

GRÖBNER BASES AND DETERMINANTAL IDEALS

An introduction

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Abstract. We give an introduction to the theory of determinantal ideals and rings, their Gröbner bases, initial ideals and algebras, respectively. The approach is based on the straightening law and the Knuth-Robinson-Schensted correspondence. The article contains a section treating the basic results about the passage to initial ideals and algebras.

Key words: determinantal ideal, Knuth-Robinson-Schensted correspondence, Gröbner basis, initial ideal, Sagbi basis, initial algebra, Hilbert function, Cohen-Macaulay ring, Gorenstein ring, canonical module, Rees algebra, algebra of minors

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Let K be a field and X an $m \times n$ matrix of indeterminates over K . For a given positive integer $t \leq \min(m, n)$, we consider the ideal $I_t = I_t(X)$ generated by the t -minors (i. e. the determinants of the $t \times t$ submatrices) of X in the polynomial ring $K[X]$ generated by all the indeterminates X_{ij} .

From the viewpoint of algebraic geometry $K[X]$ should be regarded as the coordinate ring of the variety of K -linear maps $f: K^m \rightarrow K^n$. Then $V(I_t)$ is just the variety of all f such that $\text{rank } f < t$, and $K[X]/I_t$ is its coordinate ring.

The study of the determinantal ideals I_t and the objects related to them has numerous connections with invariant theory, representation theory, and combinatorics. For a detailed account we refer the reader to Bruns and Vetter [17]. A large part of the theory of determinantal ideals can be developed over the ring \mathbb{Z} of integers (instead of a base field K) and then transferred to arbitrary rings B of coefficients (see [17]). For simplicity we restrict ourselves to fields.

This article follows the line of investigation started by Sturmfels' article [66] in which he applied the Knuth-Robinson-Schensted correspondence KRS (Knuth [50]) to the study of the determinantal ideals I_t . The "witchcraft" (Knuth [51, p. 60]) of the KRS saves one from tracing the Buchberger algorithm through tedious inductions.

Later on the method was extended by Herzog and Trung [44] to the so-called 1-cogenerated ideals, ladder determinantal ideals and ideals of pfaffians. They follow the important principle to derive properties of $K[X]/I_t$ from the analogous properties of $K[X]/\text{in}(I_t)$: the two rings appear as the generic and special fiber of a flat 1-parameter deformation. (By $\text{in}(I_t)$ we denote the ideal of initial forms with respect to a suitable term order.) The ring $K[X]/\text{in}(I_t)$ is the Stanley-Reisner ring of a shellable simplicial complex and amenable to combinatorial methods (see Stanley [68] and Bruns-Herzog [13]). In contrast to the otherwise very elegant ASL approach, one does not replace the indeterminates of $K[X]$ by a system of algebra generators containing elements of degree > 1 . This is a major advantage if one wants to investigate the Hilbert function and related invariants.

The principle of deriving properties of ideals and algebras from their initial counterparts was followed by the authors in [22], [9] and [12] for the investigation of algebras defined by minors, like the Rees algebra and the subalgebra of $K[X]$ generated by the t -minors. This requires the determination of Gröbner bases and initial ideals of powers and products of determinantal ideals. On the KRS side the necessary results are given by the theorem of Greene [42] and its variant found in [9].

In Section 1 we treat the straightening law of Doubilet, Rota and Stein [32] in the approach of De Concini, Eisenbud, and Procesi [27]. It is an indispensable tool. Moreover, we show that the residue class rings $K[X]/I_t$ are normal domains. Section 2 contains the description of the symbolic powers of the I_t and the primary decomposition of products $I_{t_1} \cdots I_{t_u}$ given in [27] and [17]. While these two sections form the introduction to determinantal ideals, Section 3 gives a fairly self-contained treatment of initial ideals and algebras. Despite the title of the article, the emphasis is on initial ideals and not on Gröbner bases.

Section 4 gives a short introduction to KRS (in the "dual" version of [50]) and the theorems of Schensted [64] and Greene [42]. These results are exploited in Section 5 for determinantal ideals, their powers and their products.

All the lines of development are brought together in Section 6 where we deal with the properties of $K[X]/I_t$, especially with its Hilbert function, following Conca and Herzog [24]. At the end of this section we have inserted some remarks that point out the extension to ladder determinantal ideals and the variants for symmetric and alternating matrices.

The last two sections deal with algebras of minors, their normality and Cohen-Macaulayness (Section 7) and their canonical modules and Gorensteinness (Section 8).

1. Determinantal ideals and the straightening law

Almost all of the approaches one can choose for the investigation of determinantal rings use standard bitableaux, to be defined below, and the straightening law. In this approach one considers all the minors of X (and not just the 1-minors X_{ij}) as generators of the K -algebra $K[X]$ so that products of minors appear as “monomials”. The price to be paid, of course, is that one has to choose a proper subset of all these “monomials” as a linearly independent K -basis: we will see that the standard bitableaux form a basis, and the straightening law tells us how to express an arbitrary product of minors as a K -linear combination of the basis elements. (In the literature standard bitableaux are often called *standard monomials*; however, we will have to use the ordinary monomials in $K[X]$ so often that we reserve the term “monomial” for products of the X_{ij} .)

Below we must often consider sequences of integers with a monotonicity property. We say that a sequence (r_i) is *increasing* if $r_i < r_{i+1}$ for all i . It is *non-increasing* if $r_i \geq r_{i+1}$ for all i .

Apart from Section 6 the letter Δ always denotes a product $\delta_1 \cdots \delta_w$ of minors, and we assume that the sizes $|\delta_i|$ (i. e. the number of rows of the submatrix X'_i of X such that $\delta_i = \det(X'_i)$) are non-increasing, $|\delta_1| \geq \cdots \geq |\delta_w|$. By convention, the value of the empty minor $[\]$ is 1. The *shape* $|\Delta|$ of Δ is the sequence $(|\delta_1|, \dots, |\delta_w|)$. If necessary we may add factors $[\]$ at the right hand side of the products, and accordingly extend the shape by a sequence of 0.

We denote the set of all non-empty minors of X by $\mathcal{M}(X)$ and the subset of minors of size t by $\mathcal{M}_t(X)$. If no confusion about the underlying matrix is possible, we will simply write \mathcal{M} or \mathcal{M}_t . Moreover,

$$[a_1 \dots a_t | b_1 \dots b_t]$$

stands for the determinant of the matrix $(X_{a_i b_j} : i = 1, \dots, t, j = 1, \dots, t)$. While we do not impose any condition on the indices a_i, b_i of $[a_1 \dots a_t | b_1 \dots b_t]$ in general, we require that $a_1 < \cdots < a_t$ and $b_1 < \cdots < b_t$ if we speak of a minor.

A product of minors is also called a *bitableau*. The choice of the term bitableau is motivated by the graphical description of a product Δ as a pair of Young tableaux as in Figure 1. Every product of minors is represented by a bitableau and, conversely, every bitableau stands for a product of minors:

$$\Delta = \delta_1 \cdots \delta_w, \quad \delta_i = [a_{i1} \dots a_{it_i} | b_{i1} \dots b_{it_i}], \quad i = 1, \dots, w.$$

According to our convention above, the indices in each row of the bitableau are increasing from the middle to both ends. Sometimes it is necessary to separate the two tableaux from which Δ is formed; we then write $\Delta = (R | C)$.

For formal correctness one should consider the bitableaux as purely combinatorial objects (as we will do in Section 4) and distinguish them from the

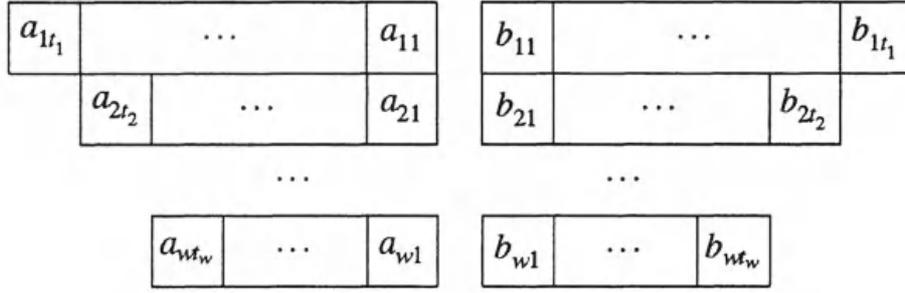


Figure 1. A bitableau

ring-theoretic objects represented by them, but since there is no real danger of confusion, we use the same terminology for both classes of objects.

Whether Δ is a standard bitableau is controlled by a partial order on $\mathcal{M}(X)$, namely

$$[a_1 \dots a_t \mid b_1 \dots b_t] \preceq [c_1 \dots c_u \mid d_1 \dots d_u] \\ \iff t \geq u \text{ and } a_i \leq c_i, b_i \leq d_i, \quad i = 1, \dots, u.$$

A product $\Delta = \delta_1 \dots \delta_w$ is called a *standard bitableau* if

$$\delta_1 \preceq \dots \preceq \delta_w,$$

in other words, if in each column of the bitableau the indices are non-decreasing from top to bottom. The letter Σ is reserved for standard bitableaux. (The empty product is also standard.)

The fundamental straightening law of Doubilet–Rota–Stein [32] says that every element of R has a unique presentation as a K -linear combination of standard bitableaux:

Theorem 1.1. (a) *The standard bitableaux are a K -vector space basis of $K[X]$.*
(b) *If the product $\gamma\delta$ of minors is not a standard bitableau, then it has a representation*

$$\gamma\delta = \sum x_i \varepsilon_i \eta_i, \quad x_i \in K, x_i \neq 0,$$

where $\varepsilon_i \eta_i$ is a standard bitableau, $\varepsilon_i \prec \gamma, \delta \prec \eta_i$ (here we must allow that $\eta_i = [] = 1$).

(c) *The standard representation of an arbitrary bitableau Δ , i.e. its representation as a linear combination of standard bitableaux Σ , can be found by successive application of the straightening relations in (b).*

For the proof of the theorem we can assume that $m \leq n$, passing to the transpose of X if necessary. We derive the theorem from its “restriction” to the subalgebra $K[\mathcal{M}_m]$ generated by the m -minors. Each m -minor is determined by its

column indices, and for simplicity we set

$$[b_1 \dots b_m] = [1 \dots m | b_1 \dots b_m].$$

The algebra $K[\mathcal{M}_m]$ is the homogeneous coordinate ring of the Grassmann variety of the m -dimensional vector subspaces of K^n . The m -minors satisfy the famous *Plücker relations*. In their description we use $\sigma(i_1 \dots i_s)$ to denote the sign of the permutation of $\{1, \dots, s\}$ represented by the sequence i_1, \dots, i_s .

Lemma 1.2. *For all indices $a_1, \dots, a_p, b_q, \dots, b_m, c_1, \dots, c_s \in \{1, \dots, n\}$ such that $s = m - p + q - 1 > m$ and $t = m - p > 0$ one has*

$$\sum_{\substack{i_1 < \dots < i_t \\ i_{t+1} < \dots < i_s \\ \{i_1, \dots, i_s\} = \{1, \dots, s\}}} \sigma(i_1 \dots i_s) [a_1 \dots a_p c_{i_1} \dots c_{i_t}] [c_{i_{t+1}} \dots c_{i_s} b_q \dots b_m] = 0.$$

Proof. Let V be the K -vector space generated by the columns X_j of X . We define $\alpha: V^s \rightarrow K(X)$ by

$$\alpha(y_1, \dots, y_s) = \sum \sigma(i_1 \dots i_s) \det(X_{a_1}, \dots, X_{a_p}, y_{i_1}, \dots, y_{i_t}) \cdot \det(y_{i_{t+1}}, \dots, y_{i_s}, X_{b_q}, \dots, X_{b_m}).$$

where the sum has the same range as above. It is straightforward to check that α is an alternating multilinear form on V^s . Since $s > \dim V = m$, one has $\alpha = 0$. \square

Let $[a_1 \dots a_m], [b_1 \dots b_m]$ be elements of $\mathcal{M}_m(X)$ such that $a_i \leq b_i$ for $i = 1, \dots, p$, but $a_{p+1} > b_{p+1}$. We put

$$q = p + 2, \quad s = m + 1, \quad (c_1, \dots, c_s) = (a_{p+1}, \dots, a_m, b_1, \dots, b_{p+1}).$$

Then, in the Plücker relation with these data, all the non-zero terms

$$[d_1 \dots d_m][e_1 \dots e_m] \neq [a_1 \dots a_m][b_1 \dots b_m]$$

have the following properties (after their column indices have been arranged in ascending order):

$$[d_1 \dots d_m] \preceq [a_1 \dots a_m], \quad d_1 \leq e_1, \dots, d_{p+1} \leq e_{p+1}.$$

By induction on p it follows that a product $\gamma\delta$ of maximal minors γ and δ is a linear combination of standard bitableaux $\alpha\beta$, $\alpha \preceq \beta$, such that $\alpha \preceq \gamma$. Note that α and β arise from γ and δ by an exchange of indices.

Let $\gamma_1 \dots \gamma_u$ be a product of maximal minors of length $u > 2$. If it is not a standard bitableau, then we find an index i with $\gamma_i \not\preceq \gamma_{i+1}$. As just seen, $\gamma_i \gamma_{i+1}$ can be expressed as a linear combination of standard bitableaux. Substitution of this

expression for $\gamma_i \gamma_{i+1}$ yields a representation of $\gamma_1 \cdots \gamma_u$ as a linear combination of bitableaux. In each of these bitableaux, indices from the $(i+1)$ st row of the bitableau of $\gamma_1 \cdots \gamma_u$ have been exchanged with *larger* indices from its i th row. An iteration of this procedure must eventually yield a linear combination of standard bitableaux, since the exchange of indices can only be repeated finitely many times.

This proves that products of maximal minors are linear combinations of standard bitableaux. In the next step we transfer this partial result to the full set of minors. We extend X by m columns of further indeterminates, obtaining

$$X' = \begin{pmatrix} X_{11} & \cdots & X_{1n} & X_{1,n+1} & \cdots & X_{1,n+m} \\ \vdots & & \vdots & \vdots & & \vdots \\ X_{m1} & \cdots & X_{mn} & X_{m,n+1} & \cdots & X_{m,n+m} \end{pmatrix}.$$

Then $K[X']$ is mapped onto $K[X]$ by substituting for each entry of X' the corresponding entry of the matrix

$$\begin{pmatrix} X_{11} & \cdots & X_{1n} & 0 & \cdots & \cdots & 0 & 1 \\ & & & \vdots & & \ddots & \ddots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & 0 & \ddots & \ddots & & \vdots \\ X_{m1} & \cdots & X_{mn} & 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Let $\varphi : K[\mathcal{M}(X')] \rightarrow K[X]$ be the induced K -algebra homomorphism. Then

$$\varphi([b_1 \dots b_m]) = \pm [a_1 \dots a_t | b_1 \dots, b_t]$$

where $t = \max\{i : b_i \leq n\}$ and a_1, \dots, a_t have been chosen such that

$$\{a_1, \dots, a_t, n+m+1-b_m, \dots, n+m+1-b_{t+1}\} = \{1, \dots, m\}.$$

Evidently φ is surjective, and furthermore it sets up a bijective correspondence between the set $\mathcal{M}_m(X')$ of m -minors of X' and $\mathcal{M}(X) \cup \{\pm 1\}$; on $\mathcal{M}_m(X') \setminus \{\tilde{\mu}\}$ the correspondence is an isomorphism of partially ordered sets. Note that the maximal element $\tilde{\mu} = [n+1 \dots n+m]$ of $\mathcal{M}_m(X')$ is mapped to ± 1 by φ , and, up to sign, standard monomials go to standard monomials. (We leave the verification of this fact to the reader; the details can also be found in [17], (4.9).)

In order to represent an arbitrary element of $K[X]$ as a linear combination of bitableaux, we lift it to $K[\mathcal{M}_m(X')]$ via φ . Then the preimage is “straightened”, and an application of φ yields the desired expression in $K[X]$.

For part (a) of Theorem 1.1 it remains to show the linear independence of the standard bitableaux. We know already that they generate the vector space $K[X]$. Moreover, they are homogeneous elements with respect to total degree, and as we will see in Section 4, there are as many standard bitableaux in every degree as

there are ordinary monomials. This implies the linear independence of the standard bitableaux and finishes the proof of Theorem 1.1(a). Therefore we may now speak of the straightening law in $K[X]$. As we have seen, arbitrary products of minors can be straightened by the successive straightening of products with two factors, and so part (c) has also been proved.

For part (b) we notice that the Plücker relations are homogeneous of degree 2. Therefore there are exactly two factors ε_i and η_i in each term on the right hand side of

$$\gamma\delta = \sum x_i \varepsilon_i \eta_i$$

if $|\gamma|, |\delta| = m$. Since the straightening law in $K[X]$ is a specialization via φ of that in $K[\mathcal{M}_m(X')]$, there can be at most 2 factors in each of the summands on the right hand side.

It follows easily from the straightening procedure that $\varepsilon_i \preceq \gamma$. In fact, the inequality holds for all intermediate expressions that arise in the successive application of the Plücker relations, as observed above. In order to see that $\varepsilon_i \preceq \delta$ as well, we simply straighten $\delta\gamma = \gamma\delta$. By the linear independence of the standard bitableaux, the result is the same. Also for $\gamma, \delta \preceq \eta_i$ one has to use the linear independence of the standard monomials (the intermediate expressions in the straightening procedure may violate it). It is enough to prove this relation in $K[\mathcal{M}_m(X')]$. On the set $\mathcal{M}_m(X')$ we consider the reverse partial order, arising from rearranging the columns of the matrix in the order $m+n, m+n-1, \dots, 1$. It has the same set of standard monomials as \preceq , at least up to sign. By linear independence, straightening with respect to the reverse partial order must have the same result as that with respect to \preceq (up to sign). This concludes the proof of Theorem 1.1(b).

Corollary 1.3. (a) *The kernel of $\varphi : K[\mathcal{M}_m(X')] \rightarrow K[X]$ is generated by $\bar{\mu} + 1$ or $\bar{\mu} - 1$.*
(b) $\dim K[\mathcal{M}_m(X)] = m(n-m) + 1$.

Proof. (a) Every element x of $K[\mathcal{M}_m(X')]$ has a unique representation $x = \sum p_\Delta(\bar{\mu})\Delta$ as a linear combination of standard bitableaux Δ over $\mathcal{M}_m(X') \setminus \{\bar{\mu}\}$ with coefficients $p_\Delta(\bar{\mu}) \in K[\bar{\mu}]$. Clearly $\varphi(x) = 0$ if and only if $\varphi(p_\Delta(\bar{\mu})) = p_\Delta(\pm 1) = 0$ for all Δ .

(b) X plays the role of X' for an $m \times (n-m)$ matrix of indeterminates. Now one applies (a). \square

Remark 1.4. The straightening law for $K[\mathcal{M}_m(X)]$ is due to Hodge [48], [49]. The proof of the straightening law for $K[X]$, with the exception of the linear independence of the standard bitableaux, follows De Concini, Eisenbud, and Procesi [27]. It has also been reproduced in [17]. The linear independence can be proved without the KRS; see [27] (or [17]) and [49] (or [13]) for two alternative proofs.

A third alternative: one proves the linear independence of the standard bitableaux in $\mathcal{M}_m(X)$ by a Gröbner basis argument (see Remark 5.1), shows 1.3(b), deduces 1.3(a), and concludes the linear independence of all standard bitableaux.

The straightening law can be refined. Let e_1, \dots, e_m and f_1, \dots, f_n denote the canonical \mathbb{Z} -bases of \mathbb{Z}^m and \mathbb{Z}^n respectively. Clearly $K[X]$ is a $\mathbb{Z}^m \oplus \mathbb{Z}^n$ -graded algebra if we give X_{ij} the “vector bidegree” $e_i \oplus f_j$. All minors and bitableaux are homogeneous with respect to this grading. The coordinates of the vector bidegree of a bitableau Δ , usually called the *content* of Δ , just count the multiplicities with which the rows and columns of X appear. The homogeneity of bitableaux implies that *straightening preserves content*.

Next we can compare the shapes of the tableaux appearing on the left and the right hand side of a straightening relation. For a sequence $\sigma = (s_1, \dots, s_u)$ we set

$$\alpha_k(\sigma) = \sum_{i \leq k} s_i,$$

and define $\sigma \leq \tau$ by $\alpha_k(\sigma) \leq \alpha_k(\tau)$ for all k . It follows from Theorem 1.1(b) and (c) that *straightening does not decrease shape*.

It is a natural question whether at least one standard bitableau in a straightening relation must have the same shape as the left hand side. This is indeed true, and for a more precise statement we introduce the *initial tableau* $I(\sigma)$ of shape $\sigma = (s_1, \dots, s_u)$: in its k th row it contains the numbers $1, \dots, s_k$.

Theorem 1.5. *Let Δ be a bitableau of shape σ , with row tableau R and column tableau C .*

- (a) *Every bitableau in the standard representation of Δ has the same content as Δ .*
- (b) *$(R | I(\sigma))$ has a standard representation $\sum \alpha_i (R_i | I(\sigma))$, $\alpha_i \neq 0$, and $(I(\sigma) | C)$ has a standard representation $\sum \beta_j (I(\sigma) | C_j)$, $\beta_j \neq 0$.*
- (c) *$\Delta - \sum \alpha_i \beta_j (R_i | C_j)$ is a linear combination of standard bitableaux of size $> \sigma$.*

Part (a) has been justified above. Part (b) follows from the fact that there is no tableau of shape $\geq \sigma$ that has the same content as $I(\sigma)$. It is much more difficult to show (c), and we forego a proof; for example, see [17, (11.4)].

An ideal in a partially ordered set (M, \leq) is a subset N such that N contains all elements $x \leq y$ if $y \in N$. Let \mathcal{N} be an ideal in the partially ordered set $\mathcal{M}(X)$ and consider the ideal $I = \mathcal{N}K[X]$ generated by \mathcal{N} . Every element of I is a K -linear combination of elements δx with $x \in K[X]$ and $\delta \in \mathcal{N}$. It follows from Theorem 1.1 that every standard bitableaux $\Sigma = \gamma_1 \cdots \gamma_u$ in the standard representation of δx has $\gamma_1 \leq \delta$, and therefore $\gamma_1 \in \mathcal{N}$. This shows

Proposition 1.6. *Let \mathcal{N} be an ideal in the partially ordered set $\mathcal{M}(X)$. Then the standard bitableaux $\Sigma = \gamma_1 \cdots \gamma_u$ with $\gamma_1 \in \mathcal{N}$ form a K -basis of the ideal*

$I = \mathcal{N}K[X]$ in the ring $K[X]$, and the (images of the) the standard bitableaux $\Sigma' = \delta_1 \cdots \delta_v$ with $\delta_j \notin \mathcal{N}$ for all j form a K -basis of $K[X]/I$.

Corollary 1.7. *The standard bitableaux $\Sigma = \gamma_1 \cdots \gamma_u$ such that $|\gamma_1| \geq t$ form a K -basis of I_t , and (the images of) the standard bitableaux $\Sigma' = \delta_1 \cdots \delta_v$ with $|\delta_j| \leq t - 1$ for all j form a K -basis of $K[X]/I_t$.*

In fact, I_t is generated by all minors of size $\geq t$, and these form the ideal

$$\{\delta \in \mathcal{M}(X) : \delta \not\prec [1 \dots t - 1 \mid 1 \dots t - 1]\}.$$

The ideals I_t are special instances of the so-called *1-cogenerated ideals* I_γ , $\gamma \in \mathcal{M}(X)$, that are generated by all minors $\delta \not\prec \gamma$.

Remark 1.8. The straightening law and its refined version in Theorem 1.5 can be used in various approaches to the theory of the determinantal ideals and rings:

(a) The straightening law implies that $K[X]$ and $K[\mathcal{M}_m]$ are *algebras with straightening law* on the partially ordered sets $\mathcal{M}(X)$ and $\mathcal{M}_m(X)$, resp. This property is passed on the residue class rings modulo the class of ideals considered in Proposition 1.6. See [28] and [17].

(b) The “filtration by shapes” as indicated in Theorem 1.5 can be used for a deformation process. This allows one to deduce properties of the determinantal ring $K[X]/I_t$ from the “semigroup of shapes” occurring in it. See [10].

(c) (Related to (b).) The refined form of the straightening law is the basis for the investigation of the determinantal rings via representation theory. See Akin, Buchsbaum and Weyman [3], [27] or [17].

The straightening law allows us to prove basic properties about the ideals I_t without much effort. Additionally we need the following extremely useful *induction lemma*. It uses that an ideal of type $I_t(A)$ remains unchanged if one applies elementary row and column operations to the matrix A .

Lemma 1.9. *Let $X = (X_{ij})$ and $Y = (Y_{ij})$ be matrices of indeterminates over K of sizes $m \times n$ and $(m - 1) \times (n - 1)$, resp. Then the substitution*

$$\begin{aligned} X_{ij} &\rightarrow Y_{ij} + X_{mj}X_{in}X_{mn}^{-1}, & 1 \leq i \leq m - 1, & & 1 \leq j \leq n - 1, \\ X_{mj} &\rightarrow X_{mj}, X_{in} &\rightarrow X_{in}, & & 1 \leq i \leq m, & & 1 \leq j \leq n, \end{aligned}$$

induces an isomorphism

$$K[X][X_{mn}^{-1}] \cong K[Y][X_{m1}, \dots, X_{mn}, X_{1n}, \dots, X_{m-1,n}][X_{mn}^{-1}] \quad (*)$$

under which the extension of $I_t(X)$ is mapped to the extension of $I_{t-1}(Y)$. Therefore

one has an isomorphism

$$(K[X]/I_t(X))[x_{mn}^{-1}] \cong (K[Y]/I_{t-1}(Y))[X_{m1}, \dots, X_{mn}, X_{1n}, \dots, X_{m-1,n}][X_{mn}^{-1}].$$

(here x_{mn} denotes the residue class of X_{mn} in $K[X]/I_t(X)$).

Proof. The substitution has an inverse, namely $Y_{ij} \rightarrow X_{ij} - X_{mj}X_{in}X_{mn}^{-1}$, $X_{mj} \rightarrow X_{mj}$, $X_{in} \rightarrow X_{in}$, and so induces the isomorphism (*).

Since X_{mn} is invertible in $K[X][X_{mn}^{-1}]$, one can apply elementary row and column operations to the matrix X with pivot element X_{mn} . Now one just “writes” Y_{ij} for the entries in the rows $1, \dots, m-1$ and the columns $1, \dots, n-1$ of the transformed matrix. With this identification one has $I_t(X) = X_{mn}I_{t-1}(Y) = I_{t-1}(Y)$ in the ring $K[X][X_{mn}^{-1}]$. \square

Theorem 1.10. *The ring $K[X]/I_t(X)$ is a normal domain of dimension $(m+n-t+1)(t-1)$. Its singular locus is the variety of the ideal I_{t-1}/I_t .*

Proof. The case $t=1$ is trivial. Suppose that $t > 1$. Set $R = K[X]/I_t$. We claim that the residue class x_{mn} of X_{mn} is a non-zero-divisor on R . In fact, the product of $X_{mn} = [m|n]$ and a standard bitableau is a standard bitableau, and if x is a linear combination of standard bitableaux without a factor of size $\geq t$, then so is $X_{mn}x$. Corollary 1.7 now implies our claim. The argument shows even more: X_{mn} is a non-zero-divisor modulo every ideal of $K[X]$ generated by an ideal \mathcal{N} in the partially ordered set $\mathcal{M}(X)$ such that $X_{mn} \notin \mathcal{N}$.

In order to verify that R is a domain, it suffices to prove this property for $R[x_{mn}^{-1}]$, and to the latter ring we can apply induction via Proposition 1.9.

The dimension formula follows by the same induction, since an affine domain does not change dimension upon the inversion of a non-zero element.

Let \mathfrak{m} be the maximal ideal of R generated by the residue classes x_{ij} of the indeterminates. Clearly $R_{\mathfrak{m}}$ is not regular if $t \geq 2$. Every other prime ideal \mathfrak{p} does not contain one of the x_{ij} , and by symmetry we can assume $x_{mn} \notin \mathfrak{p}$. Then $R_{\mathfrak{p}}$ is of the form $S[X_{in}, X_{mj}, X_{mn}^{-1}]_{\mathfrak{q}}$ with $S = K[Y]/I_{t-1}(Y)$. It follows that $R_{\mathfrak{p}}$ is regular if and only if $S_{\mathfrak{q} \cap S}$ is regular. Moreover, \mathfrak{p} contains $I_{t-1}(X)/I_t(X)$ if and only if $\mathfrak{q} \cap S$ contains $I_{t-2}(Y)/I_{t-1}(Y)$. Again we can apply induction to prove the claim about the singular locus.

For normality we use Serre’s normality criterion. We know the singular locus, and by the dimension formula it has codimension ≥ 2 . Now it is enough to show that $\text{depth } R_{\mathfrak{p}} \geq 2$ if $\dim R_{\mathfrak{p}} \geq 2$. If $\mathfrak{p} \neq \mathfrak{m}$, we obtain this by induction as above, and it remains to show that $\text{depth } R_{\mathfrak{m}} \geq 2$. The set of minors δ of size $< t$ has a smallest element with respect to \preceq , namely $\varepsilon = [1 \dots t-1 | 1 \dots t-1]$. The same argument that we have applied to X_{mn} shows that ε is a non-zero-divisor modulo I_t . Moreover, the ideal $J = I_t + (\varepsilon)$ is generated by an ideal in $\mathcal{M}(X)$, and so X_{mn}

is a non-zero-divisor modulo J . It follows that \mathfrak{m} contains a regular R -sequence of length 2. \square

We will show in Section 6 that the rings $K[X]/I_t$ are Cohen–Macaulay.

2. Powers and products of determinantal ideals

In this section we want to determine the primary decomposition of powers and, more generally, products $J = I_{t_1} \cdots I_{t_u}$, $t_1 \geq \cdots \geq t_u$, of determinantal ideals. It is easy to see that only the ideals I_t with $t \leq t_1$ can be associated to J . In fact, suppose \mathfrak{p} is a prime ideal in $R = K[X]/J$ different from the irrelevant maximal ideal \mathfrak{m} . Then \mathfrak{p} does not contain one of the x_{ij} , and so we may invert x_{ij} without losing the extension of \mathfrak{p} as an associated prime ideal of $(R/J)[x_{ij}^{-1}]$. But now the Induction Lemma 1.9 applies: by symmetry we can assume $(i, j) = (m, n)$.

Thus we have to find primary components of J with respect to the ideals I_t . Immediate, and as we will see, optimal candidates are the symbolic powers of the ideals I_t . We determine them first.

The ideal $\mathfrak{p} = I_t$ is a prime ideal in the regular ring $A = K[X]$. With each such prime ideal one associates a valuation on the quotient field of A as follows. We pass to the localization $P = A_{\mathfrak{p}}$ and let $\mathfrak{q} = \mathfrak{p}P$. Now we set

$$v_{\mathfrak{p}}(x) = \max\{i : x \in \mathfrak{q}^i\}$$

for all $x \in P$, $x \neq 0$, and $v_{\mathfrak{p}}(0) = \infty$. The associated graded ring $\bigoplus_{i=0}^{\infty} \mathfrak{q}^i / \mathfrak{q}^{i+1}$ is a polynomial ring over the field P/\mathfrak{q} (we only use that it is an integral domain). This implies $v_{\mathfrak{p}}(xy) = v_{\mathfrak{p}}(x) + v_{\mathfrak{p}}(y)$ for all $x, y \in P$, and that $v_{\mathfrak{p}}(x+y) \geq \min(v_{\mathfrak{p}}(x), v_{\mathfrak{p}}(y))$ is clear anyway. To sum up: $v_{\mathfrak{p}}$ is a discrete valuation on P and can be extended to the quotient field $\text{QF}(P) = \text{QF}(A)$. By definition, the i th symbolic power is $\mathfrak{p}^{(i)} = \mathfrak{p}^i P \cap A$. With the help of $v_{\mathfrak{p}}$ we can also describe it as $\mathfrak{p}^{(i)} = \{x \in A : v_{\mathfrak{p}}(x) \geq i\}$. If $\mathfrak{p} = \mathfrak{m}$ is the maximal irrelevant ideal of $K[X]$ and f is a homogeneous polynomial, then $v_{\mathfrak{p}}(f)$ is the ordinary total degree of f .

For the choice $A = K[X]$, $\mathfrak{p} = I_t$, we denote $v_{\mathfrak{p}}$ by γ_t . We claim that

$$\gamma_t(\delta) = \begin{cases} 0, & |\delta| < t, \\ |\delta| - t + 1, & |\delta| \geq t. \end{cases}$$

For $t = 1$, this follows immediately, since $\gamma_1(\delta)$ is its total degree $|\delta|$. Let $t > 1$. If $|\delta| < t$, then $\delta \notin I_t$, and so $\gamma_t(\delta) = 0$. Suppose that $|\delta| \geq t$. It is useful to note that $\gamma_t(\delta)$ does only depend on $|\delta|$: minors δ and δ' of the same size are conjugate under an isomorphism of $K[X]$ that leaves \mathfrak{p} invariant. We can therefore assume that $\delta = [m - t + 1 \dots m | n - t + 1 \dots n]$. The substitution in the induction lemma 1.9 maps δ to a minor of size $|\delta| - 1$, and reduces t by 1, too. So induction finishes the proof.

We transfer the function γ_t to sequences of integers:

$$\gamma_t(s_1, \dots, s_u) = \sum_{i=1}^u \max(s_i - t + 1, 0).$$

For instance, if $\lambda = (4, 3, 3, 1)$, then $\gamma_4(\lambda) = 1$, $\gamma_3(\lambda) = 4$, $\gamma_2(\lambda) = 7$, $\gamma_1(\lambda) = 11$.

It is clear that $\gamma_t(s_1, \dots, s_u) = \gamma_t(\delta_1 \cdots \delta_u)$ for minors δ_i with $|\delta_i| = s_i$, $i = 1 \dots, u$. In Section 1 we have introduced the partial order of shapes based on the functions α_k . We can also use the γ_t for such a comparison, but it yields the same partial order:

Lemma 2.1. *Let $\rho = (r_1, \dots, r_u)$ and $\sigma = (s_1, \dots, s_v)$ be non-increasing sequences of integers. Then $\rho \leq \sigma$ if and only if $\gamma_t(\rho) \leq \gamma_t(\sigma)$ for all t .*

Proof. We use induction on u . The case $u = 1$ is trivial. Furthermore, there is nothing to show if $r_i \leq s_i$ for all i . It remains the case in which $r_j > s_j$ for some j . Next, if $r_i = s_i > 0$ for some i , then we can remove r_i and s_i and compare the shortened sequences by induction. Thus we may assume that $r_i \neq s_i$ for all i with $r_i > 0$. Let k be the smallest index with $s_k < r_k$. It is easy to see that $k \geq 2$ if $\rho \leq \sigma$ or $\gamma_{s_1}(\rho) \leq \gamma_{s_1}(\sigma)$. Extending σ by 0 if necessary we may assume that $k \leq v$.

We have $s_{k-1} > r_{k-1} \geq r_k > s_k$; in particular, $s_{k-1} \geq s_k + 2$. Define $s'_i = s_i$ for $i \neq k-1, k$, $s'_{k-1} = s_{k-1} - 1$, $s'_k = s_k + 1$, and $\sigma' = (s'_1, \dots, s'_v)$.

Suppose first that $\rho \leq \sigma$. Then it follows easily that $\rho \leq \sigma'$. Moreover, $\gamma_t(\sigma') \leq \gamma_t(\sigma)$ for all t , and a second inductive argument allows us to assume that $\gamma_t(\rho) \leq \gamma_t(\sigma')$ for all t .

Conversely, let $\gamma_t(\rho) \leq \gamma_t(\sigma)$ for all t . Since $\sigma' \leq \sigma$, it is enough to show that $\gamma_t(\rho) \leq \gamma_t(\sigma')$ for all t . One has $\gamma_t(\sigma') = \gamma_t(\sigma)$ for $t \leq s_k + 1$, and $\gamma_t(\sigma') = \gamma_t(\sigma) - 1$ for $t = s_k + 2, \dots, s_{k-1}$. Since obviously $\gamma_t(\rho) < \gamma_t(\sigma)$ for $t > r_k$, the critical range is $s_k + 2 \leq t \leq r_k$. Suppose that $\gamma_t(\rho) = \gamma_t(\sigma)$ for some t in this range.

Exactly s_1, \dots, s_{k-1} contribute to $\gamma_{t-1}(\sigma)$, but at least r_1, \dots, r_k contribute to $\gamma_{t-1}(\rho)$. It follows that

$$k - 1 = \gamma_{t-1}(\sigma) - \gamma_t(\sigma) \geq \gamma_{t-1}(\rho) - \gamma_t(\rho) \geq k,$$

a contradiction finishing the proof. □

Setting

$$I^\sigma = I_{s_1} \cdots I_{s_u}.$$

for $\sigma = (s_1, \dots, s_u)$ we now describe the symbolic powers of the I_t .

Proposition 2.2. *One has*

$$I_t^{(k)} = \sum_{\substack{\sigma=(s_1, \dots, s_u) \\ \gamma_t(\sigma) \geq k}} I^\sigma.$$

$I_t^{(k)}$ has a K -basis of the standard bitableaux Σ with $\gamma_t(\Sigma) \geq k$.

Proof. If $t = 1$, the right hand side is just the ideal of elements of total degree $\geq k$, and there is nothing to prove. Suppose that $t > 1$. Then X_{mn} is a non-zero-divisor modulo $I_t^{(k)}$ (by definition of the symbolic power). But it is also a non-zero-divisor modulo the right hand side, as we will see. Therefore we can invert X_{mn} . This transforms all sizes in the right way, and equality follows by induction on t .

We have noticed in Theorem 1.5 that straightening does not decrease shape. This implies $\gamma_t(\Sigma) \geq \gamma_t(\Delta)$ for all standard bitableaux in the standard representation of a bitableau Δ . Thus the right hand side has a K -basis by all Σ with $\gamma_t(\Sigma) \geq k$. Since multiplication by X_{mn} does not affect values under γ_t , and maps standard bitableaux to standard bitableaux, the result follows. \square

Of course, only finitely many summands are needed for $I_t^{(k)}$. The simplest non-trivial case is $t = k = 2$:

$$I_2^{(2)} = I_2^2 + I_3,$$

since every other summand is contained in I_2^2 or I_3 . If $m \leq 2$ or $n \leq 2$, then $I_2^2 = I_2^{(2)}$. This observation is easily generalized to the following result of Trung [70].

Corollary 2.3. *The symbolic powers of the ideal of maximal minors coincide with the ordinary ones.*

Proof. Suppose that $m = \min(m, n)$. Then a bitableau Δ has $\gamma_m(\Delta) \geq k$ if and only if the first k factors have size exactly m . \square

The primary decomposition of products of the ideals I_t depends on characteristic. This indicates that the straightening law alone is not sufficient to prove it. Actually the straightening law enters the proof only via Theorem 2.2.

Theorem 2.4. *Let $\rho = (r_1, \dots, r_u)$ be a non-increasing sequence of integers and suppose that $\text{char } K = 0$ or $\text{char } K > \min(r_i, m - r_i, n - r_i)$ for $i = 1, \dots, u$. Then*

$$I^\rho = \bigcap_{t=1}^{r_1} I_t^{(\gamma_t(\rho))}.$$

The theorem was proved by De Concini, Eisenbud and Procesi [27] in characteristic 0 and generalized in [17]. The inclusion \subseteq is a triviality (and independent of the hypothesis on characteristic): $\gamma_t(x) \geq \gamma_t(\rho)$ for all $x \in I^\rho$. Before we indicate the proof of the converse inclusion, let us have a look at the first non-trivial case, namely

$$I_2^2 = I_1^4 \cap (I_3 + I_2^2).$$

For equality we must prove that the degree ≥ 4 elements in I_3 , namely those of $I_3 I_1$, are contained in I_2^2 . This type of containment is the crucial point in the proof

of the theorem. It is the ideal-theoretic analogue of the passage from σ to σ' in the proof of Lemma 2.1.

Lemma 2.5. *Let u, v be integers, $0 \leq u \leq v - 2$. Suppose that $\text{char} K = 0$ or $\text{char} K > \min(u + 1, m - (u + 1), n - (u + 1))$. Then $I_u I_v \subseteq I_{u+1} I_{v-1}$.*

For reasons of space we refer the reader to [17, (10.10)] for a proof of the lemma. It is based on a symmetrization argument, and that explains the condition on characteristic. Note that symmetrization is, in a sense, the opposite of straightening.

In view of Theorem 2.2 it is enough for the proof of Theorem 2.4 to show that a product $\Delta = \delta_1 \cdots \delta_p$ is in I^ρ if $\gamma_t(\Delta) \geq \gamma_t(\rho)$ for all t . For the analogy to the proof of Lemma 2.1 set $\sigma = |\Delta|$. Then the same induction works, since Lemma 2.5 implies in the critical case that Δ is a linear combination of bitableaux Δ' , $|\Delta'| = \sigma'$.

Remark 2.6. (a) Since we are mainly interested in asymptotic properties, we do not discuss when the decomposition in 2.4 is irredundant; see [17, (10.12)] and [17, (10.13)] for a precise result. Roughly speaking, the I_t -primary component is irredundant if I_t appears in the product (evidently) or if the number of factors I_u with $t < u < \min(m, n)$ is sufficiently large. In particular, the decomposition of I_t^k is irredundant for $k \gg 0$ if $t < \min(m, n)$.

(b) In characteristic 2 one has $I_3 I_1 \not\subseteq I_2^2$ if $m, n \geq 4$; see [17, (10.14)].

(c) Independently of the characteristic the intersection in Theorem 2.4 is the integral closure of I^σ ; see Bruns [8].

Because of Lemma 2.1 we can replace the γ -functions by α -functions in the description of the standard bases of products and powers. For powers one obtains a very simple statement:

Proposition 2.7. *Suppose that $\text{char} K = 0$ or $\text{char} K > \min(t, m - t, n - t)$. Then I_t^k has a basis consisting of all standard bitableaux Σ with $\alpha_k(\Sigma) \geq kt$.*

It is possible to derive this proposition from Theorem 2.4 by “diagram arithmetic”, but it is easier to prove it directly. Let V be the K -vector space generated by all standard bitableaux $\Sigma = \delta_1 \cdots \delta_u$ with $\alpha_k(\Sigma) \geq kt$. That $I_t^k \subseteq V$ follows again from the fact that straightening does not decrease shape. The converse inclusion follows immediately (and much easier than Theorem 2.4) from Lemma 2.5: if $|\delta_i| < t$ for some i , $1 \leq i \leq k$, then there is also an index j in this range such that $|\delta_j| > t$. Lemma 2.5 allows us to increase $|\delta_i|$ at the expense of $|\delta_j|$.

3. Gröbner bases, initial ideals and initial algebras

The aim of this section is to recall the definitions and some important properties of Gröbner bases, monomial orders, initial ideals and initial algebras. For further

information on the theory of Gröbner bases we refer the reader to the books by Eisenbud [33], Kreuzer and Robbiano [56], Sturmfels [67] and Vasconcelos [71]. For the so-called Sagbi bases and initial algebras one should consult Conca, Herzog and Valla [26], Robbiano and Sweedler [62], and [67, Chapter 11].

Throughout this section let K be a field, and let R be the polynomial ring $K[X_1, \dots, X_n]$. A *monomial* (or power product) of R is an element of the form $X^\alpha = \prod_{i=1}^n X_i^{\alpha_i}$ with $\alpha \in \mathbb{N}^n$. A *term* is an element of the form λm where λ is a non-zero element of K and m is a monomial. Let $M(R)$ be the K -basis of R consisting of all the monomials of R . Every polynomial $f \in R$ can be written as a sum of terms. The only lack of uniqueness in this representation is the order of the terms. If we impose a total order on the set $M(R)$, then the representation is uniquely determined, once we require that the monomials are written according to the order, from the largest to the smallest. The set $M(R)$ is a semigroup (naturally isomorphic to \mathbb{N}^n) and a total order on the set $M(R)$ is not very useful unless it respects the semigroup structure.

Definition 3.1. A *monomial order* τ is a total order $<_\tau$ on the set $M(R)$ which satisfies the following conditions:

- (a) $1 <_\tau m$ for all the monomials $m \in M(R) \setminus \{1\}$.
- (b) If $m_1, m_2, m_3 \in M(R)$ and $m_1 <_\tau m_2$, then $m_1 m_3 <_\tau m_2 m_3$.

From the theoretical as well as from the computational point of view it is important that descending chains in $M(R)$ terminate:

Remark 3.2. A *monomial order* on the set $M(R)$ is a well-order, i.e. every non-empty subset of $M(R)$ has a minimal element. Equivalently, there are no infinite descending chains in $M(R)$.

This follows from the fact that every (monomial) ideal in R is finitely generated. Therefore a subset N of $M(R)$ has only finitely many elements that are minimal with respect to divisibility. One of them is the minimal element of N .

We list the most important monomial orders.

Example 3.3. For monomials $m_1 = X_1^{\alpha_1} \dots X_n^{\alpha_n}$ and $m_2 = X_1^{\beta_1} \dots X_n^{\beta_n}$ one defines

- (a) the *lexicographic order* (Lex) by $m_1 <_{\text{Lex}} m_2$ iff for some k one has $\alpha_k < \beta_k$ and $\alpha_i = \beta_i$ for $i < k$;
- (b) the *degree lexicographic order* (DegLex) by $m_1 <_{\text{DegLex}} m_2$ iff $\deg(m_1) < \deg(m_2)$ or $\deg(m_1) = \deg(m_2)$ and $m_1 <_{\text{Lex}} m_2$;
- (c) the *(degree) reverse lexicographic order* (RevLex) by $m_1 <_{\text{RevLex}} m_2$ iff $\deg(m_1) < \deg(m_2)$ or $\deg(m_1) = \deg(m_2)$ and for some k one has $\alpha_k > \beta_k$ and $\alpha_i = \beta_i$ for $i > k$.

These three monomial orders satisfy $X_1 > X_2 > \dots > X_n$. More generally, for every total order on the indeterminates one can consider the Lex, DegLex and

RevLex orders extending the order of the indeterminates; just change the above definition correspondingly.

From now on we fix a monomial order τ on (the monomials of) R . Whenever there is no danger of confusion we will write $<$ instead of $<_\tau$. Every polynomial $f \neq 0$ has a unique representation

$$f = \lambda_1 m_1 + \lambda_2 m_2 + \cdots + \lambda_k m_k$$

where $\lambda_i \in K \setminus \{0\}$ and m_1, \dots, m_k are distinct monomials such that $m_1 > \cdots > m_k$. The *initial monomial* of f with respect to τ is denoted by $\text{in}_\tau(f)$ and is, by definition, m_1 . Clearly one has

$$\text{in}_\tau(fg) = \text{in}_\tau(f) \text{in}_\tau(g) \quad (1)$$

and $\text{in}_\tau(f + g) \leq \max_\tau\{\text{in}_\tau(f), \text{in}_\tau(g)\}$. For example, the initial monomial of the polynomial $f = X_1 + X_2 X_4 + X_3^2$ with respect to the Lex order is X_1 , with respect to Deglex it is $X_2 X_4$, and with respect to RevLex it is X_3^2 .

Given a K -subspace $V \neq 0$ of R , we define

$$M_\tau(V) = \{\text{in}_\tau(f) : f \in V\}$$

and set

$$\text{in}_\tau(V) = \text{the } K\text{-subspace of } R \text{ generated by } M_\tau(V).$$

The space $\text{in}_\tau(V)$ is called the *space of the initial terms of V* . Whenever there is no danger of confusion we suppress the reference to the monomial order and use the notation $\text{in}(f)$, $M(V)$ and $\text{in}(V)$.

Any positive integral vector $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ induces a graded structure on R , called the *a -grading*. With respect to the a -grading the indeterminate X_i has degree $a(X_i) = a_i$. Every monomial X^α is a -homogeneous of a -degree $\sum \alpha_i a_i$, and the a -degree $a(f)$ of a non-zero polynomial $f \in R$ is the largest a -degree of a monomial in f . Then $R = \bigoplus_{i=0}^{\infty} R_i$ where R_i is the *a -graded component* of R of degree i , i.e. the span of the monomials of a -degree i . With respect to this decomposition R has the structure of a positively graded K -algebra [13, Section 1.5]. The elements of R_i are *a -homogeneous* of a -degree i . We say that a vector subspace V of R is *a -graded* if it is generated, as a vector space, by homogeneous elements. This amounts to the decomposition $V = \bigoplus_{i=0}^{\infty} V_i$ where $V_i = V \cap R_i$.

Proposition 3.4. *Let V be a K -subspace of R .*

- (a) *If $m \in M(V)$ then there exists $f_m \in V$ such that $\text{in}(f) = m$. The polynomial f_m is uniquely determined if we further require that the support of f_m intersects $M(V)$ exactly in m and that f_m has leading coefficient 1.*
- (b) *$M(V)$ is a K -basis of $\text{in}(V)$.*
- (c) *The set $\{f_m : m \in M(V)\}$ is a K -basis of V .*
- (d) *If V has finite dimension, then $\dim(V) = \dim(\text{in}(V))$.*

- (e) Let $a \in \mathbb{N}^r$ a positive weight vector. Suppose V is a -graded, say $V = \bigoplus_{i=0}^{\infty} V_i$. Then $\text{in}(V) = \bigoplus_{i=0}^{\infty} \text{in}(V_i)$. In particular, V and $\text{in}(V)$ have the same Hilbert function, i.e. $\dim(V_i) = \dim(\text{in}(V)_i)$ for all $i \in \mathbb{N}$.
- (f) Let $V_1 \subseteq V_2$ be K -subspaces of R . Then $\text{in}(V_1) \subseteq \text{in}(V_2)$ and the (residue classes of the) elements in $M(V_2) \setminus M(V_1)$ form a K -basis of the quotient space $\text{in}(V_2)/\text{in}(V_1)$. Furthermore the set of the (residue classes of the) f_m with $f_m \in V_2$ and $m \in M(V_2) \setminus M(V_1)$ is a K -basis of V_2/V_1 (regardless of the choice of the f_m).
- (g) The set of the (residue classes of the) elements in $M(R) \setminus M(V)$ is a K -basis of R/V .
- (h) Let $V_1 \subseteq V_2$ be K -subspaces of R . If $\text{in}(V_1) = \text{in}(V_2)$, then $V_1 = V_2$.
- (i) Let V be a K -subspace of R and σ, τ monomial orders. If $\text{in}_{\tau}(V) \subseteq \text{in}_{\sigma}(V)$, then $\text{in}_{\tau}(V) = \text{in}_{\sigma}(V)$.

Proof. (a) and (b) follow easily from the fact that the monomials form a K -basis of R . For (a) we have to use that descending chains in $M(R)$ terminate.

To prove (c) one notes that the f_m are linearly independent since they have distinct initial monomials. To show that they generate V , we pick any non-zero $f \in V$ and set $m = \text{in}(f)$. Then $m \in M(V)$ and we may subtract from f a suitable scalar multiple of f_m , say $g = f - \lambda f_m$, so that $\text{in}(g) < \text{in}(f)$, unless $g = 0$. Since $g \in V$, we may repeat the procedure with g and go on in the same manner. By Remark 3.2, after a finite number of steps we reach 0, and f is a linear combination of the polynomials f_m collected in the subtraction procedure.

(d) and (e) follow from (b) and (c) after the observation that the element f_m can be taken a -homogeneous if V is a -graded.

The first two assertions in (f) are easy. For the last we note that f_m can be chosen in V_1 if $m \in \text{in}(V_1)$.

The residue classes of the f_m with $m \in M(V_2) \setminus M(V_1)$ are linearly independent modulo V_1 since otherwise there would be a non-trivial linear combination $g = \sum \lambda_m f_m \in V_1$. But then $\text{in}(g) \in \text{in}(V_1)$, a contradiction since $\text{in}(g)$ is one of the monomials m which by assumption do not belong to $M(V_1)$.

To show that the f_m with $m \in M(V_2) \setminus M(V_1)$ generate V_2/V_1 take some non-zero element $f \in V_2$ and set $m = \text{in}(f)$. Subtracting a suitable scalar multiple of f_m from f we obtain a polynomial in V_2 with smaller initial monomial than f (or 0). If $m \in M(V_1)$, then $f_m \in V_1$. Repeating the procedure we reach 0 after finitely many steps. So f can be written as a linear combination of elements of the form f_m with $m \in M(V_2) \setminus M(V_1)$ and elements of V_1 , which is exactly what we want.

(g) is a special case of (f) with $V_2 = R$ since in this case we can take $f_m = m$ for all $m \in M(R) \setminus M(V)$.

(h) follows from (f) since $\text{in}(V_1) = \text{in}(V_2)$ implies that the empty set is a basis of V_2/V_1 .

Finally, (i) follows from (g) because an inclusion between the two bases $\{m \in M(R) : m \notin M_\tau(V)\}$ and $\{m \in M(R) : m \notin M_\sigma(V)\}$ of the space R/V implies that they are equal. \square

Remark/Definition 3.5. (a) If A is a K -subalgebra of R , then $\text{in}(A)$ is also a K -subalgebra of R . This follows from equation (1) and from 3.4(a). The K -algebra $\text{in}(A)$ is called the *initial algebra* of A (with respect to τ).

(b) If A is a K -subalgebra of R and J is an ideal of A , then $\text{in}(J)$ is an ideal of the initial algebra $\text{in}(A)$. This, too, follows from equation (1) and from 3.4(a).

(c) If I is an ideal of R , then $\text{in}(I)$ is also an ideal of R . This is a special case of (b) since $\text{in}(R) = R$.

Definition 3.6. Let A be K -subalgebra of R . A subset F of A is said to be a *Sagbi basis* of A (with respect to τ) if the initial algebra $\text{in}(A)$ is equal to the K -algebra generated by the monomials $\text{in}(f)$ with $f \in F$.

If the initial algebra $\text{in}(A)$ is generated, as a K -algebra, by a set of monomials G , then for every m in G we can take a polynomial f_m in A such that $\text{in}(f_m) = m$. Therefore A has a finite Sagbi basis iff $\text{in}(A)$ is finitely generated. However it may happen that A is finitely generated, but $\text{in}(A)$ is not; see [62].

Definition 3.7. Let A be a K -subalgebra of R and J be an ideal of A . A subset F of J is said to be a *Gröbner basis* of J with respect to τ if the initial ideal $\text{in}(J)$ is equal to the ideal of $\text{in}(A)$ generated by the monomials $\text{in}(f)$ with $f \in F$.

If the initial ideal $\text{in}(J)$ is generated, as an ideal of $\text{in}(A)$, by a set of monomials G , then for every m in G we can take a polynomial f_m in J such that $\text{in}(f_m) = m$. Therefore J has a finite Gröbner basis iff $\text{in}(J)$ is finitely generated. In particular, if $\text{in}(A)$ is a finitely generated K -algebra, then it is Noetherian and so all the ideals of A have a finite Gröbner basis. Evidently, all the ideals of R have a finite Gröbner basis.

There is an algorithm to determine a Gröbner basis of an ideal of R starting from any (finite) system of generators, the famous *Buchberger algorithm*. Similarly there is an algorithm that decides whether a given (finite) set of generators for a subalgebra A is a Sagbi basis. There also exists a procedure that completes a system of generators to a Sagbi basis of A , but it does not terminate if the initial algebra is not finitely generated. If a finite Sagbi basis for an algebra A is known, a generalization of Buchberger's algorithm finds Gröbner bases for ideals of A . We will not use these algorithms in this article and so we refer the interested readers to the literature quoted at the beginning of this section..

Initial objects with respect to weights. In order to present the deformation theory for initial ideals and algebras we need to further generalize these notions and

consider initial objects with respect to weights. As pointed out above, any positive integral weight vector $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ induces a structure of positively graded algebra on R . Let t be a new variable and set

$$S = R[t].$$

For $f = \sum \gamma_i m_i \in R$ with $\gamma_i \in K$ and monomials m_i one defines the a -homogenization $\text{hom}_a(f)$ of f to be the polynomial

$$\text{hom}_a(f) = \sum \gamma_i m_i t^{a(f) - a(m_i)}.$$

Let $a' = (a_1, \dots, a_n, 1) \in \mathbb{N}^{n+1}$. Clearly, for every $f \in R$ the element $\text{hom}_a(f) \in S$ is a' -homogeneous, and $f = \text{hom}_a(f)$ iff f is a -homogeneous. One has

$$\begin{aligned} \text{in}_a(fg) &= \text{in}_a(f) \text{in}_a(g) \\ \text{hom}_a(fg) &= \text{hom}_a(f) \text{hom}_a(g) \end{aligned} \quad \text{for all } f, g \in R \quad (2)$$

For every K -subspace V of R we set

$$\begin{aligned} \text{in}_a(V) &= \text{the } K\text{-subspace of } R \text{ generated by } \text{in}_a(f) \text{ with } f \in V, \\ \text{hom}_a(V) &= \text{the } K[t]\text{-submodule of } S \text{ generated by } \text{hom}_a(f) \text{ with } f \in V. \end{aligned}$$

If A is a K -subalgebra of R and J is an ideal of A , then it follows from (2) that $\text{in}_a(A)$ is a K -subalgebra of R and $\text{in}_a(J)$ is an ideal of $\text{in}_a(A)$. Furthermore $\text{hom}_a(A)$ is a $K[t]$ -subalgebra of S and $\text{hom}_a(J)$ is an ideal of $\text{hom}_a(A)$. As for initial objects with respect to monomial orders, $\text{in}_a(A)$ and $\text{hom}_a(A)$ need not be finitely generated K -algebras, even when A is finitely generated. But if $\text{in}_a(A)$ is finitely generated, we may find generators of the form $\text{in}_a(f_1), \dots, \text{in}_a(f_k)$ with $f_1, \dots, f_k \in A$. It is easy to see that the f_i generate A . This follows from the next lemma in which we use the notation $f^\alpha = \prod f_i^{\alpha_i}$ for a vector $\alpha \in \mathbb{N}^k$ and the list $f = f_1, \dots, f_k$.

Lemma 3.8. *Let A be K -subalgebra of R . Assume that $\text{in}_a(A)$ is finitely generated by $\text{in}_a(f_1), \dots, \text{in}_a(f_k)$ with $f_1, \dots, f_k \in A$. Then every $F \in A$ has a representation*

$$F = \sum \lambda_i f^{\beta_i}$$

where $\lambda_i \in K \setminus \{0\}$ and $a(F) \geq a(f^{\beta_i})$ for all i .

Proof. By decreasing induction on $a(F)$. The case $a(F) = 0$ being trivial, we assume $a(F) > 0$. Since $F \in A$ we have $\text{in}_a(F) \in \text{in}_a(A) = K[\text{in}_a(f_1), \dots, \text{in}_a(f_k)]$. Since $\text{in}_a(F)$ is an a -homogeneous element of the a -graded algebra $\text{in}_a(A)$, we may write

$$\text{in}_a(F) = \sum \lambda_i \text{in}_a(f^{\alpha_i})$$

where $a(\text{in}_a(f^{\alpha_i})) = a(\text{in}_a(F))$ for all i . We set $F_1 = F - \sum \lambda_i f^{\alpha_i}$ and conclude by induction since $a(F_1) < a(F)$ if $F_1 \neq 0$. \square

The following lemma contains a simple but crucial fact:

Lemma 3.9. *Let A be a K -subalgebra of R and J be an ideal of A . Assume that $\text{in}_a(A)$ is finitely generated by $\text{in}_a(f_1), \dots, \text{in}_a(f_k)$ with $f_1, \dots, f_k \in A$. Let $B = K[Y_1, \dots, Y_k]$ and take presentations*

$$\varphi_1 : B \rightarrow A/J \quad \text{and} \quad \varphi : B \rightarrow \text{in}_a(A)/\text{in}_a(J)$$

defined by the substitutions $\varphi_1(Y_i) = f_i \bmod (J)$ and $\varphi(Y_i) = \text{in}_a(f_i) \bmod (\text{in}_a(J))$. Set $b = (a(f_1), \dots, a(f_k)) \in \mathbb{N}_+^k$. Then

$$\text{in}_b(\text{Ker } \varphi_1) = \text{Ker } \varphi.$$

Proof. As a vector space, $\text{in}_b(\text{Ker } \varphi_1)$ is generated by the elements $\text{in}_b(p)$ with $p \in \text{Ker } \varphi_1$. Set $u = b(p)$. Then we may write $p = \sum \lambda_i Y^{\alpha_i} + \sum \mu_j Y^{\beta_j}$ where $b(Y^{\alpha_i}) = u$ and $b(Y^{\beta_j}) < u$. The image $F = \sum \lambda_i f^{\alpha_i} + \sum \mu_j f^{\beta_j}$ belongs to J , and, hence, $\text{in}_a(F) \in \text{in}_a(J)$. Since $b(Y^\gamma) = a(f^\gamma)$, it follows that $\text{in}_a(F) = \sum \lambda_i \text{in}_a(f^{\alpha_i})$. Thus $\text{in}_b(p) \in \text{Ker } \varphi$, and this proves the inclusion \subseteq .

For the other inclusion we lift φ_1 and φ to presentations

$$\rho_1 : B \rightarrow A \quad \text{and} \quad \rho : B \rightarrow \text{in}_a(A),$$

mapping Y_i to f_i and to $\text{in}_a(f_i)$, respectively. Take a system of b -homogeneous generators G_1 of the ideal $\text{Ker } \rho$ of B and a system of a -homogeneous generators G_2 of the ideal $\text{in}_a(J)$ of $\text{in}_a(A)$. Every $g \in G_2$, being a -homogeneous of degree $u = a(g)$, is of the form $g = \text{in}_a(g')$, with $g' \in J$. Then $g' = \sum \gamma_i f^{\alpha_i} + \sum \mu_j f^{\beta_j}$ with $a(f^{\alpha_i}) = u$ and $a(f^{\beta_j}) < u$. Therefore $g = \sum \gamma_i \text{in}_a(f^{\alpha_i})$.

We choose the canonical preimage of the given representation of g , i.e. $h_g = \sum \gamma_i Y^{\alpha_i}$. Then the set $G_1 \cup \{h_g : g \in G_2\}$ generates the ideal $\text{Ker } \varphi$. For all $g \in G_2$ and g' as above, the canonical preimage of the given representation of g' , i.e. $h = \sum \gamma_i Y^{\alpha_i} + \sum \mu_j Y^{\beta_j}$ is in $\text{Ker } \varphi_1$, and one has $\text{in}_b(h) = h_g$.

It remains to show that $g \in \text{in}_b(\text{Ker } \varphi_1)$ for $g \in G_1$. Every $g \in G_1$ is homogeneous, say of degree u , and hence $g = \sum \lambda_i Y^{\alpha_i}$ with $b(Y^{\alpha_i}) = u$. It follows that $\sum \lambda_i \text{in}_a(f^{\alpha_i}) = 0$. Therefore $\sum \lambda_i f^{\alpha_i} = \sum \mu_j f^{\beta_j}$ with $a(f^{\beta_j}) < u$ by Lemma 3.8. That is, $g' = \sum \lambda_i Y^{\alpha_i} - \sum \mu_j Y^{\beta_j}$ is in $\text{Ker } \rho$. In particular, $g' \in \text{Ker } \varphi_1$ and $\text{in}_b(g') = g$. \square

A weight vector a and a monomial order τ on R define a new monomial order τa that “refines” the weight a by τ :

$$m_1 >_{\tau a} m_2 \iff \begin{cases} a(m_1) > a(m_2) \text{ or} \\ a(m_1) = a(m_2) \text{ and } m_1 >_{\tau} m_2. \end{cases}$$

We extend τa to $S = R[t]$ by setting:

$$m_1 t^i >_{\tau a'} m_2 t^j \iff \begin{cases} a'(m_1 t^i) > a'(m_2 t^j) \text{ or} \\ a'(m_1 t^i) = a'(m_2 t^j) \text{ and } i < j \text{ or} \\ a'(m_1 t^i) = a'(m_2 t^j) \text{ and } i = j \text{ and } m_1 >_{\tau} m_2. \end{cases}$$

By construction one has

$$\text{in}_{\tau a}(f) = \text{in}_{\tau a'}(\text{hom}_a(f)) \quad \text{for all } f \in R, f \neq 0.$$

Given a K -subspace V of R , we let $VK[t]$ denote the $K[t]$ -submodule of S generated by the elements in V .

Proposition 3.10. *Let a be a positive integral vector and τ be a monomial order on R . For every K -subspace V of R one has:*

- (a) $\text{in}_{\tau a}(V) = \text{in}_{\tau a}(\text{in}_a(V)) = \text{in}_{\tau}(\text{in}_a(V))$,
- (b) *If either $\text{in}_{\tau}(V) \subseteq \text{in}_a(V)$ or $\text{in}_{\tau}(V) \supseteq \text{in}_a(V)$, then $\text{in}_{\tau}(V) = \text{in}_a(V)$,*
- (c) $\text{in}_{\tau a}(V)K[t] = \text{in}_{\tau a'}(\text{hom}_a(V))$,
- (d) *The quotient $S/\text{hom}_a(V)$ is a free $K[t]$ -module.*

Proof. (a) Note that $\text{in}_{\tau a}(f) = \text{in}_{\tau a}(\text{in}_a(f)) = \text{in}_{\tau}(\text{in}_a(f))$ holds for every $f \in R$. It follows that the first space is contained in the second and in the third. On the other hand, since $\text{in}_a(V)$ is a -homogeneous, the monomials in its initial space are initial monomials of a -homogeneous elements. But every a -homogeneous element in $\text{in}_a(V)$ is of the form $\text{in}_a(f)$ with $f \in V$. This gives the other inclusions.

(b) If one of the two inclusions holds, then an application of $\text{in}_{\tau}(\dots)$ to both sides yields that $\text{in}_{\tau}(V)$ either contains or is contained in $\text{in}_{\tau}(\text{in}_a(V))$. By (a) the latter is $\text{in}_{\tau a}(V)$. Then by Proposition 3.4(i) we have that $\text{in}_{\tau}(V) = \text{in}_{\tau a}(V)$. Next we may apply 3.4(h) and conclude that $\text{in}_{\tau}(V) = \text{in}_a(V)$.

(c) For every $f \in R$ one has $\text{in}_{\tau a'}(\text{hom}_a(f)) = \text{in}_{\tau a}(f)$. Thus $\text{in}_{\tau a}(V)K[t] \subseteq \text{in}_{\tau a'}(\text{hom}_a(V))$. On the other hand, $\text{hom}_a(V)$ is an a' -homogeneous space. Therefore its initial space is generated by the initial monomials of its a' -homogeneous elements. An a' -homogeneous element of degree, say, u in $\text{hom}_a(V)$ has the form $g = \sum_{i=1}^k \lambda_i t^{\alpha_i} \text{hom}_a(f_i)$ where $f_i \in V$ and $\alpha_i + a(f_i) = u$. If $\alpha_i = \alpha_j$ then $a(f_i) = a(f_j)$ and $\text{hom}_a(f_i + f_j) = \text{hom}_a(f_i) + \text{hom}_a(f_j)$. In other words, we may assume that the α_i are all distinct and, after reordering if necessary, that $\alpha_i < \alpha_{i+1}$. Then $\text{in}_{\tau a'}(g) = t^{\alpha_1} \text{in}_{\tau a'}(\text{hom}(f_1)) = t^{\alpha_1} \text{in}_{\tau a}(f_1)$. This proves the other inclusion.

(d) By (c) and Proposition 3.4(b) the (classes of the) elements $t^{\alpha} m$, $\alpha \in \mathbb{N}$, $m \in M(R) \setminus M(V)$, form a K -basis of $S/\text{hom}_a(V)$. This implies that the set $M(R) \setminus M(V)$ is a $K[t]$ -basis of $S/\text{hom}_a(V)$. \square

The next proposition connects the structure of R/I with that of $R/\text{in}_a(R)$:

Proposition 3.11. *For every ideal I of R the ring $S/\text{hom}_a(I)$ is a free $K[t]$ -module. In particular $t - \alpha$ is a non-zero divisor on $S/\text{hom}_a(I)$ for every $\alpha \in K$. Furthermore $S/(\text{hom}_a(I) + (t)) \cong R/\text{in}_a(I)$ and $S/(\text{hom}_a(I) + (t - \alpha)) \cong R/I$ for all $\alpha \neq 0$.*

Proof. The first assertion follows from 3.10(c). It implies that every non-zero element of $K[t]$ is a non-zero divisor on $S/\text{hom}_a(I)$. For $S/(\text{hom}_a(I) + (t)) \cong R/\text{in}_a(I)$ it is enough that $\text{hom}_a(I) + (t) = \text{in}_a(I) + (t)$. This is easily seen since for every $f \in R$ the polynomials $\text{in}_a(f)$ and $\text{hom}_a(f)$ differ only by a multiple of t . To prove that $S/(\text{hom}_a(I) + (t - \alpha)) \cong R/I$ for every $\alpha \neq 0$, we consider the graded isomorphism $\psi : R \rightarrow R$ induced by $\psi(X_i) = \alpha^{-a_i} X_i$. One checks that $\psi(m) = \alpha^{-a(m)} m$ for every monomial m of R and that $\text{hom}_a(f) - \alpha^{a(f)} \psi(f)$ is a multiple of $t - \alpha$ for all the $f \in R$. So $\text{hom}_a(I) + (t - \alpha) = \psi(I) + (t - \alpha)$, which implies the desired isomorphism. \square

Now we use Proposition 3.11 for comparing R/I with $R/\text{in}_a(I)$.

Proposition 3.12. (a) *R/I and $R/\text{in}_a(I)$ have the same Krull dimension.*
 (b) *The following properties are passed from $R/\text{in}_a(I)$ on to R/I : being reduced, a domain, a normal domain, Cohen-Macaulay, Gorenstein.*
 (c) *Suppose that I is graded with respect to some positive weight vector b . Then $\text{in}_a(I)$ is b -graded, too, and the Hilbert functions of R/I and $R/\text{in}_a(I)$ coincide.*

Proof. Let us start with (b). The K -algebra $A = S/\text{hom}_a(I)$ is positively graded. Let \mathfrak{m} denote its maximal ideal generated by the residue classes of the indeterminates. Then A has one of the properties mentioned if and only if the localization $A' = A_{\mathfrak{m}}$ does so. In fact, all of the properties depend only on the localizations of A with respect to graded prime ideals, and such localizations are localizations of A' (see [13, Section 1.5 and Chapter 2]). The element t is a non-zero-divisor in the maximal ideal of the local ring A' . Moreover $A'/(t)$ is a localization of $R/\text{in}_a(I)$, and the properties under consideration are inherited by localizations. As just pointed out, they ascend from A' to A . Therefore it remains to prove that they also ascend from $A'/(t)$ to A' .

It is elementary to show that A' is reduced or an integral domain if $A'/(t)$ has this property. For the Cohen-Macaulay and Gorenstein property the same conclusion is contained in [13, 2.1.3 and 3.1.9].

It remains to consider normality. We show that A' has the Serre properties (R_1) and (S_2) if these hold for $A'/(t)$. Let \mathfrak{p} be a prime ideal of A' with height $\mathfrak{p} \leq 1$. If $t \in \mathfrak{p}$, then $\bar{\mathfrak{p}} = \mathfrak{p}/(t)$ is a minimal prime ideal of $A'/(t)$, and the regularity of $(A'/(t))_{\bar{\mathfrak{p}}} = A'_{\mathfrak{p}}/(t)$ implies that of A' . If $t \notin \mathfrak{p}$, we choose a minimal prime overideal \mathfrak{q} of $\mathfrak{p} + (t)$. Since A' is an integral domain and a localization of an

affine K -algebra, we must have $\text{height } \mathfrak{q} = \text{height } \mathfrak{p} + 1$. Moreover, $\text{height } \mathfrak{q}/(t) = \text{height } \mathfrak{q} - 1 = \text{height } \mathfrak{p}$. It follows that $(A'/(t))_{\mathfrak{q}}$ is regular. So $A'_{\mathfrak{q}}$ and its localization $A'_{\mathfrak{p}}$ are regular. Suppose now that $\text{height } \mathfrak{p} \geq 2$. We must show that $\text{depth } A'_{\mathfrak{p}} \geq 2$. If $t \in \mathfrak{p}$, then we certainly have $\text{depth}(A'/(t))_{\mathfrak{p}} \geq 1$, since $(A'/(t))_{\mathfrak{p}}$ is regular or has depth at least 2. Otherwise we take \mathfrak{q} as above. Then $\text{depth}(A'/(t))_{\mathfrak{q}} \geq 2$, and $\text{depth } A'_{\mathfrak{q}} \geq 3$. We choose $u \neq 0$ in \mathfrak{p} . If $\text{depth } A'_{\mathfrak{p}} = 1$, then $\mathfrak{p}/(u)$ is an associated prime ideal of $A'/(u)$. Moreover, we have $\text{depth } A'_{\mathfrak{q}}/(u) \geq 2$, and $\dim A'_{\mathfrak{q}}/\mathfrak{p}A'_{\mathfrak{q}} = 1$. This is a contradiction to [13, 1.2.13]: for a local ring R one has $\text{depth } R \leq \dim R/\mathfrak{p}$ for all associated prime ideals \mathfrak{p} of R .

It remains to transfer the properties listed in (b) to $A'' = A/(t-1) \cong R/I$, the dehomogenization of A with respect to the degree 1 element t . So A'' is the degree 0 component of the graded ring $A[t^{-1}]$, and $A[t^{-1}]$ is just the Laurent polynomial ring in the variable t over A'' . (This is not hard to see; cf. [13, Section 1.5]. The main point is that the surjection $A \rightarrow A''$ factors through $A[t^{-1}]$ and that the latter ring has a homogeneous unit of degree 1.) Finally, each of the properties descends from the Laurent polynomial ring to A'' .

For (a) one follows the same chain of descents and ascents: $\dim R/I = \dim A'' = \dim A''[t, t^{-1}] - 1 = \dim A[t^{-1}] = \dim A - 1$. For the very last equality one has to use that t is a non-zero-divisor in an affine K -algebra.

(c) First one should note that $\text{in}_a(I)$ is b -graded, since the initial form of a b -homogeneous element is b -homogeneous, too. We refine the weight a by a monomial order τ and derive the chain of equations

$$H(R/\text{in}_a(I)) = H(R/\text{in}_{\tau}(\text{in}_a(I))) = H(R/\text{in}_{\tau a}(I)) = H(R/I).$$

for the Hilbert function $H(\dots)$ from 3.4(e) and 3.10(a). \square

Very often one wants to compare finer invariants of $R/\text{in}_a(I)$ and R/I , for example if I is a graded ideal of R with respect to some other weight vector b . The next proposition shows that the comparison is possible for graded components of Tor-modules. One can prove an analogous inequality for Ext-modules.

Proposition 3.13. *Let a, b positive integral vectors and let J, J_1, J_2 be b -homogeneous ideals of R with $J \subseteq J_1$ and $J \subseteq J_2$. Then $\text{in}_a(J), \text{in}_a(J_1), \text{in}_a(J_2)$ are also b -homogeneous ideals, and one has*

$$\dim_K \text{Tor}_i^{R/J}(R/J_1, R/J_2)_j \leq \dim_K \text{Tor}_i^{R/\text{in}_a(J)}(R/\text{in}_a(J_1), R/\text{in}_a(J_2))_j$$

where the graded structure on the Tor-modules is inherited from the b -graded structure of their arguments.

Proof. On S we introduce a bigraded structure, setting $\deg X_i = (b_i, a_i)$ and $\deg t = (0, 1)$. The ideals $I = \text{hom}_a(J)$, $I_1 = \text{hom}_a(J_1)$ and $I_2 = \text{hom}_a(J_2)$ are then bigraded and so are the algebras they define. We need a standard result in

homological algebra: if A is a ring, M, N are A -modules and x is a non-zero-divisor on A as well as on M then $\mathrm{Tor}_i^A(M, N/xN) \cong \mathrm{Tor}_i^{A/xA}(M/xM, N/xN)$. (It is difficult to find an explicit reference; for example, one can use [13, 1.1.5].) If, in addition, x is a non-zero-divisor also on N , then we have the short exact sequence $0 \rightarrow N \rightarrow N \rightarrow N/xN \rightarrow 0$. It yields the exact sequence

$$0 \rightarrow \mathrm{CoKer} \varphi_i \rightarrow \mathrm{Tor}_i^{A/xA}(M/xM, N/xN) \rightarrow \mathrm{Ker} \varphi_{i-1} \rightarrow 0$$

where φ_i is multiplication by x on $\mathrm{Tor}_i^A(M, N)$.

Set $A = S/\mathrm{hom}_a(J)$, $M = S/\mathrm{hom}_a(J_1)$, $N = S/\mathrm{hom}_a(J_2)$ and $T_i = \mathrm{Tor}_i^A(M, N)$. Since the modules involved are bigraded, so is T_i . Let T_{ij} be the direct sum of all the components of T_i of bidegree (j, k) as k varies. Since T_i is a finitely generated bigraded S -module, T_{ij} is a finitely generated and graded $K[t]$ -module (with respect to the standard grading of $K[t]$). So we may decompose it as

$$T_{ij} = F_{ij} \oplus G_{ij}$$

where F_{ij} is the free part and G_{ij} is the torsion part, which, being $K[t]$ -graded, is a direct sum of modules of the form $K[t]/(t^a)$ for various $a > 0$. Denote the minimal number of generators of F_{ij} and G_{ij} as $K[t]$ -modules by f_{ij} and g_{ij} , respectively. Now we consider the b -homogeneous component of degree j of the above short exact sequence with $x = t$, which is a non-zero-divisor by Proposition 3.10(d). It follows that

$$\dim_K \mathrm{Tor}_i^{R/\mathrm{in}_a(J)}(R/\mathrm{in}_a(J_1), R/\mathrm{in}_a(J_2))_j = f_{ij} + g_{ij} + g_{i-1, j}.$$

If we take $x = t - 1$ instead of x , then we have

$$\dim_K \mathrm{Tor}_i^{R/J}(R/J_1, R/J_2)_j = f_{ij}$$

and this shows the desired inequality. \square

Note that one can also use Proposition 3.13 to transfer the Cohen-Macaulay and Gorenstein properties from $R/\mathrm{in}_a(I)$ to R/I if I is b -graded.

If I is graded with respect to the ordinary weight $(1, \dots, 1)$ then it makes sense to ask for the Koszul property of R/I . By definition, R/I is Koszul if $\mathrm{Tor}_i^{R/I}(R/\mathfrak{m}, R/\mathfrak{m})_j$ is non-zero only for $i = j$. Backelin and Fröberg [4] give a detailed discussion of this class of rings.

Corollary 3.14. *Suppose that I is a graded ideal with respect to the weight $(1, \dots, 1)$. If, for some positive weight a , $\mathrm{in}_a(I)$ is generated by degree 2 monomials, then R/I is Koszul.*

Proof. By a theorem of Fröberg [35] the algebra $R/\mathrm{in}_a(I)$ is Koszul. so that the corollary follows from 3.13. \square

In order to apply the previous results to initial objects defined by monomial orders we have to approximate such orders by weight vectors. This is indeed possible, provided only finitely many monomials have to be considered.

Proposition 3.15. *Let τ be a monomial order on R .*

- (a) *Let $\{(m_1, n_1), \dots, (m_k, n_k)\}$ be a finite set of pairs of monomials such that $m_i >_{\tau} n_i$ for all i . Then there exists a positive integral weight $a \in \mathbb{N}_+^n$ such that $a(m_i) > a(n_i)$ for all i .*
- (b) *Let A be a K -subalgebra of R and I_1, \dots, I_h be ideals of A . Assume that $\text{in}_{\tau}(A)$ is finitely generated as a K -algebra. Then there exists a positive integral weight $a \in \mathbb{N}_+^n$ such that $\text{in}_{\tau}(A) = \text{in}_a(A)$ and $\text{in}_{\tau}(I_i) = \text{in}_a(I_i)$ for all $i = 1, \dots, h$.*

Proof. (a) Set $m_i = X^{\alpha_i}$ and $n_i = X^{\beta_i}$ and $\gamma_i = \alpha_i - \beta_i \in \mathbb{Z}^n$. Let Γ be the $k \times n$ integral matrix whose rows are the vectors γ_i . We are looking for a positive column vector a such that all the coefficients of the vector Γa are > 0 . Suppose, by contradiction, there is no such a . Then (one version of the famous) Farkas Lemma (see Schrijver [65, Section 7.3]) says that there exists a linear combination $v = \sum c_i \gamma_i$ with non-negative integral coefficients $c_i \in \mathbb{N}$ such that $v \leq 0$, that is $v = (v_1, \dots, v_n)$ with $v_i \leq 0$. Then it follows that $\prod_i m_i^{c_i} X^{-v} = \prod_i n_i^{c_i}$, which contradicts our assumptions because the monomial order is compatible with the semigroup structure.

(b) Let F_0 be a finite Sagbi basis of A , let F_i be a finite Gröbner basis of I_i and set $F = \bigcup_i F_i$. Consider the set U of pairs of monomials $(\text{in}(f), m)$ where $f \in F$ and m is any non-initial monomial of f . Since U is finite, by (a) there exists $a \in \mathbb{N}_+^n$ such that $\text{in}_a(f) = \text{in}_{\tau}(f)$ for every $f \in F$. We show a has the desired property. Set $V_0 = A$ and $V_i = I_i$. By construction the (algebra for $i = 0$ and ideal for $i > 0$) generators of the $\text{in}_{\tau}(V_i)$ belong to $\text{in}_a(V_i)$ so that $\text{in}_{\tau}(V_i) \subseteq \text{in}_a(V_i)$. But then, by Proposition 3.10(c), we may conclude that $\text{in}_{\tau}(V_i) = \text{in}_a(V_i)$. \square

The main theorem of this section summarizes what we can say about the transfer of ring-theoretic properties from initial objects. For the Koszul property of subalgebras we must allow a “normalization” of degree. Suppose that b is a positive weight vector b , and suppose that a subalgebra A is generated by elements f_1, \dots, f_s of the same b -degree $e \in \mathbb{N}$. Then every element g of A has b -degree divisible by e , and dividing the b -degree by e we obtain the e -normalized b -degree of g .

Theorem 3.16. *Let $\text{in}(\dots)$ denote the initial objects with respect to a positive integral vector $a \in \mathbb{N}^n$ or to a monomial order τ on R . Let A be a K -subalgebra of R and J be an ideal of A . Suppose that $\text{in}(A)$ is finitely generated.*

- (a) *One has $\dim A/J = \dim \text{in}(A) / \text{in}(J)$.*

- (b) If $\text{in}(A)/\text{in}(J)$ is reduced, a domain, a normal domain, Cohen-Macaulay, or Gorenstein, then so is A/J .
- (c) Let b be a positive weight vector, and suppose that A and J are b -graded. Then A/J and $\text{in}(A)/\text{in}(J)$ have the same Hilbert function.
- (d) If, in addition to the hypothesis of (c), $\text{in}(A)/\text{in}(J)$ is Koszul with respect to e -normalized b -degree for some e , then so is A/J .

Proof. If the initial objects are formed with respect to a monomial order then, by 3.15, we may represent them as initial objects with respect to a suitable positive integral weight vector. Therefore in both cases the initial objects are taken with respect to a positive integral weight a . By Lemma 3.9 there exist a polynomial ring, say B , an ideal H , and a positive weight c such that $B/H \cong A/J$ and $B/\text{in}_c(H) \cong \text{in}(A)/\text{in}(J)$. Furthermore, under the hypothesis of (c), the weight b can be lifted from the generators of $\text{in}(A)$ to the indeterminates of B . Now the theorem follows from Proposition 3.12 and Lemma 3.14. \square

The theorem is usually applied in two extreme cases. In the first case $A = R$, so that $\text{in}(A) = R$, and in the second case $H = 0$, so that $\text{in}(J) = 0$. There is a special instance that deserves a separate statement.

Corollary 3.17. *Let A be K -subalgebra of R , and suppose that $\text{in}(A)$ is finitely generated. If it is generated by monomials (e.g. if the initial algebra is taken with respect to a monomial order) and normal, then A is normal and Cohen-Macaulay.*

Proof. By a theorem of Hochster [13, 6.3.5] the normal semigroup algebra $\text{in}(A)$ is Cohen-Macaulay. \square

Sometimes one of the implications in Theorem 3.16 can be reversed:

Corollary 3.18. *Let b be a positive weight vector, and suppose that the K -subalgebra A is b -graded and has a Cohen-Macaulay initial algebra $\text{in}(A)$. Then A is Gorenstein iff $\text{in}(A)$ is Gorenstein.*

Proof. Since $\text{in}(A)$ is Cohen-Macaulay, A is Cohen-Macaulay as well. So both algebras are positively graded Cohen-Macaulay domains. By a theorem of Stanley [13, 4.4.6], the Gorenstein property of such rings depends only on their Hilbert function, and both algebras have the same Hilbert function. \square

We want to extend Theorem 3.16 in such a way that it allows us to determine the canonical module of A/I .

Theorem 3.19. *Let A be a subalgebra of R as in Theorem 3.16, and $I \subseteq J$ ideals of A . Suppose that $\text{in}(A)/\text{in}(I)$ and, hence, A/I are Cohen-Macaulay.*

- (a) If $\text{in}(J)/\text{in}(I)$ is the canonical module of $\text{in}(A)/\text{in}(I)$, then J/I is the canonical module of A/I .
- (b) Suppose in addition that A, I, J are b -graded with respect to a positive weight b and $\text{in}(J)/\text{in}(I)$ is the canonical module of $\text{in}(A)/\text{in}(I)$ (up to a shift). Then J/I is the graded canonical module (up to the same shift).

Proof. For the sake of simplicity and since it is sufficient for our applications, we restrict ourselves to the graded case in (b) and $I = 0$. Since A is a Cohen-Macaulay positively graded K -algebra which is a domain, to prove that J is the canonical module of A it suffices to show that J is a maximal Cohen-Macaulay module whose Hilbert series satisfies the relation $H_J(t) = (-1)^d t^k H_A(t^{-1})$ for some integer k and $d = \dim A$ [13, Thm. 4.4.5, Cor. 4.4.6].

The relation $H_J(t) = (-1)^d t^k H_A(t^{-1})$ holds since, by hypothesis, the corresponding relation holds for the initial objects, and Hilbert series do not change by taking initial terms; see Theorem 3.16(c). So it is enough to show that J is a maximal Cohen-Macaulay module. But $\text{in}(J)$ is a height 1 ideal since it is the canonical module [13, Prop. 3.3.18], and hence J , too, has height 1. Therefore it suffices that A/J is a Cohen-Macaulay ring. But this follows again from 3.16(b) since $\text{in}(A)/\text{in}(J)$ is Cohen-Macaulay (even Gorenstein) by [13, Prop. 3.3.18]. \square

In order to prove the general version, one chooses representations $A/I \cong B/I_1$, $A/J \cong B/I_2$, $\text{in}(A)/\text{in}(I) \cong B/\text{in}(I_1)$, $\text{in}(A)/\text{in}(J) \cong B/\text{in}(I_2)$ as in Lemma 3.9. For the application of 3.11 one notes that t is a non-zero-divisor on all the residue class rings to be considered and that C is the canonical module of a positively graded ring R if C/tC is the canonical module of $R/(t)$ for a homogeneous non-zero-divisor of R and C . ([13] contains all the tools one needs to prove this claim.)

4. The Knuth–Robinson–Schensted correspondence

The Knuth–Robinson–Schensted correspondence (in our context) sets up a bijection between standard bitableaux and monomials in the ring $K[X]$. The passage from bitableaux to monomials is based on the *deletion* algorithm.

Definition 4.1. *Deletion* takes a standard tableau $A = (a_{ij})$, say of shape (s_1, s_2, \dots) , and an index p such that $s_p > s_{p+1}$, and constructs from them a standard tableau B and a number x , determined as follows:

- (1) Define the sequence k_p, k_{p-1}, \dots, k_1 by setting $k_p = s_p$ and choosing k_i for $i < p$ to be the largest integer $\leq s_i$ such that $a_{ik_i} \leq a_{i+1, k_{i+1}}$.
- (2) Define B to be the standard tableau obtained from A by
 - removing a_{ps_p} from the p th row, and

- replacing the entry a_{ik_i} of the i th row by $a_{i+1,k_{i+1}}$, $i = 1, \dots, p-1$
- (3) Set $x = a_{1k_1}$.

The reader should check that B is again a standard tableau. It has the same shape as A , except that its row p is shorter by one entry. Deletion has an inverse:

Definition 4.2. *Insertion* takes a standard tableau $A = (a_{ij})$, say of shape (s_1, s_2, \dots) , and an integer x , and constructs from them a standard tableau B and an index p determined as follows:

- (1) Set $i = 1$ and $B = A$.
- (2) If $s_i = 0$ or $x > a_{is_i}$, then add x at the end of the i th row of B , set $p = i$ and terminate.
- (3) Otherwise let k_i be the smallest j such that $x \leq a_{js_i}$, replace $b_{k_i s_i}$ with x , set $x = a_{k_i s_i}$ and $i = i + 1$. Then go to (2).

Again it is easily checked that B is a standard tableau whose shape coincides with that of A , except that the row p of B is longer by one entry.

Deletion and Insertion are clearly inverse to each other: if Deletion applied to input (A, p) gives output (B, x) , then Insertion applied to (B, x) gives output (A, p) and viceversa.

The *Knuth–Robinson–Schensted correspondence*, KRS for short, is at first defined as a bijective correspondence between the set of the standard bitableaux (as combinatorial objects) and the set of the two-line arrays of a certain type. The two-line array $\text{KRS}(\Sigma)$ is constructed from the standard bitableau Σ by an iteration of the following KRS-step:

Definition 4.3. Let $\Sigma = (A|B) = (a_{ij}|b_{ij})$ be a non-empty standard bitableau. Then *KRS-step* constructs a pair of integers (ℓ, r) and a standard bitableau Σ' as follows.

- (1) Choose the largest entry ℓ in the *left* tableau of Σ ; suppose that $\{(i_1, j_1), \dots, (i_u, j_u)\}$, $i_1 < \dots < i_u$, is the set of indices (i, j) such that $\ell = a_{ij}$. Set $p = i_u$ and $q = j_u$. (We call (p, q) the *pivot position*.)
- (2) Let A' be the standard tableau obtained by removing a_{pq} from A .
- (3) Apply Deletion to the pair (B, p) . The output is a standard tableau B' and an element r .
- (4) set $\Sigma' = (A', B')$.

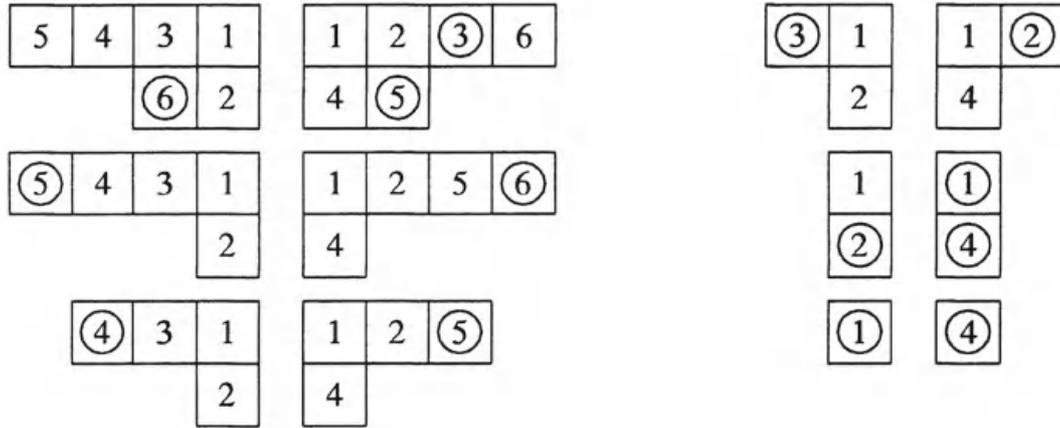
Now $\text{KRS}(\Sigma)$ is constructed from the outputs of a sequence of KRS-steps:

Definition 4.4. Let Σ be a non-empty standard bitableau of shape s_1, s_2, \dots, s_p . Set $k = s_1 + \dots + s_p$ and define the two-line array

$$\text{KRS}(\Sigma) = \begin{pmatrix} \ell_1 & \ell_2 & \dots & \ell_{k-1} & \ell_k \\ r_1 & r_2 & \dots & r_{k-1} & r_k \end{pmatrix}$$

as follows. Starting from $\Sigma_k = \Sigma$, the KRS-step 4.3, applied to Σ_i for $i = k, k - 1, \dots, 1$, produces the bitableau Σ_{i-1} and the pair (ℓ_i, r_i) .

We give an example in Figure 2. The circles in the left tableau mark the pivot position, those in the right mark the chains of “bumps” given in 4.1(2):



$$\text{KRS}(\Sigma) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 2 & 5 & 6 & 3 \end{pmatrix} \longleftrightarrow X_{14}X_{21}X_{32}X_{45}X_{56}X_{63}.$$

Figure 2. The KRS algorithm

The two-line array $\text{KRS}(\Sigma)$ has the following properties:

- (a) $\ell_i \leq \ell_{i+1}$ for all i ,
- (b) if $\ell_i = \ell_{i+1}$ then $r_i \geq r_{i+1}$.

Property (a) is clear since the algorithm chooses $\ell_{i+1} \geq \ell_i$. If $\ell_i = \ell_{i+1}$ then the pivot position of the $(i+1)$ th deletion step lies left of (or above) the pivot position of the i th deletion step. Now it is easy to see that the element pushed out by the $(i+1)$ th step is not larger than that pushed out by the i th step.

KRS gives a correspondence between standard bitableaux and two-line arrays with properties (a) and (b). It is bijective since it has an inverse. For the inversion of KRS one just applies the Insertion algorithm to the bottom line of the array to build the right tableau: at step i it inserts r_i in the tableau obtained after the $(i-1)$ th step. Simultaneously one accumulates the left tableau by placing the element ℓ_i in the position which is added to the right tableau by the i th insertion.

It remains to explain how we can interpret any two-line array satisfying (a) and (b) as a monomial: we associate the monomial

$$X_{\ell_1 r_1} X_{\ell_2 r_2} \cdots X_{\ell_k r_k},$$

to it, clearly establishing the desired bijection. To sum up, we have constructed a bijective correspondence between standard bitableaux and monomials. If we

restrict our attention to standard bitableaux and monomials where the entries and the indeterminates come from an $m \times n$ matrix, we get

Theorem 4.5. *The map KRS is a bijection between the set of standard bitableaux on $\{1, \dots, m\} \times \{1, \dots, n\}$ and the monomials of $K[X]$.*

This theorem proves the second half of the straightening law: the KRS correspondence says that in every degree d there are as many standard bitableaux as monomials. Since we know already that the standard bitableaux generate the space of homogeneous polynomials of degree d , we may conclude that they must be linearly independent.

Remark 4.6. In the fundamental paper [50] Knuth extensively treats the KRS correspondence for *column standard* bitableaux with increasing columns and non-decreasing rows. (Deletion still bumps the entries row-wise, but the condition $a_{ik_i} \leq a_{i+1, k_{i+1}}$ in 4.1(1) must be replaced by $a_{ik_i} < a_{i+1, k_{i+1}}$). The same point of view is taken in Fulton [36], Knuth [51], Sagan [63] and Stanley [69]. The version we are using is the “dual” one (see [50, Section 5 and p. 724]). The notes to Chapter 7 of [69] contain a detailed historical discussion of the correspondence.

Below we will consider decompositions of sequences into increasing and non-increasing sequences. For column standard bitableaux these attributes must be exchanged.

We have just seen that KRS is a bijection between two bases of the same vector space. Therefore we can *extend KRS to a K -linear automorphism of $K[X]$* . The automorphism KRS does not only preserve the total degree, but even the $\mathbb{Z}^m \oplus \mathbb{Z}^n$ degree introduced above: in fact, no column or row index gets lost. However, note that KRS is not a K -algebra isomorphism: it acts as the identity on polynomials of degree 1 but it is not the identity map. It would be interesting to have some insight in the properties of KRS as a linear map, like, for instance, its eigenvalues and eigenspaces.

Remark 4.7. We note some important properties of KRS:

(a) KRS commutes with transposition of the matrix X : Let X' be a $n \times m$ matrix of indeterminates, and let $\tau : K[X] \rightarrow K[X']$ denote the K -algebra isomorphism induced by the substitution $X_{ij} \mapsto X'_{ji}$; then $\text{KRS}(\tau(f)) = \tau(\text{KRS}(f))$ for all $f \in K[X]$. Note that it suffices to prove the equality when f is a standard bitableau. Herzog and Trung [44, Lemma 1.1] point out how to translate Knuth’s argument from column to row standard tableaux.

(b) All the powers Σ^k of a standard bitableau are again standard, and one has

$$\text{KRS}(\Sigma^k) = \text{KRS}(\Sigma)^k.$$

This is not hard to check: k successive deletion steps on Σ^k act like a single deletion step on k copies of Σ .

(c) If Σ is a minor $[a_1 a_2 \dots a_t | b_1 b_2 \dots b_t]$ then $\text{KRS}(\Sigma)$ is just (the product of the elements on) the main diagonal of Σ . More generally, if one the two tableaux of Σ is “nested”, i.e. the set of entries in each row contains the entries in the next row, then $\text{KRS}(\Sigma)$ is the product of the main diagonals of Σ . (This is easy to see if the right tableau is nested; one uses (a) for transposition.)

Note, however, that in general $\text{KRS}(\Sigma)$ need not to be one of the monomials which appear in the expansion of Σ . In other words, KRS does not simply select a monomial of the polynomial Σ ; there is no algebraic relation between $\text{KRS}(\Sigma)$ and the monomials appearing in Σ .

In the application of KRS to Gröbner bases of determinantal ideals it will be important to relate the shape of a standard bitableau Σ to “shape” invariants of $\text{KRS}(\Sigma)$. In the KRS correspondence the right tableau, and hence its shape, is determined solely by the bottom line of the corresponding two-line array. Therefore we are lead to the following problem: Let $r = r_1, r_2, \dots, r_k$ be a sequence of integers and let $\text{Ins}(r)$ be the standard tableau determined by the iterated insertions of the r_i . What is the relationship between the shape of $\text{Ins}(r)$ and the sequence r ?

A subsequence v of r is determined by a subset U of $\{1, \dots, k\}$: if $U = \{i_1, \dots, i_t\}$ with $i_1 < \dots < i_t$ then $v = v(U, r) = r_{i_1}, \dots, r_{i_t}$. The *length* of v is simply the cardinality of U . A first answer to the question above was given by Schensted and had indeed been a motivation of his studies.

Theorem 4.8 (Schensted). *The length of the first row of $\text{Ins}(r)$ is the length of the longest increasing subsequence of r .*

Does the length of the i th row for $i > 1$ have a similar meaning? Actually, these lengths cannot be interpreted individually, but some of their combinations reflect properties of the decompositions of the sequence r into increasing subsequences. This is the content of Greene’s extension of Schensted theorem, which we will now explain. A decomposition of r into subsequences corresponds to a partition $U = (U_1, U_2, \dots, U_s)$ of the set $\{1, \dots, k\}$. The shape of the decomposition is $(|U_1|, |U_2|, \dots, |U_s|)$: we always assume that $|U_1| \geq |U_2| \geq \dots \geq |U_s|$. An *inc-decomposition* of r is given by a partition (U_1, U_2, \dots, U_s) of the set $\{1, \dots, k\}$ such that the associated subsequences are increasing.

In Section 1 we have defined the functions α_k , namely $\alpha_k(\lambda) = \sum_{i=1}^k \lambda_i$. Now We introduce a variant $\hat{\alpha}_k$ for sequences of integers r , setting

$$\hat{\alpha}_k(r) = \max\{\alpha_k(\lambda) : r \text{ has a inc-decomposition of shape } \lambda\}.$$

Similarly we set

$$\alpha_k(P) = \alpha_k(\lambda) \quad \text{and} \quad \alpha_k(U) = \alpha_k(\lambda)$$

for every tableau P and every inc-decomposition U of shape λ .

Inc-decompositions are crucial for us since they describe realizations of a monomial as an initial monomial of a product of minors. This will be made more precise in the next section. However, in Section 7 it will turn out useful to have also a measure for decompositions into *non-increasing subsequences*. For a shape $\lambda = (s_1, \dots, s_t)$ we define λ^* to be the *dual* shape: the i th component of $\lambda^* = (s_1^*, \dots, s_{s_1}^*)$ counts the number of boxes in the i th column of a tableau of shape σ ; formally

$$s_i^* = |\{k : s_k \geq i\}|.$$

The functions α_k are dualized to

$$\alpha_k^*(\lambda) = \alpha_k(\lambda^*) \quad \text{and} \quad \alpha_k^*(P) = \alpha_k^*(\lambda)$$

if P is a tableau of shape λ . Analogously one defines $\alpha_k^*(U)$ for a non-inc-decomposition. However, the passage from inc-decompositions to non-inc-decompositions contains already a dualization, and we set

$$\widehat{\alpha}_k^*(r) = \max\{\alpha_k(\lambda) : r \text{ has a non-inc-decomposition of shape } \lambda\}.$$

We can rephrase the definition of $\widehat{\alpha}_k$ and $\widehat{\alpha}_k^*$ for sequences r as follows: $\widehat{\alpha}_k(r)$ (respectively, $\widehat{\alpha}_k^*(r)$) is the length of the longest subsequence of r that can be decomposed into k increasing (non-increasing) subsequences.

Theorem 4.9 (Greene). *For every sequence of integers r and every $k \geq 0$ we have*

$$(a) \quad \widehat{\alpha}_k(r) = \alpha_k(\text{Ins}(r)), \quad (b) \quad \widehat{\alpha}_k^*(r) = \alpha_k^*(\text{Ins}(r)).$$

For $k = 1$ one obtains Schensted's theorem from (a). (Schensted also proved (b) for $k = 1$.) The proof of Greene's theorem is based on Knuth's basic relations: two sequences of integers r and s differ by a *Knuth relation* if one of the following conditions holds:

- (1) $r = x_1, \dots, x_i, y, z, w, x_{i+4}, \dots, x_k$ and
 $s = x_1, \dots, x_i, y, w, z, x_{i+4}, \dots, x_k$,
with $z \leq y < w$;
- (2) $r = x_1, \dots, x_i, z, w, y, x_{i+4}, \dots, x_k$ and
 $s = x_1, \dots, x_i, w, z, y, x_{i+4}, \dots, x_k$,
with $z < y \leq w$.

For every standard tableau P there is a *canonical sequence* r_P associated with P such that $\text{Ins}(r_P) = P$, defined as follows: r_P is obtained by listing the rows of P from bottom to top, i.e.

$$r_P = P_{t1}P_{t2} \cdots P_{ts_t}P_{t-11}P_{t-12} \cdots P_{t-1s_{t-1}} \cdots P_{11}P_{12} \cdots P_{1s_1};$$

here (s_1, \dots, s_t) is the shape of P .

Theorem 4.10 (Knuth). *Let r and s be sequences. Then $\text{Ins}(r) = \text{Ins}(s)$ iff r is obtained from s by a sequence of Knuth relations. In particular, the canonical sequence associated with $\text{Ins}(r)$ is obtained from r by a sequence of Knuth relations.*

A detailed proof of the theorem for column standard tableaux is contained in [50]. On [50, p. 724] one finds an explanation how to modify the Knuth relations and statements for row standard tableaux.

Now Greene's theorem is proved as follows: for (a) one shows that

- (i) $\alpha_k(P) = \widehat{\alpha}_k(r_P)$ for the canonical sequence r_P associated to a standard tableau P , and
- (ii) $\widehat{\alpha}_k$ has the same value on sequences that differ by a Knuth relation.

The same scheme works for (b). We confine ourselves to (a), leaving (b) to the reader.

For the proof of (i) let $P = (p_{ij})$ and $\sigma = (s_1, \dots, s_u)$ be the shape of P . First of all, r_P has the inc-decomposition p_{i1}, \dots, p_{is_i} , $i = 1, \dots, r$. The decomposition has the same shape as P . Hence $\alpha_k(P) \leq \widehat{\alpha}_k(r_P)$. On the other hand, note that the columns of P partition r_P into non-increasing subsequences. Therefore an increasing subsequence can contain at most one element from each column, and the total number of elements in a disjoint union of k increasing subsequences is at most $\sum_{i=1}^{s_1} \min(k, s_i^*) = \alpha_k(P)$.

For the proof of (ii) we consider sequences a and b of integers that differ by a Knuth relation. In order to prove that $\widehat{\alpha}_k(a) = \widehat{\alpha}_k(b)$ it suffices to show that for every inc-decomposition G of a there exists an inc-decomposition H of b such that $\alpha_k(G) \leq \alpha_k(H)$. So let G be an inc-decomposition of a and let z, w and y as in the definition of the Knuth relations. If z and w belong to distinct subsequences in the decomposition G , then the increasing subsequences of G are not affected by the Knuth relation, and we may take H equal to G . (Strictly speaking, we must change the partition of the index set underlying the sequences by exchanging the positions of z and w .)

It remains the case in which z and w belong to the same increasing subsequence. Then a must play the role of r , and y must belong to another subsequence in its decomposition G . Let $u = p_1, z, w, p_2$ and $v = p_3, y, p_4$ denote those subsequences in g that contain z, w and y . Here the p_i are increasing subsequences of a . Assume first the Knuth relation is of type (1). We can rearrange the elements of the sequences u and v into increasing subsequences of b in three ways:

$$(I) \begin{cases} u' = p_1, z, p_4 \\ v' = p_3, y, w, p_2 \end{cases} \quad (II) \begin{cases} u' = p_1, w, p_2 \\ v' = p_3, y, p_4 \\ w' = z \end{cases} \quad (III) \begin{cases} u' = p_1, y, w, p_2 \\ v' = p_3, p_4 \\ w' = z \end{cases}$$

Suppose that both u and v contribute to $\alpha_k(G)$. Then we replace u and v by the sequences u' and v' defined in (I), obtaining an inc-decomposition H of b

with $\alpha_k(H) \geq \alpha_k(G)$: in fact, H contains k subsequences whose lengths sum up to $\alpha_k(G)$.

If u does not contribute to $\alpha_k(G)$, then we replace u and v by the sequences u', v', w' in (II). The inc-decomposition H of b consisting of the remaining subsequences of a and u', v', w' has $\alpha_k(H) = \alpha_k(G)$.

Now suppose that u contributes to $\alpha_k(G)$, but v does not. Then we replace u and v by the three sequences defined in (III), with the same result as in the previous case.

The dual argument works in the case of a Knuth relation of type (2). It completes the proof of Greene's theorem.

Another series of very useful functions is given by the γ_t introduced in Section 2, $\gamma_t(\lambda) = \sum_i \max(\lambda_i - t + 1, 0)$, where t is a non-negative integer and λ is a shape. We extend the γ_t to sequences in the same way as the α_k :

$$\widehat{\gamma}_t(r) = \max\{\gamma_t(\lambda) : r \text{ has an inc-decomposition of shape } \lambda\}.$$

Like the α_k , the γ_t are invariant under KRS:

Theorem 4.11. *For every sequence of integers r we have $\widehat{\gamma}_t(r) = \gamma_t(\text{Ins}(r))$.*

One can prove Theorem 4.11 by arguments completely analogous with those leading to Greene's theorem. This approach has been chosen in [9]. Alternatively one can use the following lemma:

Lemma 4.12. *For each shape λ the following holds:*

$$\gamma_t(\lambda) \geq u \iff \alpha_k(\lambda) \geq (t-1)k + u \text{ for some } k, 1 \leq k \leq t.$$

We leave the easy proof to the reader, as well as the dual version of Theorem 4.11.

Remark 4.13. In spite of the Theorems 4.9 and 4.11: in general there does not exist an inc-decomposition of a sequence r with the same shape as $\text{Ins}(r)$. The sequence 4, 1, 2, 5, 6, 3 as in Figure 2 has no inc-decomposition of shape (4, 2) but the shapes (4, 1, 1) and (3, 3) occur, and this is enough for the invariance of the functions α_k and γ_t .

5. KRS and Gröbner bases of ideals

Once and for all we now introduce a *diagonal term order* on the polynomial ring $K[X]$. With respect to such a term order, the initial monomial $\text{in}(\delta)$ is the product of the elements on the main diagonal of δ (for brevity we call this monomial the main diagonal). There are various choices for a diagonal term order. For example, one can take the lexicographic order induced by the total order of the X_{ij} that coincides with the lexicographic order of the positions (i, j) .

Remark 5.1. It is not hard to show that distinct standard bitableaux Σ of maximal minors have distinct initial monomials with respect to a diagonal term order on $K[X]$. This proves the linear independence of these standard bitableaux by 3.4(d). Also KRS is “trivial” for such Σ , since $\text{KRS}(\Sigma) = \text{in}(\Sigma)$. See 4.7(c) for a more general statement.

The power of KRS in the study of Gröbner bases for determinantal ideals was detected by Sturmfels [66]. His simple, but fundamental observation is the following. Assume that I is an ideal of $K[X]$ which has a K -basis of standard bitableaux B . Then $\text{KRS}(I)$ is a vector space of $K[X]$ that has two of the properties of an initial ideal: it has a basis of monomials and it has the same Hilbert function as I . Can we conclude that $\text{KRS}(I)$ is the initial ideal of I ? Not in general, but if $\text{KRS}(I) \subseteq \text{in}(I)$ or the other way round, then equality is forced by the Hilbert function. This argument yields

Lemma 5.2. (a) *Let I be an ideal of $K[X]$ which has a K -basis, say B , of standard bitableaux, and let S be a subset of I . Assume that for all $\Sigma \in B$ there exists $s \in S$ such that $\text{in}(s) \mid \text{KRS}(\Sigma)$. Then S is a Gröbner basis of I and $\text{in}(I) = \text{KRS}(I)$.*
(b) *Let I and J be homogeneous ideals such that $\text{in}(I) = \text{KRS}(I)$ and $\text{in}(J) = \text{KRS}(J)$. Then $\text{in}(I) + \text{in}(J) = \text{in}(I + J) = \text{KRS}(I + J)$ and $\text{in}(I) \cap \text{in}(J) = \text{in}(I \cap J) = \text{KRS}(I \cap J)$.*

Proof. (a) Let J be the ideal generated by the monomials $\text{in}(s)$ with $s \in S$. The hypothesis implies that $\text{KRS}(I) \subseteq J \subseteq \text{in}(I)$. Since the first and the third term have the same Hilbert function it follows that $\text{KRS}(I) = J = \text{in}(I)$. For (b) one uses

$$\begin{aligned} \text{KRS}(I + J) &= \text{KRS}(I) + \text{KRS}(J) = \text{in}(I) + \text{in}(J) \subseteq \text{in}(I + J), \\ \text{KRS}(I \cap J) &= \text{KRS}(I) \cap \text{KRS}(J) = \text{in}(I) \cap \text{in}(J) \supseteq \text{in}(I \cap J), \end{aligned}$$

and concludes equality from the Hilbert function argument. \square

Sturmfels applied Schensted’s theorem to prove

Theorem 5.3. *The t -minors of X form a Gröbner basis of I_t , and $\text{KRS}(I_t) = \text{in}(I_t)$.*

Proof. The set B of standard bitableaux whose first row has length $\geq t$ is a K -basis of I_t . Let S be the set of the minors of size t . By 5.2, it is enough to show that for every $\Sigma \in B$ there exists $\delta \in S$ such that $\text{in}(\delta) \mid \text{KRS}(\Sigma)$. Let ℓ and r be the top and bottom vector of the KRS image of Σ so that $\text{KRS}(\Sigma) = \prod X_{\ell_i r_i}$. By Schensted’s theorem 4.8 we can find an increasing subsequence r of length t , say $r_{i_1} < r_{i_2} < \dots < r_{i_t}$ with $i_1 < i_2 < \dots < i_t$. But then $\ell_{i_1} < \ell_{i_2} < \dots < \ell_{i_t}$ follows from property (b) of the KRS-image. In other words, the factor $\prod X_{\ell_{i_j} r_{i_j}}$ of $\text{KRS}(\Sigma)$ is the main diagonal and hence the initial monomial of a t -minor. \square

Remark 5.4. That the t -minors of X form a Gröbner basis of I_t has been proved by several authors. To the best of our knowledge, the result was first published by Narasimhan [61]. Independently, a proof was given by Caniglia, Guccione, and Guccione [18]. The result was re-proved by Ma [59].

The ideal of maximal minors has better properties than the I_t in general, not only in regard to its primary decomposition (see Corollary 2.3), but also in regard to the Gröbner basis: the maximal minors form a universal Gröbner basis, i.e. a Gröbner basis for every term order on $K[X]$. This difficult result was proved by Bernstein and Zelevinsky [6].

Also in the case $t = 2$ a universal Gröbner basis of I_t is known. It consists of binomials; see Sturmfels [67].

We fix an important observation, already used in the proof above:

Remark 5.5. Let Σ is a standard bitableau, ℓ and r be the top and bottom vector of the KRS image of Σ . Then any decomposition of the monomial $\text{KRS}(\Sigma)$ into product of main diagonals corresponds to a decomposition of r into increasing subsequences.

Consequently we extend the definition of $\widehat{\alpha}_k$ and $\widehat{\gamma}_t$ to monomials by setting

$$\widehat{\alpha}_k(M) = \widehat{\alpha}_k(r) \quad \text{and} \quad \widehat{\gamma}_t(M) = \widehat{\gamma}_t(r)$$

where r denotes the bottom row in the two line array representing the monomial M , as discussed between Definition 4.4 and Theorem 4.5.

Lemma 5.2 leads us to introduce the following definition:

Definition 5.6. Let I be an ideal with a basis of standard bitableaux. Then we say that I is *in-KRS* if $\text{in}(I) = \text{KRS}(I)$; if, in addition, the bitableaux (standard or not) $\Delta \in I$ form a Gröbner basis, then I is *G-KRS*. In slightly different words, an ideal I with a basis of standard bitableaux is *in-KRS* if for each $\Sigma \in I$ there exists $x \in I$ with $\text{KRS}(\Sigma) = \text{in}(x)$; it is *G-KRS*, if x can always be chosen as a bitableau.

As a consequence of Lemma 5.2 one obtains

Lemma 5.7. *Let I and J be ideals with a basis of standard bitableaux.*

- (a) *If I and J are G-KRS, then $I + J$ is also G-KRS.*
- (b) *If I and J are in-KRS, then $I + J$ and $I \cap J$ are also in-KRS.*

In general the property of being G-KRS is not inherited by intersections as we will see below.

Now we are in the position to use the information we have accumulated on determinantal ideals and on the KRS map to describe Gröbner bases and/or initial ideals of powers, products and symbolic powers of determinantal ideals.

Theorem 5.8. *For every $k \in \mathbb{N}$ the symbolic power $I_t^{(k)}$ of I_t is a G-KRS ideal. Its initial ideal is generated, as a vector space, by the monomials M with $\widehat{\gamma}_t(M) \geq k$. In particular, a Gröbner basis of $I_t^{(k)}$ is given by the set of bitableaux Σ with $\gamma_t(\Sigma) = k$ and no factor of size $< t$.*

Proof. Let S_1 be the set of the products of minors Δ with $\gamma_t(\Delta) \geq k$. One has $S_1 \subseteq I_t^{(k)}$. By virtue of Theorem 4.11 and Remark 5.5 we know that for all standard bitableau Σ with $\gamma_t(\Sigma) \geq k$ there exists Δ in S_1 with $\text{in}(\Delta) \mid \text{KRS}(\Sigma)$. Thus it follows from 5.2 that S_1 is a Gröbner basis of $I_t^{(k)}$ and $I_t^{(k)}$ is G-KRS.

It remains to show that the initial term of any product of minors Δ with $\gamma_t(\Delta) \geq k$ is divisible by the initial term of a product of minors Δ_1 without factors of size $< t$ and with $\gamma_t(\Delta_1) = k$. If Δ has factors of size $< t$, we simply get rid of them. If $\gamma_t(\Delta) > k$, then we cancel $\gamma_t(\Delta) - k$ boxes in the bitableau with the corresponding entries. In this way we get Δ_1 . \square

Another important consequence of Lemma 5.7 and Theorem 2.4 is:

Theorem 5.9. *Let t_1, \dots, t_r be positive integers and set $I = I_{t_1} \cdots I_{t_r}$ and $g_i = \gamma_i(t_1, \dots, t_r)$. If $\text{char } K = 0$ or $\text{char } K > \min(t_i, m - t_i, n - t_i)$ for all i , then I is in-KRS and $\text{in}(I)$ is generated, as a K -vector space, by the monomials M with $\widehat{\gamma}_i(M) \geq g_i$ for all i .*

Theorem 5.9 is satisfactory if one only wants to determine the initial ideal of the product $I_{t_1} \cdots I_{t_r}$, but it does not tell us how to find a Gröbner basis. A natural guess is that any such ideal is G-KRS, i.e. a Gröbner basis of $I_{t_1} \cdots I_{t_r}$ is given by the products of minors (standard or not) which are in $I_{t_1} \cdots I_{t_r}$. Unfortunately this is wrong in general.

Example 5.10. Suppose that $m \geq 4$ and $\text{char } K = 0$ or > 3 , and consider the ideal $I_4 I_2$. The monomial $M = X_{11} X_{13} X_{22} X_{34} X_{43} X_{45}$ has $\widehat{\gamma}_4(M) = 1$, $\widehat{\gamma}_3(M) = 2$, $\widehat{\gamma}_2(M) = 4$, $\widehat{\gamma}_1(M) = 6$. (We have seen a similar example already in Remark 4.13.) Hence, by virtue of 5.9, we know that $M \in \text{in}(I_4 I_2)$. The products of minors of degree 6 in $I_4 I_2$ have the shapes 6 or (5, 1), or (4, 2). Clearly M is not the initial monomial of a product of minors of shape 6 or of shape 5, 1. The only initial monomial of a 4-minor that divides M is $X_{11} X_{22} X_{34} X_{45}$ but the remaining factor $X_{13} X_{43}$ is not the initial monomial of a 2-minor. Hence M is not the initial monomial of a product of minors that belongs to $I_4 I_2$.

Nevertheless, if we confine our attention to powers of determinantal ideals, the result is optimal.

Theorem 5.11. *Suppose that $\text{char } K = 0$ or $\text{char } K > \min(t, m - t, n - t)$. Then I_t^k is G-KRS and $\text{in}(I_t^k)$ is generated, as a K -vector space, by the monomials M with $\widehat{\alpha}_k(M) \geq kt$. In particular, a Gröbner basis of I_t^k is given by the products of minors*

Δ such that Δ has at most k factors, $\alpha_k(\Delta) = kt$, and $\deg \Delta = kt$. Therefore I_t^k has a minimal system of generators which is a Gröbner basis.

Proof. Let S_1 be the set of the products of minors Δ with $\alpha_k(\Delta) \geq kt$. By Proposition 2.7 we know that $S_1 \subseteq I_t^k$. Greene's theorem 4.9 and Remark 5.5 imply that for all standard bitableau Δ with $\alpha_k(\Delta) \geq kt$ there exists Δ in S_1 with $\text{in}(\Delta) \mid \text{KRS}(\Sigma)$. Thus it follows from Theorem 5.2 that S_1 is a Gröbner basis of I_t^k and I_t^k is G-KRS.

It remains to show that for every product of minors Δ with $\alpha_k(\Delta) \geq kt$ there exists a product of minors Δ_1 with at most k factors, degree kt and $\alpha_k(\Delta_1) = kt$ such that $\text{in}(\Delta_1) \mid \text{in}(\Delta)$. This step is as easy as the corresponding one in the proof of Theorem 5.8: Δ_1 is obtained from Δ by skipping the rows of index $> k$ (if any) and deleting $\alpha_k(\Delta) - kt$ boxes from the first k rows (in any way). \square

Remark 5.12. (a) We can obviously generalize 5.9 and 5.11 as follows: Let $c_1, \dots, c_t \in \mathbb{N}$, $t = \min(m, n)$, and V be the vector space spanned defined by all standard monomials Σ with $\gamma_i(\Sigma) \geq c_i$ (or $\alpha_i(\Sigma) \geq c_i$) for all i ; then V is an ideal and in-KRS. In fact, each of the inequalities defines a G-KRS ideal in $K[X]$. We can even intersect V with a homogeneous component of $K[X]$ (with respect to the total degree or the $\mathbb{Z}^m \oplus \mathbb{Z}^n$ -grading) to obtain an in-KRS vector space.

(b) In [11] we have further analyzed the properties of being G-KRS or in-KRS. If $\text{char} K = 0$ or $> \min(m, n)$, then all ideals I defined by shape have a standard monomial basis and are in-KRS: that I is defined by shape means that it is generated by products of minors and, for such a product Δ , it depends only on $|\Delta|$ whether Δ belongs to I .

Furthermore, an ideal defined by shape is G-KRS exactly if is the sum of ideals $J(k, d) \cap I_1^d$ and, if $m = n$, $(J(k, d) \cap I_1^d) I_m^v$ where the $J(k, d)$ play the same role for the α -functions as the symbolic powers do for the γ -functions: $J(k, d)$ is generated by all bitableaux Δ with $\alpha_k(\Delta) \geq d$.

Remark 5.13. Let T be a new indeterminate, and consider the polynomial ring $R' = K[X, T]$ where X , as usually, is an $m \times n$ matrix of indeterminates with $m \leq n$. The KRS-invariance of the functions α_k has found another application to the ideal

$$J = I_m + I_{m-1}T + \dots + I_1T^{m-1} + (T^m).$$

and its powers; see Bruns and Kwieciński [14]. With the results accumulated so far, the reader can easily show that

$$J^k = R' \left(\sum_{d=0}^{km} J(k, d) T^{km-d} \right).$$

Let us extend the diagonal term order from $K[X]$ to R' by first comparing total degrees and, in the case of equal total degree, the X -factors of the monomials. It

follows that $\text{in}(J^k) = \text{in}(J)^k$ and J^k has a Gröbner basis of products ΔT^{km-d} where Δ is a bitableaux of total degree d such that $\alpha_k(\Delta) \geq d$. The technique by which we explore the Rees algebra of the ideal I_t in Section 7 can also be applied to the Rees algebra of J ; see [14].

6. Cohen-Macaulayness and Hilbert series of determinantal rings

Hochster and Eagon [47] proved that the determinantal ring $K[X]/I_t$ is Cohen-Macaulay. Their proof is based on the notion of principal radical system; for this and several other approaches see [17]. Abhyankar [1] presented a formula for the Hilbert function of $K[X]/I_t$ obtained by enumerating the standard bitableaux in the standard basis of $K[X]/I_t$.

The goal of this section is to show how these results can be proved by Gröbner deformation, i.e. by the study of the ring $K[X]/\text{in}(I_t)$. By 5.3 we know that $\text{in}(I_t)$ is a square-free monomial ideal. There are special techniques available for the study of such ideals. We briefly recall the main properties and notions to be used; for more details we refer the reader to [13, Chapter 5].

A simplicial complex on a set of vertices $V = \{1, \dots, n\}$ is a set Δ of subsets F of V such that $G \in \Delta$ whenever $G \subseteq F$ and $F \in \Delta$. To any square-free monomial ideal I in a polynomial ring $R = K[X_1, \dots, X_n]$ one can associate the (abstract) simplicial complex

$$\Delta = \{F \subseteq \{1, \dots, n\} : X_F \notin I\}$$

where $X_F = \prod_{i \in F} X_i$. Conversely, to any simplicial complex Δ on the vertices $\{1, \dots, n\}$ one associates a square-free monomial ideal I by setting

$$I = (X_F : F \notin \Delta).$$

The ring $K[\Delta] = K[X]/I$ is called the *Stanley-Reisner ring* associated to Δ . One can study the homological properties and the numerical invariants of $K[\Delta]$ by analyzing the combinatorial properties and invariants of Δ . An element F of Δ is called a *face*; its cardinality is denoted by $|F|$. The *dimension* of F is $|F| - 1$ and the dimension of Δ is $\max\{\dim F \mid F \in \Delta\}$. By $\mathbf{F}(\Delta)$ we denote the set of the facets of Δ , i.e. the maximal elements of Δ under inclusion. Then Δ is said to be *pure* if every facet has maximal dimension, in other words, if $\dim F = \dim \Delta$ for all $F \in \mathbf{F}(\Delta)$.

Lemma 6.1. *The Krull dimension of $K[\Delta]$ is $\dim \Delta + 1$, and the multiplicity of $K[\Delta]$ equals the number of facets of maximal dimension of Δ .*

Proof. This follows from the fact that the defining ideal of $K[\Delta]$ is radical and its minimal primes are of the form $(X_i : i \notin F)$ where F is a facet of Δ . \square

Definition 6.2. A simplicial complex Δ is said to be *shellable* if it is pure and if its facets can be given a total order, say F_1, F_2, \dots, F_e , so that the following condition holds: for all i and j with $1 \leq j < i \leq e$ there exist $v \in F_i \setminus F_j$ and an index k , $1 \leq k < i$, such that $F_i \setminus F_k = \{v\}$. A total order of the facets satisfying this condition is called a *shelling* of Δ .

Shellability is a strong property, very well suited for inductive arguments. Suppose that F_1, \dots, F_e is a shelling of a simplicial complex Δ , let Δ_i denote the smallest simplicial complex containing F_1, \dots, F_i , and Δ_i^* the smallest simplicial complex containing $F_j \cap F_i$ for all $j < i$. Then Δ_i is obviously shellable and one has a short exact sequence

$$0 \rightarrow K[\Delta_i] \rightarrow K[\Delta_{i-1}] \oplus K[F_i] \rightarrow K[\Delta_i^*] \rightarrow 0$$

where $K[F_i]$ is the Stanley-Reisner ring of the simplex defined by F_i , i.e. $K[F_i]$ is the polynomial ring on the set of vertices of F_i . The fact that the given order of the facets is a shelling translates immediately into an algebraic property: $K[\Delta_i^*]$ is defined by a single monomial. Its degree is the cardinality of the set

$$c(F_i) = \{v \in F_i : \text{there exists } k < i \text{ such that } F_i \setminus F_k = \{v\}\}.$$

This implies

Theorem 6.3. *Let Δ be a shellable simplicial complex of dimension $d - 1$ with shelling F_1, \dots, F_e . Then:*

- (a) *The Stanley-Reisner ring $K[\Delta]$ is Cohen-Macaulay.*
- (b) *The Hilbert series of $K[\Delta]$ has the form $h(z)/(1-z)^d$ with $h(z) = \sum h_j z^j \in \mathbb{Z}[x]$, $h_0 = 1$ and $h_j = |\{i \in \{1, \dots, e\} : |c(F_i)| = j\}|$.*

Proof. In view of the exact sequence above, one easily proves both statements by induction on e , using the behavior of Cohen-Macaulayness and Hilbert series along short exact sequences. \square

Let us return to determinantal rings. As pointed out, the ideal $\text{in}(I_t)$ is generated by square free monomials, namely the main diagonal monomials of the t -minors of X . The corresponding simplicial complex Δ_t consists of all the subsets of

$$V = \{1, \dots, m\} \times \{1, \dots, n\}$$

that do not contain a t -diagonal. (Note that the usual notation of matrix positions differs from Cartesian coordinates by a 90° rotation!) The facets of Δ_t can be described in terms of family of non-intersecting paths. To do this, we give V a poset structure (certainly not the most natural one). We set

$$(i, j) \leq (h, k) \iff i \leq h \text{ and } j \geq k.$$

A subset A of V is said to be a *chain* if each two elements of A are comparable in the poset V , and it is an *antichain* if it does not contain a pair of comparable elements. It is easy to see that an antichain with t elements corresponds to the main diagonal of a t -minor. For $t = 2$ the simplicial complex coincides with the *order* (or *chain*) complex of V : its faces are the chains and its facets are the maximal chains of the poset V . For general t the simplicial complex Δ_t is the set of those subsets of V which do not contain antichains of t elements; it is called the *step t order complex* of V .

A maximal chain of V can be described as a path in V . A *path* P in V from point A to point B , with $A \leq B$, is, by definition, an unrefinable chain with minimum A and maximum B . It can be written as a sequence

$$P: A = (a_1, b_1), (a_2, b_2), \dots, (a_d, b_d) = B$$

where

$$(a_{i+1}, b_{i+1}) - (a_i, b_i) = (1, 0) \text{ or } (0, -1) \quad \text{for all } i.$$

A point (a_k, b_k) is said to be a *right-turn* of the path P if $1 < k < d$ and

$$(a_{k+1}, b_{k+1}) - (a_k, b_k) = (0, -1), \quad \text{and} \quad (a_k, b_k) - (a_{k-1}, b_{k-1}) = (1, 0).$$

If one describes the lattice V using either the Cartesian or the matrix notation, then a right-turn of P is exactly a point where the path turns to the right. Given two sets of s points $\mathcal{A} = A_1, \dots, A_s$ and $\mathcal{B} = B_1, \dots, B_s$ of V , a set F is said to be a *family of non-intersecting paths* from \mathcal{A} to \mathcal{B} if it can be decomposed as $F = P_1 \cup P_2 \cdots \cup P_s$ where P_i is a path from A_i to B_i and $P_i \cap P_j = \emptyset$ if $i \neq j$. (We identify a family of paths with the set of points on its paths. In the present setting this is allowed because the decomposition above is unique.) We will then say that a point $C \in F$ is a right turn of F if it is a right turn of the path to which it belongs. The *length* of a path from $A = (x_1, x_2)$ to a point $B = (y_1, y_2)$ depends only on A

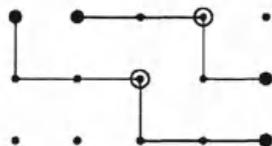


Figure 3. A pair of non-intersecting paths with 2 right-turns ($m = 5, n = 3, t = 3$)

and B and is equal to $y_1 - x_1 + x_2 - y_2 + 1$.

Proposition 6.4. *The facets of Δ_t are exactly the families of non-intersecting paths from $(1, n), (2, n), \dots, (t-1, n)$ to $(m, 1), (m, 2), \dots, (m, t-1)$.*

Proof. A family of non-intersecting paths is in Δ_t since an antichain intersects a chain in at most one point. That it is a facet can be easily proved directly, but

follows also from the fact that any such family has dimension $(m+n-t+1)(t-1)-1$, which is, by 1.10, the dimension of Δ_t since $\dim \Delta_t = \dim K[\Delta] - 1 = \dim K[X]/I_t - 1$.

It remains to show that every facet $G \in \Delta_t$ is a family of non-intersecting paths from $(1, n), (2, n), \dots, (t-1, n)$ to $(m, 1), (m, 2), \dots, (m, t-1)$. The points of

$$W = \{(a, b) : b - a \geq n - t + 1 \text{ or } a - b \geq m - t + 1\}$$

do not belong to any t -antichain: there is not enough room. So G must contain W . We put

$$(a, b) \prec (c, d) \iff a < c \text{ and } b < d.$$

By construction, distinct points P, Q are either comparable with respect to $<$ or comparable with respect to \prec , but not both. So a chain with respect to \prec is an antichain with respect to $<$, and viceversa. For a set of points A we denote by $\text{Min}_{\prec}(A)$ the set of the elements of A which are minimal with respect to \prec , i.e. the elements $P \in A$ such that there is no $Q \in A$ with $Q \prec P$. We then define:

$$G_1 = \text{Min}_{\prec}(G) \quad \text{and} \quad G_i = \text{Min}_{\prec}\left(G \setminus \bigcup_{j=1}^{i-1} G_j\right) \text{ for } i > 1.$$

This is called the *light and shadow decomposition* (the light here comes from the point $(1, 1)$). The family of the G_i satisfies the following conditions:

- (a) Every G_i is a chain since otherwise G_i would contain two $<$ -incomparable elements P, Q and then either $P \prec Q$ or $Q \prec P$ which is impossible.
- (b) For every P in G_i there exist $P_1 \in G_1, \dots, P_{i-1} \in G_{i-1}$ such that $P_1 \prec P_2 \prec \dots \prec P_{i-1} \prec P$. This is clear by construction.
- (c) For $i = 1, \dots, t-1$ the set G_i contains the points (i, n) and (m, i) . This follows from the fact that $W \subseteq G$.
- (d) G_i is empty for $i > t-1$ since otherwise we would get a t -antichain in G by (b).

It follows that G is the disjoint union of the chains G_1, \dots, G_{t-1} and that each chain G_i contains (i, n) and (m, i) . We prove that each G_i is indeed a path from (i, n) and (m, i) . Clearly G_i cannot contain points which are smaller than (i, n) or larger than (m, i) since those points belong already to the G_j with $j < i$. So it remains to show that G_i is saturated. Recall that Q is said to be an upper neighbor of P if $P < Q$ and there is not an H with $P < H < Q$. We have to show the following

Claim. If $P = (a, b)$ and $Q = (c, d)$ belong to G_i and $Q > P$, but Q is not an upper neighbor, then there exists $H_0 \in G_i$ such that $P < H_0 < Q$.

We set

$$H = \begin{cases} (c, b) & \text{if } c \neq a \text{ and } b \neq d, \\ (c, d+1) & \text{if } c = a \text{ and } b \neq d, \\ (a+1, b) & \text{if } c \neq a \text{ and } b = d. \end{cases}$$

Since $P < H < Q$, if $H \in G_i$, then we are done; just set $H_0 = H$. If $H \notin G_i$, then there are three possible cases:

- (1) If $H \in G_j$ for some $j < i$, then, by (b), we may find $T_1, T_2 \in G_j$ such that $T_1 \prec P$ and $T_2 \prec Q$. But T_1 and T_2 must be $<$ -comparable with H . This is a contradiction because, by the choice of H , either $T_1 \prec H$ or $T_2 \prec H$.
- (2) If $H \in G_j$ for some $j > i$, then by (b) there is a $T \in G_i$ such that $T \prec H$. If $P < T < Q$, then we set $H_0 = T$. Otherwise either $T \leq P$ or $T \geq Q$. This is impossible since $P < H < Q$.
- (3) If $H \notin G$ then $G \cup \{H\}$ does not contain a t -antichain since it has a decomposition into $t - 1$ chains; just add H to G_i . This contradicts the maximality of G and concludes the proof. \square

It follows from the above description that every facet of Δ_t has exactly $(m + n + 1 - t)(t - 1)$ elements, so that Δ_t is pure. Therefore the multiplicity of $K[\Delta_t]$ and, hence, that of $K[X]/I_t$ is given by the number of families of non-intersecting paths from $(1, n), (2, n), \dots, (t - 1, n)$ to $(m, 1), (m, 2), \dots, (m, t - 1)$. This number can be computed by the Gessel-Viennot determinantal formula [37]: given two sets of points $\mathcal{A} = A_1, \dots, A_s$ and $\mathcal{B} = B_1, \dots, B_s$, the number $\text{Paths}(\mathcal{A}, \mathcal{B})$ of families of non-intersecting paths from \mathcal{A} to \mathcal{B} is

$$\text{Paths}(\mathcal{A}, \mathcal{B}) = \det(\text{Paths}(A_i, B_j))_{i,j=1,\dots,s}$$

provided there is no family of non-intersecting paths from \mathcal{A} to any non-trivial permutation of \mathcal{B} . Here $\text{Paths}(A_i, B_j)$ denotes the number of paths from A_i to B_j .

For $A_i = (i, n)$ and $B_j = (m, j)$ a simple inductive argument gives $\text{Paths}(A_i, B_j) = \binom{m-i+n-j}{m-i}$ and hence it yields the formula

$$e(K[X]/I_t) = \det \left(\binom{m-i+n-j}{m-i} \right)_{i,j=1,\dots,t-1}.$$

After some row and column operations one can evaluate the determinant using Vandermonde's formula to obtain

Theorem 6.5.

$$e(K[X]/I_t) = \prod_{i=0}^{n-t} \frac{\binom{m+i}{t-1}}{\binom{t+i-1}{t-1}}$$

The formula for $e(K[X]/I_t)$ is due to Giambelli (1903). The proof above has been given by Herzog and Trung [44]. They have generalized this approach (including Schensted's theorem 4.8) to the 1-cogenerated ideals introduced in Section 1. (See Harris and Tu [43] for a different approach to Theorem 6.5.)

Next we show that Δ_t is shellable. This is a special case of a more general theorem due to Björner [7, Thm.7.1].

Theorem 6.6. *The simplicial complex Δ_t is shellable. More precisely, the facets of Δ_t can be given a total order such that $c(F)$ is the set of right-turns of F for each facet F of Δ_t .*

Proof. First we give a partial order to the set of paths connecting points A, B with $A \leq B$. For two paths P_1 and Q_1 from A to B we write $P_1 < Q_1$ if P_1 is “on the right” of Q_1 as one goes from A to B (in Cartesian as well as in matrix notation). This is a partial order.

Let $A_i = (i, n)$ and $B_i = (m, i)$ for $i = 1, \dots, t-1$. Given two families of non-intersecting paths $P = P_1, \dots, P_{t-1}$ and $Q = Q_1, \dots, Q_{t-1}$ from A_1, \dots, A_{t-1} to B_1, \dots, B_{t-1} we set $P < Q$ if $P_i < Q_i$ for the largest i such that $P_i \neq Q_i$. We extend this partial order on the set of families arbitrarily to a total order.

To prove that the resulting total order is indeed a shelling, one takes two families Q and P with $P < Q$ and lets i denote the largest index j with $P_j \neq Q_j$. Then Q_i is not on the right of P_i . It is easy to see (just draw a picture) that there exists a right-turn, say H , of Q_i which is (strictly) on left of P_i . By the choice of i , the point H does not belong to P_j for all j . So it suffices to show that for every right-turn $H = (x, y)$ of Q_i there is a family R which is $< Q$ in the total order such that $Q \setminus R = \{H\}$.

This is easy if either $(x-1, y-1)$ does not belong to Q_{i-1} or $i = 1$: just replace (x, y) with $(x-1, y-1)$ in Q_i to get a path Q'_i , and then set $R = R_1, \dots, R_{t-1}$ with $R_j = Q_j$ if $j \neq i$ and $R_i = Q'_i$. By construction $R < Q$ in the total order.

It is a little more complicated to define R when (by bad luck) the element $(x-1, y-1)$ belongs to P_{i-1} . But if this is the case, then $(x-1, y-1)$ must be a right-turn of P_{i-1} (draw a picture). If $(x-2, y-2)$ does not belong to P_{i-2} , we may repeat the construction above: define R as the family obtained from Q by replacing (x, y) with $(x-1, y-1)$ in Q_i and $(x-1, y-1)$ with $(x-2, y-2)$ in Q_{i-1} . The general case follows by the same construction. \square

Theorem 6.6 has two important consequences. The first is

Theorem 6.7. *The algebras $K[\Delta_t]$ and $K[X]/I_t$ are Cohen-Macaulay.*

Proof. By Theorem 6.6 Δ_t is shellable, and hence $K[\Delta_t]$ is Cohen-Macaulay by Theorem 6.3. Since $K[\Delta_t]$ is $K[X]/\text{in}(I_t)$, it follows from Theorem 3.16 that $K[X]/I_t$ is Cohen-Macaulay as well. \square

The second consequence is a combinatorial interpretation of the Hilbert series of determinantal rings. We need some more notation for it. Given two sets of s points \mathcal{A} and \mathcal{B} , let $\text{Paths}(\mathcal{A}, \mathcal{B})_k$ denote the numbers of families of non-intersecting paths from \mathcal{A} to \mathcal{B} with exactly k right turns, and set $\text{Paths}(\mathcal{A}, \mathcal{B}, z) = \sum_k \text{Paths}(\mathcal{A}, \mathcal{B})_k z^k$. In the case of just one starting and one ending point, say A and B , we denote this polynomial simply by $\text{Paths}(A, B, z)$. We have

Theorem 6.8. *The Hilbert series $H_t(z)$ of $K[\Delta_t]$ and $K[X]/I_t$ is of the form*

$$H_t(z) = \frac{\text{Paths}(\mathcal{A}, \mathcal{B}, z)}{(1-z)^d}$$

where $d = (m+n+1-t)(t-1)$ is the Krull dimension, $\mathcal{A} = (1, n), (2, n), \dots, (t-1, n)$ and $\mathcal{B} = (m, 1), (m, 2), \dots, (m, t-1)$.

For $t = 2$, i.e. one starting and one end point, the polynomial $\text{Paths}(\mathcal{A}, \mathcal{B}, z)$ can be easily computed by induction on m and n and this yields the following formula:

$$H_2(z) = \frac{\sum_k \binom{m-1}{k} \binom{n-1}{k} z^k}{(1-z)^{m+n-1}}.$$

It can be obtained also from the interpretation of $K[X]/I_2$ as the Segre product of two polynomial rings.

By analogy with the Gessel-Viennot formula one may wonder whether the polynomial $\text{Paths}(\mathcal{A}, \mathcal{B}, z)$ has a determinantal expression as

$$\det \left(\text{Paths}(A_i, B_j, z) \right)_{i,j=1,\dots,t-1} \quad (3)$$

This is obviously true if there is just one starting and ending point, but cannot be true in general since $\text{Paths}(\mathcal{A}, \mathcal{B}, 0) = 1$ and $\text{Paths}(A_i, B_j, 0) = 1$ for all i, j . But, very surprisingly, equality holds after a shift of degree if the starting points are consecutive integral points on a vertical line and the end points are consecutive integral points on a horizontal line. This is essentially the content of

Theorem 6.9. *The Hilbert series $H_t(z)$ of $K[\Delta_t]$ and $K[X]/I_t$ is*

$$H_t(z) = \frac{\det \left(\sum_k \binom{m-i}{k} \binom{n-j}{k} z^k \right)_{i,j=1,\dots,t-1}}{z^{\binom{t-1}{2}} (1-z)^{(m+n+1-t)(t-1)}}.$$

Proof. Krattenthaler [53] proved a determinantal formula for $\text{Paths}(\mathcal{A}, \mathcal{B}, z)$ for general \mathcal{A}, \mathcal{B} . If the starting and end points are those specified in 6.8, one can show that the determinant in Krattenthaler's formula is equal to

$$z^{-\binom{t-1}{2}} \det \left(\sum_k \binom{m-i}{k} \binom{n-j}{k} z^k \right)$$

For the proof one has to describe the transformations in the corresponding matrices. Details are to be found in [24]. \square

Krattenthaler's determinantal formula for the enumeration of families of non-intersection paths with a given number of right turns holds no matter how the

starting and end points are located. But this is not equal to the polynomial (3) even if one allows a shift in degree. So the formula of 6.9 should be regarded as an “accident” while the combinatorial description of 6.8 holds more generally, for instance, for algebras defined by 1-cogenerated ideals. On the other hand Krattenthaler and Prohaska [54] were able to show that the same “accident” takes place if the paths are restricted to certain subregions called one-sided ladders. This proves a conjecture of Conca and Herzog on the Hilbert series of one-sided ladder determinantal rings; see [24].

Remark 6.10. Since the rings $K[X]/I_t$ are Cohen-Macaulay domains, the Gorenstein ones among them are exactly those with a symmetric numerator polynomial in the Hilbert series [13, 4.4.6]. By a tedious analysis of the formula for the Hilbert series (see [24]) one can prove that $K[X]/I_t$, $t \geq 2$, is Gorenstein if and only if $m = n$, a result due to Svanes. It is however more informative to determine the canonical module of $K[X]/I_t$ for all shapes of matrices; see [17, Section 8] or [13, 7.3].

Remark 6.11. It follows from Theorem 3.16 that the ring $K[X]/I_2$ is Koszul. We can also conclude that the homogeneous coordinate ring $K[\mathcal{M}_m]$ (with $m \leq n$) of the Grassmannian is Koszul. To this end we represent it as the residue class ring of a polynomial ring S whose indeterminates are mapped to the m -minors of X . Then we refine the partial order $<$ of m -minors to a linear order, lift that order to the indeterminates of S , and choose the Revlex term order on S . The elements of S representing the Plücker relations form a Gröbner basis of the ideal defining $K[\mathcal{M}_m]$.

Remark 6.12. Ladder determinantal rings are an important generalization of the classical determinantal rings. They are defined by the minors coming from certain subregions, called *ladders*, of a generic matrix. These objects have been introduced by Abhyankar in his study of the singularities of Schubert varieties of flag manifolds and have been investigated by many authors, including Conca, Ghorpade, Gonciulea, Herzog, Knutson, Krattenthaler, Kulkarni, Lakshmibai, Miller, Mulay, Narasimhan, Prohaska, Rubey and Trung [20, 21, 25, 39, 40, 41, 52, 44, 54, 55, 57, 60]. Ladder determinantal rings share many property with classical determinantal rings, for instance they are Cohen-Macaulay normal domains, the Gorenstein ones are completely characterized in terms of the shape of the ladder, and there are determinantal formulas for the Hilbert series and functions. Many of these results are derived from the combinatorial structure of the Gröbner bases of the ideals of definition.

Remark 6.13. The ideal of t -minors of a symmetric matrix of indeterminates and the ideal Pf_t of $2t$ -pfaffians of an alternating matrix of indeterminates can also be treated by Gröbner basis methods based on suitable variants of KRS.

For pfaffians the method was introduced by Herzog and Trung [44]. They used it to compute the multiplicity of $K[X]/\text{Pf}_t$. A determinantal formula for the Hilbert

series can be found in De Negri [31]; see also Ghorpade and Krattenthaler [38]. Baeřica has extended the results of Section 7 to the pfaffian case.

Conca [19] has transferred the method of Herzog and Trung to the symmetric case, introducing a suitable version of KRS. He derived formulas for the Hilbert series and the multiplicity (see [19] for the latter).

Conca [23] has attacked another class of determinantal ideals by Gröbner basis methods, the ideals of minors of a Hankel matrix. This case, like that of maximal minors, is “easy” since different standard products of minors have different initial terms so that the essential point is to define the standard products.

In addition to the generic case, Harris and Tu [43] give formulas of type 6.5 also in the symmetric and the alternating case.

7. Algebras of minors: Cohen-Macaulayness and normality

In this section we consider three types of algebras: the Rees algebra $\mathcal{R}(I)$ of a product $I = I_{t_1} \cdots I_{t_u}$ of determinantal ideals, the symbolic Rees algebra $\mathcal{R}^{\text{symb}}(I_t)$ of I_t , and the algebra of t -minors A_t , namely the K -subalgebra $K[\mathcal{M}_t]$ of $K[X]$ generated by the t -minors. By studying their initial algebras we will show that these algebras are normal and Cohen-Macaulay (under a suitable hypothesis on the characteristic of K). In all the cases the initial algebra is a finitely generated normal semigroup ring and its description as well as its normality are essentially a translation of the results of Section 5 into the algebra setting.

It is convenient to embed all these algebras into a common polynomial ring S , obtained by adjoining a variable T to $K[X]$,

$$S = K[X, T] = K[X][T].$$

For an ideal I of $K[X]$ the Rees algebra $\mathcal{R}(I)$ of I can be described as $\mathcal{R}(I) = \bigoplus_k I^k T^k \subseteq S$. The symbolic Rees algebra of I_t is $\mathcal{R}^{\text{symb}}(I_t) = \bigoplus_k I_t^{(k)} T^k \subseteq S$ and the algebra of minors A_t can be realized as the subalgebra of S generated by the elements of the form δT where δ is a minor of size t . (One only uses that all t -minors have the same degree as elements of $K[X]$.)

Let us first discuss some simple and/or classical cases. (They are included in the general discussion below.) The Rees algebra of the polynomial $K[X]$ with respect to the ideal I_1 , its irrelevant maximal ideal, can be represented as a determinantal ring. In fact, let $R = K[X_1, \dots, X_n]$ where the X_i are pairwise different indeterminates. Then the substitution $X_i \mapsto X_i, Y_i \mapsto X_i T, i = 1, \dots, n$, yields the isomorphism $\mathcal{R}(X_1, \dots, X_n) \cong K[X, Y]/I_2(U)$ where

$$U = \begin{pmatrix} X_1 & \cdots & X_n \\ Y_1 & \cdots & Y_n \end{pmatrix}.$$

For the isomorphism it is enough to note that the 2-minors of U are mapped to 0 by the substitution and that $I_2(U)$ is a prime ideal of height $n - 1$ so that

$\dim K[X, Y]/I_2(U) = n + 1 = \dim \mathcal{R}(X_1, \dots, X_n)$. It follows that the Rees algebra is a normal Cohen-Macaulay domain. It is Gorenstein only in the cases $n = 1, 2$.

The other extreme case $t = \min(m, n)$ is also much simpler than the general one. Eisenbud and Huneke [34] have shown that $\mathcal{R}(I_t)$ is an algebra with straightening law on a wonderful poset. In particular it is Cohen-Macaulay. By Proposition 2.3 $\mathcal{R}(I_t) = \mathcal{R}^{\text{symb}}(I_t)$. This implies normality since symbolic powers of primes in $K[X]$ are integrally closed. See [17, Section 9] or Bruns, Simis and Trung [16] for generalizations.

For A_t the case $t = 1$ is completely trivial, since $A_1 = K[X]$. In the opposite extreme case $t = \min(m, n)$, say $t = m \leq n$, the algebra $K[\mathcal{M}_m] = A_m$ is the homogeneous coordinate ring of the Grassmannian of m -dimensional subspaces of the vector space K^n , as discussed in Section 1. This algebra is a factorial Gorenstein ring; see [17].

We have seen in Corollary 1.3 that $\dim A_m = m(n - m) + 1$. However, if $t < \min(m, n)$, then $\dim A_t = \dim K[X] = mn$. Indeed, the indeterminates X_{ij} are algebraic over the quotient field of A_t . It is enough to show this for a $(t + 1) \times (t + 1)$ matrix X . The entries of the adjoint matrix \tilde{X} of X are in A_t . Therefore $(\det X)^t \in A_t$. It follows that the entries of $X^{-1} = (\det X)^{-1} \tilde{X}$ are algebraic over $\text{QF}(A_t)$, and playing the same game again, we conclude algebraicity for the entries of $X = (X^{-1})^{-1}$.

Incidentally, this discussion has revealed another simple case: If $t = m - 1 = n - 1$, then A_t is generated by $mn = \dim A_t$ elements, and so is isomorphic to a polynomial ring over K .

We turn to the general case. Powers and products of determinantal ideals are intersections of symbolic powers; see Theorem 2.4. It follows immediately that the Rees algebra of $I_{t_1} \cdots I_{t_u}$ is the intersection of symbolic Rees algebras of the various I_t and their Veronese subalgebras. The representation as an intersection is passed on to the initial algebras: this is a consequence of the in-KRS property. To sum up: the key part is the description of the initial algebra of the symbolic Rees algebra of I_t . The rest, at least as far as normality and Cohen-Macaulayness are concerned, will follow at once.

So let us start with the symbolic Rees algebra of I_t . The description of the symbolic powers in Proposition 2.2 yields the following description of the symbolic Rees algebra:

$$\mathcal{R}^{\text{symb}}(I_t) = K[X] [I_t T, I_{t+1} T^2, \dots, I_m T^{m-t+1}].$$

Consider a diagonal term order on $K[X]$ and extend it arbitrarily to a term order on $K[X, T]$. The initial algebra $\text{in}(\mathcal{R}^{\text{symb}}(I_t))$ of $\mathcal{R}^{\text{symb}}(I_t)$ is then $\bigoplus_k \text{in}(I_t^{(k)}) T^k$. The description of $\text{in}(I_t^{(k)})$ in Theorem 5.8 yields the following

Lemma 7.1. *The initial algebra $\text{in}(\mathcal{R}^{\text{symb}}(I_t))$ of the symbolic Rees algebra $\mathcal{R}^{\text{symb}}(I_t)$ is equal to*

$$K[X] [\text{in}(I_t)T, \text{in}(I_{t+1})T^2, \dots, \text{in}(I_m)T^{m-t+1}].$$

In particular, a monomial MT^k is in $\text{in}(\mathcal{R}^{\text{symb}}(I_t))$ if and only if $\widehat{\gamma}_t(M) \geq k$.

The next step is to show that $\text{in}(\mathcal{R}^{\text{symb}}(I_t))$ is normal. This can be done directly by using the convexity of the function $\widehat{\gamma}_t$ as in [9]. Instead we give a longer, but more informative argument which involves the description of the initial algebra by linear inequalities (for the exponent vectors of the monomials in it). This description will be used in the next section to identify the canonical modules of various algebras. The crucial fact is the primary decomposition of $\text{in}(I_t^{(k)})$:

Lemma 7.2. *Let \mathbf{F}_t denote the set of facets of Δ_t , and, for every $F \in \mathbf{F}_t$, let P_F be the ideal generated by the indeterminates X_{ij} with $X_{ij} \notin F$. Then*

$$\text{in}(I_t^{(k)}) = \bigcap_{F \in \mathbf{F}_t} P_F^k.$$

We have seen in Theorem 5.8 that $\text{in}(I_t^{(k)})$ is generated by the monomials M with $\widehat{\gamma}_t(M) \geq k$. A monomial $M = \prod_{i=1}^s X_{a_i b_i}$ is in P_F^k if and only if the cardinality of $\{i : (a_i, b_i) \notin F\}$ is $\geq k$. Equivalently, M is in P_F^k if and only if the cardinality of $\{i : (a_i, b_i) \in F\}$ is $\leq \deg(M) - k$. As a measure we introduce

$$w_t(M) = \max\{|A| : A \subseteq [1, \dots, s] \text{ and } \{(a_i, b_i) : i \in A\} \in \Delta_t\}.$$

Then a monomial M is in $\bigcap_{F \in \mathbf{F}_t} P_F^k$ if and only if $w_t(M) \leq \deg(M) - k$, or, equivalently, $\deg(M) - w_t(M) \geq k$. Now Proposition 7.2 follows from

Lemma 7.3. *Let M be a monomial. Then $\widehat{\gamma}_t(M) + w_t(M) = \deg(M)$.*

We reduce this lemma to a combinatorial statement on sequences of integers. For such a sequence b we set

$$w_t(b) = \max\{\text{length}(c) : c \text{ is a subsequence of } b \text{ and } \widehat{\gamma}_t(c) = 0\}.$$

Let $M = \prod_{i=1}^s X_{a_i b_i}$ be a monomial. We order the indices as in the KRS correspondence, namely $a_i \leq a_{i+1}$ for every i and $b_{i+1} \geq b_i$ whenever $a_i = a_{i+1}$. By Remark 5.5 we have $w_t(M) = w_t(b)$. To sum up, it suffices to prove

Lemma 7.4. *One has $\widehat{\gamma}_t(b) + w_t(b) = \text{length}(b)$ for every sequence b of integers.*

Proof. We use part (b) of Greene's theorem 4.9: the sum $\alpha_k^*(\text{Ins}(b))$ of the lengths of the first k columns of the insertion tableau $\text{Ins}(b)$ of b is the length

of the longest subsequence of b that can be decomposed into k non-increasing subsequences.

It follows that a sequence a has no increasing subsequence of length t if and only if it can be decomposed into $t - 1$ non-increasing subsequences. In fact, the sufficiency of the condition is obvious, whereas its necessity follows from Schensted's theorem 4.8 and the just quoted result of Greene: if a has no increasing subsequence of length t , then all the rows in the insertion tableau $\text{Ins}(a)$ have length at most $t - 1$. So $\alpha_{t-1}^*(\text{Ins}(a))$ is the length of a , and a can be decomposed into $t - 1$ non-increasing subsequences by Greene's theorem.

Consequently $w_t(b)$ is the maximal length of a subsequence of b that can be decomposed into $t - 1$ decreasing subsequences. Applying Greene's theorem once more, we see that $w_t(b) = \alpha_{t-1}^*(\text{Ins}(b))$. On the other hand, $\widehat{\gamma}_t(b) = \gamma_t(\text{Ins}(b))$ by Theorem 4.11. Since $\gamma_t(\text{Ins}(b))$ is the sum of the lengths of the columns of P of index $\geq t$, one has $w_t(b) + \widehat{\gamma}_t(b) = \text{length}(b)$. \square

Now we are ready to describe the linear inequalities supporting $\text{in}(\mathcal{R}^{\text{symb}}(I_t))$. To simplify notation we identify monomials of S with their exponent vectors in \mathbb{R}^{mn+1} . For every subset F of $\{1, \dots, m\} \times \{1, \dots, n\}$ we define a linear form ℓ_F on \mathbb{R}^{mn} by setting $\ell_F(X_{ij}) = 1$ if $(i, j) \notin F$ and 0 otherwise.

Theorem 7.5. *We extend ℓ_F for $F \in \mathbf{F}_t$ to a linear form L_F on $\mathbb{R}^{mn} \oplus \mathbb{R}$ by setting $L_F(T) = -1$. Then:*

- (a) *A monomial MT^k is in the initial algebra $\text{in}(\mathcal{R}^{\text{symb}}(I_t))$ iff it has non-negative exponents and $L_F(MT^k) \geq 0$ for all $F \in \mathbf{F}_t$.*
- (b) *The initial algebra $\text{in}(\mathcal{R}^{\text{symb}}(I_t))$ is normal and Cohen-Macaulay.*
- (c) *The symbolic Rees algebra $\mathcal{R}^{\text{symb}}(I_t)$ is normal and Cohen-Macaulay.*

Proof. (a) is a restatement of 7.2. Part (b) follows from (a) and [13, 6.1.2, 6.1.4, 6.3.5]. Finally (c) follows from (b) and Theorem 3.16. \square

As already mentioned, Theorem 7.5 has several consequences. The first is

Theorem 7.6. *Suppose that $\text{char } K = 0$ or $\text{char } K > \min(t_i, m - t_i, n - t_i)$ for all i . Then*

- (a) *$\text{in}(\mathcal{R}(I_{t_1} \cdots I_{t_r}))$ is finitely generated and normal,*
- (b) *$\mathcal{R}(I_{t_1} \cdots I_{t_r})$ is Cohen-Macaulay and normal.*

Proof. Set $J = I_{t_1} \cdots I_{t_r}$. One has $\text{in}(\mathcal{R}(J)) = \bigoplus_{k \geq 0} \text{in}(J^k)T^k$, and, by Theorem 5.9, $\text{in}(J^k) = \bigcap_{1 \leq j \leq m} \text{in}(I_j^{(kg_j)})$. Hence

$$\text{in}(\mathcal{R}(J)) = \bigcap_{1 \leq j \leq m} \bigoplus_{k \geq 0} \text{in}(I_j^{(kg_j)})T^k.$$

The monomial algebra $\bigoplus_{k \geq 0} \text{in}(I_j^{(kg_j)})T^k$ is isomorphic to the g_j th Veronese subalgebra of the monomial algebra $\text{in}(\mathcal{R}^{\text{symb}}(I_j))$ (in the relevant case $g_j > 0$ and equal to $K[X, T]$ otherwise). By 7.5 the latter is normal and finitely generated, and therefore $\bigoplus_{k \geq 0} \text{in}(I_j^{(kg_j)})T^k$ is a normal, finitely generated monomial algebra. Thus $\text{in}(\mathcal{R}(J))$ is finitely generated and normal. In fact, the intersection of a finite number of finitely generated normal monomial algebras is finitely generated and normal. (This follows easily from standard results about normal affine semigroup rings; see Bruns and Herzog [13, 6.1.2 and 6.1.4].) For (b) one applies Corollary 3.17 again. \square

We single out the most important case.

Theorem 7.7. *Suppose that $\text{char } K = 0$ or $\text{char } K > \min(t, m - t, n - t)$. Then $\mathcal{R}(I_t)$ is Cohen–Macaulay and normal.*

Remark 7.8. (a) The Cohen-Macaulayness of the Rees algebra of I_t in the case of maximal minors has been proved by Eisenbud and Huneke in [34], as pointed out above. For arbitrary t and $\text{char } K = 0$, Bruns [8] has shown that $\mathcal{R}(I_t)$ and A_t are Cohen-Macaulay.

(b) In 7.7 the hypothesis on the characteristic is essential. If $m = n = 4$ and $\text{char } K = 2$, then $\mathcal{R}(I_2)$ has dimension 17 and depth 1; see [8].

(c) In order to obtain a version of 7.6 that is valid in arbitrary characteristic one must replace the Rees algebra by its integral closure. The integral closure is always equal to the intersection of symbolic Rees algebras that in non-exceptional characteristic gives the Rees algebra itself (see [8]).

Remark 7.9. One can describe the hyperplanes defining the initial algebra of the Rees algebra of a product of $I_{t_1} \cdots I_{t_r}$ in terms of proper extensions of the linear forms ℓ_F . For every j set $g_j = \gamma_j(t_1, \dots, t_r)$ and for every $F \in \mathbf{F}_j$ extend ℓ_F to L_F by setting $L_F(T) = -g_j$. Then the initial algebra of the Rees algebra of $I_{t_1} \cdots I_{t_r}$ is given by the inequalities $L_F(MT^k) \geq 0$ for all $F \in \mathbf{F}_j$ and for all j (and the non-negativity of the exponents of MT^k).

For the algebra of minors A_t we have

Theorem 7.10. *Suppose that $\text{char } K = 0$ or $> \min(t, m - t, n - t)$. Then the initial algebra $\text{in}(A_t)$ is finitely generated and normal. Hence A_t is a normal Cohen–Macaulay ring.*

Proof. Let V_t be the subalgebra of S generated by the monomials of the form MT with $\deg M = t$, i.e. V_t is (isomorphic to) the t -Veronese subalgebra of $K[X]$. By construction $A_t = \mathcal{R}(I_t) \cap V_t$. This clearly implies $\text{in}(A_t) = \text{in}(\mathcal{R}(I_t)) \cap V_t$. Since $\text{in}(\mathcal{R}(I_t))$ is normal by Theorem 7.6 and V_t is normal by obvious reasons, $\text{in}(A_t)$ is normal. \square

Remark 7.11. (a) As for the other cases one can give a description of the initial algebra of A_t by linear inequalities and equations.

(b) Although we have proved that the initial algebras of $\mathcal{R}(I_t)$ and A_t are finitely generated, we cannot specify a finite Sagbi basis: we do not know what largest degree occurs in a system of generators for their initial algebras.

8. Algebras of minors: the canonical module

The goal of this section is to describe the canonical modules of the algebras $\mathcal{R}(I_t)$ and A_t . The first step is to find the canonical modules of their initial algebras. The characteristic of the field K will be either 0 or $> \min(t, m-t, n-t)$ throughout.

Recall that V_t is the subalgebra of S generated by all the monomials of the form MT where M is a monomial in $K[X]$ of degree t (thus V_t is isomorphic to the Veronese subalgebra of $K[X]$). Part (a) of the following lemma is just a restatement of 2.4 and part (b) is a restatement of 5.9. However, (c) and (d) contain a somewhat surprising simplification for A_t and its initial algebra.

Lemma 8.1. (a) A K -basis of $\mathcal{R}(I_t)$ is given by the set of the elements ΣT^k where Σ is a standard bitableau with $\gamma_i(\Sigma) \geq k(t+1-i)$ for all $i = 1, \dots, t$.
 (b) A K -basis of $\text{in}(\mathcal{R}(I_t))$ is given by the set of the elements MT^k where M is a monomial of $K[X]$ with $\widehat{\gamma}_i(M) \geq k(t+1-i)$ for all $i = 1, \dots, t$.
 (c) A K -basis of A_t is given by the set of the elements ΔT^k where Δ is a standard bitableau with $\gamma_2(\Delta) \geq k(t-1)$ and $\deg(\Delta) = kt$.
 (d) A K -basis of $\text{in}(A_t)$ is given by the set of the elements MT^k where M is a monomial of $K[X]$ with $\widehat{\gamma}_2(M) \geq k(t-1)$ and $\deg(M) = kt$.

Proof. As pointed out above, only (c) and (d) still need a proof. Since $A_t = V_t \cap \mathcal{R}(I_t)$, a K basis of A_t is given by the elements ΔT^k in the basis of $\mathcal{R}(I_t)$ with $\deg(\Delta) = kt$. Similarly, since $\text{in}(A_t) = V_t \cap \text{in}(\mathcal{R}(I_t))$, a K basis of $\text{in}(A_t)$ is given by the elements MT^k in the basis of $\text{in}(\mathcal{R}(I_t))$ with $\deg(M) = kt$. Now (c) and (d) result from the following statement: if λ is a shape such that $\sum_i \lambda_i = kt$ and $\gamma_2(\lambda) \geq k(t-1)$, then $\gamma_i(\lambda) \geq k(t+1-i)$ for all $i = 1, \dots, t$. We leave the proof to the reader; it is to be found in [12]. \square

Remark 8.2. One can extend the valuations $\gamma_i, i = 1, \dots, t$, to $K[X, T]$ by choosing $\gamma_i(T) = -t + i - 1$. Therefore Lemma 8.1(a) contains a description of $\mathcal{R}(I_t)$ as an intersection of $K[X, T]$ with discrete valuation domains. This aspect is discussed in [12] and [15]. Part (c) has a similar interpretation.

However, since the equation $\widehat{\gamma}_t(MN) = \widehat{\gamma}_t(M) + \widehat{\gamma}_t(N)$ does not always hold, the functions $\widehat{\gamma}_t$ cannot be interpreted as valuations.

We know that $\text{in}(\mathcal{R}(I_t))$ and $\text{in}(A_t)$ are normal. Hence their canonical modules are the vector spaces spanned by all monomials represented by integral points

in the relative interiors of the corresponding cones (see Bruns and Herzog [13, Ch. 6]). We have seen in 7.9 how to describe the semigroup of $\text{in}(\mathcal{R}(I_t))$ in terms of linear homogeneous inequalities using the linear forms L_F defined as follows: For every $i = 1, \dots, t$ and for every facet F of Δ_i we extend ℓ_F to a linear form L_F on $\mathbb{R}^{mn} \oplus \mathbb{R}$ by setting $L_F(T) = -(t+1-i)$. We obtain

Lemma 8.3. *The canonical module of $\text{in}(\mathcal{R}(I_t))$ is the ideal of $\text{in}(\mathcal{R}(I_t))$ whose K -basis is the set of the monomials N of S with all exponents ≥ 1 and $L_F(N) \geq 1$ for every $F \in \mathbf{F}_i$ and for $i = 1, \dots, t$.*

Let \mathcal{X} denote the product of all the variables X_{ij} with $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$. We can give a description of the canonical module $\omega(\text{in}(\mathcal{R}(I_t)))$ in terms of \mathcal{X} and the functions $\widehat{\gamma}_i$:

Lemma 8.4. *The canonical module $\omega(\text{in}(\mathcal{R}(I_t)))$ of $\text{in}(\mathcal{R}(I_t))$ is the ideal of $\text{in}(\mathcal{R}(I_t))$ whose K -basis is the set of the monomials MT^k of S where M is a monomial of $K[X]$ with $\mathcal{X}T \mid MT^k$ in S and $\widehat{\gamma}_i(M) \geq (t+1-i)k+1$ for all $i = 1, \dots, t$.*

Proof. It suffices to show that the conditions given define the monomials described in 8.3. Let $N = MT^k$ be a monomial, $M \in K[X]$. Then, for a given i , one has $L_F(N) \geq 1$ for every $F \in \mathbf{F}_i$ if and only if $\ell_F(M) \geq k(t+1-i) + 1$ for every $F \in \mathbf{F}_i$. By 7.2 this is equivalent to $M \in \text{in}(I_i^{(k(t+1-i)+1)})$, which in turn is equivalent to $\widehat{\gamma}_i(M) \geq k(t+1-i) + 1$. To sum up, $L_F(N) \geq 1$ for every $F \in \mathbf{F}_i$ and $i = 1, \dots, t$ if and only if $\widehat{\gamma}_i(M) \geq (t+1-i) + 1$ for every $i = 1, \dots, t$. \square

Similarly the canonical module $\omega(\text{in}(A_t))$ has a description in terms of the function $\widehat{\gamma}_2$:

Lemma 8.5. *The canonical module $\omega(\text{in}(A_t))$ of $\text{in}(A_t)$ is the ideal of $\text{in}(A_t)$ whose K -basis is the set of the monomials MT^k of V_t where M is a monomial of $K[X]$ with $\mathcal{X}T \mid MT^k$ in S and $\widehat{\gamma}_2(M) \geq (t-1)k+1$.*

For “de-initialization” the following lemma is necessary. Its part (b) asserts that \mathcal{X} is a “linear” element for the functions $\widehat{\gamma}_i$.

Lemma 8.6. (a) $\widehat{\gamma}_i(\mathcal{X}) = (m-i+1)(n-i+1)$.
 (b) *Let M be a monomial in $K[X]$. Then $\widehat{\gamma}_i(\mathcal{X}M) = \widehat{\gamma}_i(\mathcal{X}) + \widehat{\gamma}_i(M)$ for every $i = 1, \dots, \min(m, n)$.*

Proof. Let M be a monomial in the X_{ij} 's. We know that $\widehat{\gamma}_i(M) \geq k$ if and only if $M \in \text{in}(I_i^{(k)})$. From 7.2 we deduce that

$$\widehat{\gamma}_i(M) = \inf\{\ell_F(M) : F \in \mathbf{F}_i\}.$$

Note that Δ_i is a pure simplicial complex of dimension one less the dimension of $K[X]/I_i$. Thus $\ell_F(\mathcal{X}) = (m-i+1)(n-i+1)$ for every facet F of Δ_i . In particular, $\widehat{\gamma}_i(\mathcal{X}) = (m-i+1)(n-i+1)$.

Since $\ell_F(NM) = \ell_F(N) + \ell_F(M)$ for all monomials N, M and for every F , we have $\widehat{\gamma}_i(MN) \geq \widehat{\gamma}_i(N) + \widehat{\gamma}_i(M)$. Conversely, let G be a facet of Δ_i such that $\widehat{\gamma}_i(M) = \ell_G(M)$. Then $\ell_G(\mathcal{X}M) = \ell_G(\mathcal{X}) + \ell_G(M) = \widehat{\gamma}_i(\mathcal{X}) + \widehat{\gamma}_i(M)$. Therefore $\widehat{\gamma}_i(MN) \leq \widehat{\gamma}_i(N) + \widehat{\gamma}_i(M)$, too. \square

Now we apply the above results to the canonical modules of $\mathcal{R}(I_t)$ and A_t . Assume for simplicity that $m \leq n$. Let us try to find a product of minors D such that $\text{in}(D) = \mathcal{X}$ and $\gamma_i(D) = \widehat{\gamma}_i(\mathcal{X})$ for all i . Since we have already computed $\widehat{\gamma}_i(\mathcal{X})$ (see 8.6), we can determine the shape of D , which turns out to be $1^2, 2^2, \dots, (m-1)^2, m^{(n-m+1)}$. In other words, D must be the product of 2 minors of size 1, 2 minors of size 2, \dots , 2 minors of size $m-1$ and $n-m+1$ minors of size m . Now it is not difficult to show that D is uniquely determined, the 1-minors are $[m|1]$ and $[1|n]$, the 2-minors are $[m-1, m|1, 2]$ and $[1, 2|n-1, n]$ and so on.

Theorem 8.7. *Let H be the K -subspace of S whose K -basis is the set of the elements of the form ΔT^k where Δ is a standard tableau with $\gamma_i(D\Delta) \geq (k+1)(t+1-i)$ for all $i = 1, \dots, t$.*

Let H_1 be the K -subspace of S whose K -basis is the set of the elements of the form ΔT^k where Δ is a standard tableau with $\gamma_2(D\Delta) \geq (k+1)(t-1)$ and $\deg(D\Delta) = t(k+1)$. Set $J = DTH$ and $J_1 = DTH_1$. Then we have:

- (a) *J is an ideal of $\mathcal{R}(I_t)$. Furthermore J is the canonical module of $\mathcal{R}(I_t)$.*
- (b) *J_1 is an ideal of A_t . Furthermore J_1 is the canonical module of A_t .*

Proof. That J and J_1 are indeed ideals in the corresponding algebras follows by the evaluation of shapes and by the description 8.1 of the algebras. Next we show that $\text{in}(J)$ and $\text{in}(J_1)$ are the canonical modules of $\text{in}(\mathcal{R}(I_t))$ and $\text{in}(A_t)$ respectively. It is enough to check that $\text{in}(J)$ is exactly the ideal described in 8.4 and $\text{in}(J_1)$ is the ideal described in 8.5. Note that $\text{in}(J) = \text{in}(DT)\text{in}(H) = \mathcal{X}T\text{in}(H)$ and $\text{in}(J_1) = \text{in}(DT)\text{in}(H_1) = \mathcal{X}T\text{in}(H_1)$. Furthermore, by virtue of 8.6, the canonical module of $\text{in}(\mathcal{R}(I_t))$ can be written as $\mathcal{X}TG$ where G is the space with basis the set of the monomials MT^k such that $\widehat{\gamma}_i(M) + \widehat{\gamma}_i(\mathcal{X}) \geq (k+1)(t+1-i)$ for all $i = 1, \dots, t$. Similarly, the canonical module of $\text{in}(A_t)$ can be written as $\mathcal{X}TG_1$ where G_1 is the space with basis the set of the monomials MT^k such that $\widehat{\gamma}_2(M) + \widehat{\gamma}_2(\mathcal{X}) \geq (k+1)(t-1)$ and $\deg(M\mathcal{X}) = t(k+1)$.

The spaces H and H_1 are defined by the same inequalities involving the γ -functions for bitableaux. As pointed out in Remark 5.12(a), such vector spaces are in-KRS. This implies $\text{in}(G) = H$, and similarly $\text{in}(G_1) = H_1$. As just proved, $\text{in}(J)$ and $\text{in}(J_1)$ are the canonical modules of $\text{in}(\mathcal{R}(I_t))$ and $\text{in}(A_t)$. Now the claim follows from Theorem 3.19. \square

Remark 8.8. (a) In [12] we have translated the combinatorial description of the canonical module into a divisorial one. If $t < \min(m, n)$, the divisor class group of $\mathcal{R}(I_t)$ is free of rank t , generated by the classes of prime ideals P_1, \dots, P_t where a K -basis of P_i is given by all products ΣT^k , Σ a standard bitableau with $\gamma_i(\Sigma) \geq k(t - i + 1) + 1$, $i = 1, \dots, t$. Then the canonical module of $\mathcal{R}(I_t)$ has divisor class

$$\sum_{i=1}^t (2 - (m - i + 1)(n - i + 1) + t - i) \text{cl}(P_i) = \text{cl}(I_t \mathcal{R}) + \sum_{i=1}^t (1 - \text{height } I_i) \text{cl}(P_i)$$

(b) If $t = \min(m, n)$, we may suppose that $t = m$. If even $t = m = n$, then I_t is a principal ideal, and $\mathcal{R}(I_t)$ is isomorphic to a polynomial ring over K .

Let $t = m < n$. In this case a theorem of Herzog and Vasconcelos [45] yields that the divisor class group of $\mathcal{R}(I_m)$ is free of rank 1, generated by the extension P of I_m to $\mathcal{R}(I_m)$. Moreover it implies that the canonical module has class $(2 - (n - m + 1))P$.

(c) in Section 7 we have seen that A_t is factorial and, hence, Gorenstein in the following cases: $t = 1$, $t = \min(m, n)$ (the Grassmannian), and $t = m - 1 = n - 1$.

(d) In all the cases different from those in (c), the ring A_t is not factorial. Its divisor class group is free of rank 1, generated by the class of a single prime ideal q that can be chosen as $q = (f)S[T] \cap A_t$ where f is a $(t + 1)$ -minor of X . Then

$$(mn - mt - nt) \text{cl}(q).$$

is the class of the canonical module; see [12].

(e) The expression for the class of the canonical module given in (a) can be generalized to a larger class of Rees algebras; see Bruns and Restuccia [15]. In particular one obtains results for the algebras of minors of symmetric matrices of indeterminates algebras generated by Pfaffians of alternating such matrices, and algebras of minors of Hankel matrices. The latter case has been treated by “initial methods” in [12].

As a corollary we have

Theorem 8.9. *The ring A_t is Gorenstein if and only if one of the following conditions is satisfied:*

- (a) $t = 1$; in this case $A_t = K[X]$.
- (b) $t = \min(m, n)$; in this case A_t is the coordinate ring of a Grassmannian.
- (c) $t = m - 1$ and $m = n$; in this case A_t is isomorphic to a polynomial ring.
- (d) $mn = t(m + n)$.

Proof. The cases (a), (b) and (c) are those discussed in Remark 8.8(c). So we may assume that $1 < t < \min(m, n)$ and $t \neq m - 1$ if $m = n$. Now Remark 8.8(d) completes the proof.

Since we have not discussed divisorial methods in detail, let us indicate how to prove the theorem by combinatorial methods. In view of 3.18 it makes no difference whether one works in A_t or $\text{in}(A_t)$. We choose $\text{in}(A_t)$.

Suppose that $mn = t(m+n)$. Then $\mathcal{X}T^{m+n}$ not only belongs to $\text{in}(A_t)$, but even to the ideal $\omega(\text{in}(A_t))$. Moreover, $\widehat{\gamma}_2(\mathcal{X}) - (t-1)(m+n) = 1$. Using the “linearity” of \mathcal{X} , it is now easy to see that the ideal $\omega(\text{in}(A_t))$ is generated by $\mathcal{X}T^{m+n}$. So the canonical module is isomorphic to $\text{in}(A_t)$, and $\text{in}(A_t)$ is Gorenstein.

In all the cases not covered by (a)–(d) one has to show that $\text{in}(J_1)$ is not a principal ideal. We leave this to the reader as an exercise. \square

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