other hand, since filtered colimits commute with finite limits, there is an isomorphism $H_I^0 \circ \operatorname{Hom}_{RP}(R, -) \cong \operatorname{Hom}_{RP}(R, -) \circ H_I^0$. Suppose under the assumptions of Theorem 8.4 that E is a projective resolution of K considered as an KP-module. The Grothendieck spectral sequence for the composition $\operatorname{Hom}_{RP}(R, -) \circ H_I^0$ then has E_2-term

$$\begin{aligned} E_2^{p,q}(M) &= \big(R^p \operatorname{Hom}_{RP}(R, -)\big)\big(\big(R^q H_I^0\big)(M)\big) \\ &= H_{-p} \operatorname{Hom}_{RP}\big(R \otimes_K E, H_I^q(M)\big) \\ &\cong H_{-p}\bigg(\bigoplus_{x \in P} \operatorname{Hom}_{KP}\big(E, K(x, 0)\big) \otimes_K H_I^q(M_x)\bigg). \end{aligned}$$

The statement of Theorem 8.4 implies that this spectral sequence collapses at E_2.

References

[1] P.S. Alexandrov, Combinatorial Topology, vols. 1, 2 and 3, Dover Publications, Mineola, 1998. Translated from Russian. Reprint of the 1956, 1957 and 1960 translations.

[2] H.J. Baues, G. Wirsching, Cohomology of small categories, J. Pure Appl. Algebra 38 (1985) 187–211.

[3] M. Brun, T. Römer, Subdivisions of toric complexes, J. Algebraic Combin. 21 (4) (2005) 423–448.

[4] M. Brun, T. Römer, On algebras associated to partially ordered sets, preprint, math.AC/0507372.

[5] W. Bruns, J. Gubeladze, Polytopal linear groups, J. Algebra 218 (1999) 715–737.

[6] W. Bruns, J. Gubeladze, Polyhedral algebras, arrangements of toric varieties, and their groups, in: Adv. Stud. Pure Math., vol. 33, 2001, pp. 1–51.

[7] W. Bruns, J. Herzog, Cohen–Macaulay Rings, revised ed., Cambridge Stud. Adv. Math., vol. 39, Cambridge Univ. Press, Cambridge, 1998.

[8] Z. Caijun, Cohen–Macaulay section rings, Trans. Amer. Math. Soc. 349 (11) (1997) 4659–4667.

[9] R. Hartshorne, Algebraic Geometry, corrected third printing, Grad. Texts in Math., vol. 52, Springer, New York, 1983 (in English).

[10] M.C. McCord, Singular homology groups and homotopy groups of finite topological spaces, Duke Math. J. 33 (1966) 465–474.

[11] B. Mitchell, Rings with several objects, Adv. Math. 8 (1972) 1–161.

[12] P. Schenzel, Dualisierende Komplexe in der lokalen Algebra und Buchsbaum-Ringe, Lecture Notes in Math., vol. 907, Springer, Berlin, 1982.

[13] R.P. Stanley, Generalized h-vectors, intersection cohomology of toric varieties, and related results, in: Commutative Algebra and Combinatorics, 1985, in: Adv. Stud. Pure Math., vol. 11, 1987, pp. 187–213.

[14] R.P. Stanley, Combinatorics and Commutative Algebra, second ed., Progr. Math., vol. 41, Birkhäuser, Basel, 1996.

[15] R.P. Stanley, Enumerative Combinatorics, vol. 1, paperback ed., Cambridge Stud. Adv. Math., vol. 49, Cambridge Univ. Press, Cambridge, 1999.

[16] J. Stückrad, W. Vogel, Buchsbaum Rings and Applications. An Interaction between Algebra, Geometry and Topology, Math. Monogr., vol. 21, VEB Deutscher Verlag der Wissenschaften, Berlin, 1986.

[17] C.A. Weibel, An Introduction to Homological Algebra, Cambridge Stud. Adv. Math., vol. 38, Cambridge Univ. Press, Cambridge, 1995.

[18] S. Yuzvinsky, Cohen–Macaulay rings of sections, Adv. Math. 63 (1987) 172–195.

[19] S. Yuzvinsky, Flasque sheaves on posets and Cohen–Macaulay unions of regular varieties, Adv. Math. 73 (1989) 24–42.