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Syzygies, Ideals of Height Two, and Vector Bundles

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INTRODUCTION

This paper deals with recent progress and connections between several open problems in the theory of syzygies, ideals of height two, factorial rings of small embedding codimension, and vector bundles of small rank.

Throughout the paper (unless stated otherwise) R will denote a regular local ring with maximal ideal m. All modules will be finitely generated. The central problem which we will discuss is the so-called syzygy problem. If

$$0 \to F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \to F_1 \xrightarrow{f_1} F_0 \to M \to 0$$

is a free resolution of a R-module M, then the kernel K_j of f_{j-1} is called the jth syzygy of M. The syzygy problem simply asks *whether a nonfree jth syzygy must have a rank greater than or equal to j.* Of course, a free module can be a jth syzygy for any j.

In [5] Bruns shows that if K is a jth syzygy of rank $j + s$, then there is a free submodule F of K such that K/F is a jth syzygy of rank exactly j. Bruns accomplishes this through extending techniques of Eisenbud and Evans [13] on basic elements and criteria of Auslander and Bridger [1] for a module to be a jth syzygy. One can view the syzygy problem as asking whether or not the converse of Bruns' theorem does hold.

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The main results of our paper establish equivalences between the syzygy problem, in particular its rank two case, and ring-theoretic questions. For a second syzygy M there are always exact sequences

$$\text{(A)}\quad 0 \to M \to F \to I \to 0,$$

where F is a free R-module and I an ideal [5]. For a torsion-free R-module M one has exact sequences

$$\text{(B)}\quad 0 \to F \to M \to I \to 0,$$

where again F is free and I an ideal, by a theorem of Bourbaki [4]. We will investigate the interplay between properties of M and properties of the rings R/I, I given by sequences of type (A) or (B), respectively.

Section 1 contains a compilation of the results which support a positive answer to the syzygy problem. Further we discuss its ring-theoretic formulations which arise from exact sequences of type (A). For example we will see that every third syzygy of rank two is free if and only if every unmixed ideal of height two is perfect (cf. Proposition 1.3).

In Section 2 we concentrate our attention on third syzygies of rank two and exact sequences of type (B). The rank two case proves to have very special consequences which do not appear to be shared by the higher cases. The main advantage in this situation is the self-duality of rank two third syzygies.

Section 3 is an outgrowth of two of the authors paper [14] in which they show how to construct prime ideals P of height two beginning with an arbitrary reflexive module M together with a generic construction of an exact sequence $0 \to F \to M \to P \to 0$ of type (B). In case M is a nonfree third syzygy of rank two, the ring R/P will turn out to be a non–Cohen–Macaulay factorial ring whose completion (for R containing a field of characteristic zero) as well as its power series completion $(R/P)[\![X]\!]$ are not factorial. Thus the syzygy problem has strong connections to the work of Hartshorne and Ogus [22] and touches Danilov's results [8–11] on rings having a discrete divisor class group.

In Section 4 we give a generalization of theorems of Swan [38], Lindel and Lütkebohmert [27], and Quillen [34] which provides a technical result needed in Section 3. More precisely we show that all finitely generated projective modules over the ring $K[\![X_1,..., X_n]\!][X_1^{-1}, Y_1,..., Y_m]$ are free, where K is a field.

Localizing if necessary, one may assume that a possible counterexample M to the syzygy problem is free on the punctured spectrum of R. Consequently the associated sheaf of M represents a vector bundle on the punctured spectrum of R. Thus one is looking for vector bundles of small rank which yield jth syzygies (cf. Hartshorne's article [21] on vector bundles of small rank). The known indecomposable vector bundles of small rank on $\mathbb{P}^n$ give rise to second syzygies of rank two in the case of Horrocks–Mumford ([24]) and rank three in Horrocks' example [23]. The connection between the syzygy problem and a question concerning the extension of vector bundles is shortly discussed at the end of Section 2.

Finally, we would like to explain some additional notation and terminology. The dual of a module M is denoted by M^* and is defined to be the homomorphism module $\mathrm{Hom}_R(M, R)$. There is the usual canonical homomorphism $M \to M^{**}$ which is an isomorphism whenever M is a second syzygy. The notation $\operatorname{syz} M$ indicates the largest integer m such that M is an mth syzygy unless, of course, M is free in which case we take $\operatorname{syz} M = \dim R$. The projective dimension of M is denoted by $\operatorname{pd} M$. It is well known that $\operatorname{pd} M + \operatorname{depth} M = \dim R$ (the dimension of R) and that $\operatorname{syz} M \leqslant \operatorname{depth} M$ (with equality when M is nonfree but free on the punctured spectrum). For a further explanation concerning these topics the reader is referred to the memoir of Auslander and Bridger [1]. The grade of an ideal I is the maximal length of an R-sequence contained in I. We call I unmixed if all prime ideals of R which are associated to R/I have the same grade.

An exact sequence of type (A) or (B) determines the ideal I up to isomorphism only. However, after removing the greatest common divisor the ideal I is uniquely determined and contains an R-sequence of length at least two. We will assume this throughout the paper.

If an ideal I is given by a sequence of type (A) with a nonfree module M, then necessarily $\mathrm{Ext}^1_R(I, M) = \mathrm{Ext}^2_R(R/I, M) \neq 0$. This means that an M-sequence contained in I has length at most two. If M is a third syzygy, then every R-sequence of length three is an M-sequence, and necessarily grade $I = 2$. By a similar, even simpler argument one sees that an ideal I which is given by a sequence of type (B) with a nonfree second syzygy M must also have grade exactly two. Therefore all the ideals which are of interest to us have grade two.

In connection with ideals I of grade two in R, we shall also need a few facts related to local cohomology (cf. Grothendieck's lecture notes [19] for a more complete discussion). The first nonvanishing (positive) extension group in this case is $\mathrm{Ext}^2_R(R/I, R) \cong \mathrm{Ext}^1_R(I, R)$ and is denoted by $\Omega^0_{R/I}$. This module satisfies the Serre condition (S_2) and is a divisorial ideal of R/I in case R/I is a normal domain (cf. [17, p. 54]). For R/I to be Gorenstein one must have that $\Omega^0_{R/I} \cong R/I$ as well as the vanishing of the remaining $\Omega^i_{R/I}$.

The grade and the height of an ideal in a regular local ring coincide. Whereas we preferred to use the perhaps more familiar "height" in the title and the first part of the introduction we will always use "grade" in the following because the length of a maximal R-sequence contained in an ideal is the relevant information for the problems under consideration. Some of our results remain valid if one drops the assumption of regularity on R but insists that all syzygies in question have finite projective dimension. The assumption that the syzygy has finite projective dimension is clearly necessary as the existence of periodic resolutions shows.

1. The General Case

In this section we describe the relations between rank M and syz M as far as they are known to us, and we give some equivalent formulations of the syzygy problem. The property "nth syzygy" was extensively studied in Auslander and Bridger's memoir [1]. In particular, the following results are established in [1] or can easily be derived from [1].

(a) *The module M is an nth syzygy if and only if* $\operatorname{depth} M_P \geqslant \min\{n, \operatorname{depth} R_P\}$ *for all prime ideals P of R.*

(b) *Every second syzygy M is reflexive, that is, the canonical homomorphism $M \to M^{**}$ is an isomorphism.*

(c) *A second syzygy M is an nth syzygy, for $n \geqslant 3$, if and only if* $\operatorname{Ext}_R^i(M^*, R) = 0$ *for* $i = 1,\ldots, n-2$.

(d) *One has the general inequality* $\operatorname{syz} M + \operatorname{pd} M \leqslant \operatorname{depth} R$. *Equality holds if M is locally free on the punctured spectrum of R.*

(e) *If M is an nth syzygy, F is a free R module, and if we have an exact sequence*

$$0 \to M \to F \to N \to 0,$$

then N is an $(n-1)$th syzygy if and only if N_P is a free module for all prime ideals P with $\operatorname{depth} R_P \leqslant n-1$.

(f) *With M and F as in* (e) *and the module N in the exact sequence*

$$0 \to F \to M \to N \to 0,$$

we have that N is an nth syzygy if and only if N_P is a free module for all prime ideals P with $\operatorname{depth} R_P \leqslant n$.

The following theorem contains all the relations between syz M and rank M that are known to us. References are given in the proof.

THEOREM 1.1. *Let M be a nonfree R-module of rank r.*

(a) *If $r = 1$, then* $\operatorname{syz} M \leqslant 1$.

(b) *If* $\operatorname{pd} M = 1$, *then* $\operatorname{syz} M \leqslant r$.

(c) *If R contains the field of rational numbers, then*

$$\operatorname{syz} M \leqslant \frac{(r-1)\operatorname{depth} R + 1}{r}.$$

(d) *If* M has a free resolution of the form

$$0 \to R \to R^m \to R^n \to M \to 0,$$

then $\operatorname{syz} M \leqslant r$.

(e) *If* $r = 2$ *and* M *has a free resolution of the form*

$$0 \to R^2 \to R^m \to R^m \to M \to 0,$$

then $\operatorname{syz} M \leqslant r$.

(f) *The syzygy problem has an affirmative answer if* $\dim R \leqslant 4$.

Proof. (a) Suppose $\operatorname{syz} M \geqslant 2$. Since M is a rank one torsion-free R-module, it is isomorphic to an ideal I of R which is a second syzygy and, hence is reflexive. But reflexive ideals are divisorial and R is a factorial ring. Thus M is a principal ideal. That is, M is free.

(b) This is proved in [20, 39, 7]. The fastest proof seems to be the following one. Let

$$0 \to R^m \xrightarrow{f} R^n \to M \to 0$$

be a minimal free resolution of M. If M were an $(r + 1)$th syzygy, then M_P would be a free R_P-module for all prime ideals P of grade $r + 1$. Thus the ideal J generated by the $m \times m$ minors of f cannot be contained in any prime ideal of grade $r + 1$. On the other hand, by the well-known theorem of Eagon and Northcott [19] the grade of J is less than or equal to $r + 1$.

(c) This bound was essentially given in [26, 4.2] and can be deduced from [25, Satz 1]. We sketch the argument here. If the inequality did not hold, then an easy calculation shows that

$$\operatorname{syz} M \geqslant (r - 1) \operatorname{pd} M + 2.$$

By [25, Satz 1] or [26, 3.1] this inequality implies that the rank one module $\Lambda^r M$ is a second syzygy. Thus by (a) $\Lambda^r M$ is free, and hence M is free.

(d) This is a variant on the argument given in (c). Under this assumption Lebelt's resolution of $\Lambda^r M$ shows that $\Lambda^r M$ is again a second syzygy.

(e) After localizing at a prime ideal P which is minimal among those prime ideals for which M_P is not free, we may assume that M is locally free on the punctured spectrum of R.

If $\operatorname{syz} M \geqslant 3$, then by Lebelt's argument $\Lambda^2 M$ is an ideal I with minimal resolution

$$0 \to R \to F_k \to \cdots \to F_1 \to I \to 0.$$

But M_p being locally free on the punctured spectrum implies that $\Lambda^2 M = I$ is also locally free on the punctured spectrum. So I must be primary to the maximal ideal. Thus I is a zero-dimensional Gorenstein ideal. However, Gorenstein ideals have symmetric minimal free resolutions. Lebelt's description of the free resolution of $\Lambda^2 M$ now shows that $m = 5$. Thus we get a nonfree forth syzygy of rank three and projective dimension 1. This contradicts (b).

(f) By (a) and (b) the first open case is $\operatorname{pd} M = 2$ and $\operatorname{syz} M = 3$. So $\operatorname{depth} R \geqslant \operatorname{pd} M + \operatorname{syz} M \geqslant 5$.

The syzygy problem can be described in terms of the Betti numbers of a module: It has an affirmative answer if and only if for each (minimal) free resolution

$$0 \to F_k \xrightarrow{f_k} \cdots \to F_1 \xrightarrow{f_1} F_0 \to N \to 0$$

$\operatorname{rank} F_i \geqslant 2i + 1$ for $i = 1, \ldots, k-2$ and $\operatorname{rank} F_{k-1} \geqslant \operatorname{rank} F_k + k - 1$. An affirmative answer to the syzygy problem implies the given inequalities since $\operatorname{rank} F_i = \operatorname{rank}(\ker f_i) + \operatorname{rank}(\operatorname{im} f_i)$, and if M is a counterexample to the syzygy problem, one can construct a free resolution which violates the inequalities by the method given in [5, Satz 3].

In the rest of this section we will discuss ideal-theoretic formulations of the syzygy problem.

One way to produce an ideal from a torsion-free R-module is given by a theorem of Bourbaki which we mentioned in the introduction. This connection between ideals and syzygies will be discussed in the following sections.

Another way to construct an ideal from a second syzygy is given in [5, Korollar 1]. Any module M which is a second syzygy can be made the first syzygy of an ideal by an exact sequence

(A) $0 \to M \to R^{r+1} \to I \to 0,$

where r is the rank of M. Since every dual module is a second syzygy, one can do the same for M^* to obtain

(A*) $0 \to M^* \to R^{r+1} \to I' \to 0.$

We recall that, after removing the greatest common divisor of each ideal, the grade of I and I' is exactly two. Using the exact sequence (A*) one obtains the following version of the syzygy problem:

PROPOSITION 1.2. *The syzygy problem has an affirmative answer if and only if the following assertion is true:*

Every $(r+1)$*-generated ideal* I *with* $\operatorname{Ext}^i_R(I, R) = 0$ *for* $i = 2, \ldots, r$ *and* $\operatorname{grade} I = 2$ *is perfect. That is,* $\operatorname{pd} I = 1$ *or, equivalently,* R/I *is a Cohen–Macaulay ring.*

Proof. Let I be an ideal with $\operatorname{Ext}^i_R(I, R) = 0$ for $i = 2, \ldots, r$. Let N be the kernel of an epimorphism $R^{r+1} \to I$ and let $M = N^*$. Then M^* is isomorphic to N and the exact sequence

$$0 \to N \to R^{r+1} \to I \to 0$$

yields isomorphisms $\operatorname{Ext}^{i+1}_R(I, R) \cong \operatorname{Ext}^i_R(M^*, R)$ for $i = 1, \ldots, r-1$. Since M is a second syzygy, it is an $(r+1)$th syzygy by part (c) of the results of Auslander–Bridger mentioned at the start of this section. An affirmative answer to the syzygy problem implies that M, N^*, and, hence, N are all free R-modules. Thus $\operatorname{pd} I = 1$ as desired.

Now let M be a counterexample to the syzygy problem. We form the exact sequence

$$0 \to M^* \to R^{r+1} \to I' \to 0.$$

Once again using part (c) of the Auslander–Bridger results we see that I' violates the ideal-theoretic version of the syzygy problem.

Another ideal-theoretic formulation of the syzygy problem can be derived from the exact sequence (A).

We will restrict ourselves to the case of rank $M = 2$ and leave the general case to the reader.

PROPOSITION 1.3. *The syzygy problem has an affirmative answer in rank 2 if and only if every unmixed three-generated ideal is perfect.*

Proof. Consider the exact sequence

$$0 \to M \to R^3 \to R \to R/I \to 0$$

which we obtain by resolving R modulo a three-generated ideal or by "pushing forward" a second syzygy of rank three. In both cases we may assume that the grade of I is two since unmixed ideals of grade one and three-generated ideals of grade three are always perfect.

We will show that I is unmixed if and only if M is actually a third syzygy. Let M be such a module and let P be a prime ideal of R which contains I and has $\operatorname{depth} R_P \geqslant 3$. Let $Q = P/I$. Then $\operatorname{depth}(R/I)_P = \operatorname{depth} M_P - 2 \geqslant 1$, since $\operatorname{depth} M_P \geqslant \min(3, \operatorname{depth} R_P)$ for all $P \in \operatorname{Spec} R$. So $P \notin \operatorname{Ass} R/I$ as claimed. Reading this argument backwards one obtains that $\operatorname{depth} M_P \geqslant 3$ for all $P \in \operatorname{Spec} R$ such that $\operatorname{depth} R_P \geqslant 3$. If $\operatorname{grade} P \leqslant 2$, then M_P is a free R_P-module since M is a second syzygy. Thus $\operatorname{depth} M_P \geqslant \min(3, \operatorname{depth} R_P)$, and M is a third syzygy by part (a) of the Auslander–Bridger results.

2. Third Syzygies of Rank Two

Throughout this section (as before) R will be a regular local ring. Let M be a finitely generated torsion-free R-module. Then, by Bourbaki's theorem, M is a nonsplit extension of a free module by an ideal J of R. If M is reflexive and not free, then J is necessarily isomorphic to an ideal of grade two. In our first lemma of this section we consider the reverse situation. That is, we discuss when an extension of a free R-module by an ideal of grade two yields a second syzygy.

LEMMA 2.1. *Let R be a regular local ring and J an ideal of grade two.*

(a) *The extension $0 \to R^t \to M \to J \to 0$ has M reflexive if and only if J is unmixed and* $\operatorname{Ext}^1(M, R)_P = 0$ *for each prime ideal P of grade two.*

(b) *Let J be unmixed,* $\operatorname{Ext}^i(J, R) = 0$ *for* $1 < i < j$, *and let* $0 \to R^t \to M \to J \to 0$ *be an extension with* $\operatorname{Ext}^1(M, R) = 0$. *Then M^* is a* $(j+1)$th *syzygy and is nonfree if* $\operatorname{depth} J \leqslant \operatorname{depth} R - 2$. *Moreover, if* $\operatorname{Ext}^1(J, R)$ *can be generated by t elements, then such an extension exists.*

(c) *There is an extension $0 \to R \to M \to J \to 0$ with M reflexive if and only if J is unmixed and $(R/J)_P$ is a complete intersection for each prime ideal P which is minimal with respect to containing J.*

Proof. (a) By part (a) of the criteria mentioned at the start of Section 1, a module M is reflexive if and only if $\operatorname{depth} M_Q \geqslant \min(2, \operatorname{grade} Q)$ for all prime ideals Q of R. If M is reflexive and P is a prime ideal of grade at most two, then M_P is free; hene $\operatorname{Ext}^1_R(M, R)_P = \operatorname{Ext}^1_{R_P}(M_P, R_P) = 0$. If Q is a prime ideal of grade greater than two which is associated to R/J, then $\operatorname{depth}(R/J)_Q = 0$ and $\operatorname{depth} M_Q = 1$ which contradicts the reflexivity of M.

Conversely, if J is unmixed and $\operatorname{Ext}^1(M, R)_P = 0$ for each prime ideal P of grade two and if P is such a prime ideal, then M_P is torsion-free and therefore the projective dimension of M_P is at most one. But $\operatorname{Ext}^1_{R_P}(M_P, R_P) = 0$, so M_P is free. If Q is a prime ideal of grade greater than two, then $\operatorname{depth}(R/J)_Q$ is at least one. Hence $\operatorname{depth} M_Q$ is at least two. Then M is reflexive by the criterion mentioned above.

(b) By part (a) we know that M is reflexive. Moreover, if $\operatorname{depth} J \leqslant \operatorname{depth} R - 2$, then $\operatorname{depth} M \leqslant \operatorname{depth} R - 1$ and, hence neither M nor M^* are free. Let

$$\cdots \to P_j \to P_{j-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

be a minimal projective resolution of M. Dualizing we obtain the complex

$$0 \to M^* \to P_0^* \to P_1^* \to \cdots \to P_{j-1}^* \to P_j^*$$

which is exact until the jth term because $\operatorname{Ext}^i(J,R)=\operatorname{Ext}^i(M,R)=0$ for $2\leqslant i<j$ and $\operatorname{Ext}^1(M,R)=0$. Thus M is at least a $(j+1)$th syzygy.

To establish the existence of a short exact sequence as desired we use a trick of Serre [37] which was generalized by Murthy [3]. Suppose $\operatorname{Ext}^1(J,R)$ can be generated by $\xi_1,\ldots,\xi_t$ where the class of ξ_i is represented by the extension $0\to \mathrm{Re}_i\to M_i\to J\to 0$. One constructs an extension

$$0\to \bigoplus_{i=1}^{t} \mathrm{Re}_i\to M\to J\to 0$$

in $\operatorname{Ext}^1(J,\bigoplus \mathrm{Re}_i)$ so that the pushout

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus \mathrm{Re}_i \arrow[r] \arrow[d, "\pi_i"'] & M \arrow[r] \arrow[d] & J \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Re}_i \arrow[r] & M_i \arrow[r] & J \arrow[r] & 0
\end{tikzcd}$$

along the ith projection map π_i gives the extension ξ_i in $\operatorname{Ext}^1(J,R)$. Thus the natural map $\operatorname{Hom}(R^t,R)\to\operatorname{Ext}^1(J,R)$ is an epimorphism where $R^t=\bigoplus \mathrm{Re}_i$. It follows that $\operatorname{Ext}^1(M,R)=0$.

(c) Assume there is an exact sequence $0\to R\to M\to J\to 0$ with M reflexive. By part (a) we have that J is unmixed. Thus the prime ideals P which are minimal with respect to containing J have grade two.

Since M_P is a free R_P-module of rank two for each prime ideal P with grade $P=2$, the localization of J with respect to such a prime ideal is generated by (at most) two elements. Therefore $(R/J)_P\cong R_P/J_P$ is a complete intersection if P contains J and is of grade two.

To establish the converse we assume that J is unmixed and that R/J is a complete intersection at all of its minimal prime ideals. There are only a finite numer of such ideals, say, $P_1,P_2,\ldots,P_s$, and each P_i is of grade two. Since $\operatorname{Ext}^1(J,R)\cong\Omega^0_{R/J}$ is cyclic at each P_i, we can find an element ξ in $\operatorname{Ext}^1(J,R)$ which is a local generator of $\operatorname{Ext}^1(J,R)_{P_i}$ for $1\leqslant i\leqslant s$. Let the extension $0\to R\to M\to J\to 0$ represent the class of ξ. Thus ξ goes to zero under each of the epimorphisms $\operatorname{Ext}^1(J,R)_{P_i}\to\operatorname{Ext}^1(M,R)_{P_i}$ which implies that $\operatorname{Ext}^1(M,R)_{P_i}=0$ for $1\leqslant i\leqslant s$.

Of course $\operatorname{Ext}^1(M,R)_P=0$ for all prime ideals P of height two which do not contain J. Hence by part (a) it follows that M is reflexive.

COROLLARY 2.2. *If P is a prime ideal of R of grade at most two, then there is an extension $0\to R\to M\to P\to 0$ with M reflexive.*

Proof. This assertion follows immediately from Lemma 2.1(c).

COROLLARY 2.3. *Suppose J is an unmixed ideal of grade two in R such that* $\operatorname{Ext}^1(J, R) \cong \Omega^0_{R/J}$ *is cyclic. Let the extension $0 \to R \to M \to J \to 0$ represent a generator of $\Omega^0_{R/J}$. Then M is reflexive, and M^* is a third syzygy of rank two. M is free if and only if R/J is a Cohen–Macaulay ring.*

Proof. The first statement has just been shown in the proof of part (b) of Lemma 2.1. If M is free, then $\operatorname{pd} J = 1$ and $\operatorname{depth} R/J = \operatorname{depth} R - 2 = \dim R - 2 = \dim R/J$. On the other hand, if R/J is Cohen–Macaulay, then $\operatorname{pd} J = 1$; hence $\operatorname{pd} M \leqslant 1$, and in connection with $\operatorname{Ext}^1(M, R) = 0$ this means that M is free.

The preceding corollary establishes the first part of a correspondence between (nonfree) third syzygies of rank two and a certain class of ideals. This correspondence is the main result of this section and will be given in Theorem 2.5. If M is a rank two reflexive module, one does not need Bourbaki's theorem to produce an exact sequence $0 \to R \to M \to J \to 0$ where J is an ideal. In fact, the kernel of a nonzero homomorphism $f: M \to R$ is a reflexive module of rank one, and hence free. Thus every nonzero homomorphism $f: M \to R$ gives rise to an exact sequence $0 \to R \to M \to J \to 0$.

For the proof of Theorem 2.5 we will need the following lemma:

LEMMA 2.4. *Let A be a local ring such that every associated prime ideal is minimal, and suppose there is a homomorphism $\Phi: A \to \Omega^0_A$ which is an epimorphism locally at each prime ideal P of R such that* $\operatorname{depth} A_P \leqslant 1$. *Then Φ is an isomorphism.*

Proof. We first show that Φ is a monomorphism. If $\dim A = 0$, then A and Ω^0_A have the same length, and Φ being an epimorphism, is also a monomorphism. By induction we may now assume that $\dim A > 0$, and that Φ is a monomorphism locally on the punctured spectrum of A or, equivalenty, $\ker \Phi$ has finite length. Since depth $A > 0$ by the hypothesis on A, the only finite length submodule of A is the zero ideal.

In proving that Φ is also an epimorphism, we may assume (by hypothesis on Φ) that $\operatorname{depth} A \geqslant 2$, and by induction, that coker Φ is of finite length. Since always depth $\Omega^0_A \geqslant \min(2, \dim A)$, the cokernel of the monomorphism $\Phi: A \to \Omega^0_A$ has positive depth, a contradiction unless it is zero.

THEOREM 2.5. *Let R be a regular local ring.*

(a) *A rank two R-module is a third syzygy if and only if $M \cong M^*$ and* $\operatorname{Ext}^1(M, R) = 0$.

(b) *Let M be a nonfree third syzygy of rank two. If the ideal J of grade two is given by an exact sequence $0 \to R \to M \to J \to 0$, then J has the following properties:* (i) *J is unmixed,* (ii) *$\Omega^0_{R/J} \cong R/J$, and* (iii) *R/J is not a Cohen–Macaulay ring (or a Gorenstein ring or a complete intersection).*

(c) *Conversely, let the grade two ideal J satisfy the properties* (i), (ii), *and* (iii) *of part* (b). *If the extension* $0 \to R \to M \to J \to 0$ *represents a generator of* $\operatorname{Ext}^1(J, R) \cong \Omega^0_{R/J}$, *then M is a nonfree third syzygy of rank two.*

Proof. Let us first observe that the properties "Cohen–Macaulay ring," "Gorenstein ring," and "complete intersection" are equivalent for the rings under consideration. If $\Omega^0_{R/J} \cong R/J$, then R/J is Cohen–Macaulay if and only if it is Gorenstein, and a Gorenstein ring R/J such that grade $J = 2$ is a complete intersection by a theorem of Serre [37]. (One should realize that the latter equivalence follows immediately from the last statement of Corollary 2.3. The module M in an exact sequence $0 \to R \to M \to J \to 0$, representing a generator of $\operatorname{Ext}^1(J, R) \cong \Omega^0_{R/J}$, is free.)

A self-dual module M is always reflexive, and is a third syzygy if and only if M^* is a third syzygy. M^* is a third syzygy if and only if $\operatorname{Ext}^1(M^{**}, R) = \operatorname{Ext}^1(M, R) = 0$. For part (a) it thus remains to show that a third syzygy of rank two is self-dual.

Let M be a nonfree third syzygy of rank two, and let the ideal J of grade two be given by an exact sequence $0 \to R \to M \to J \to 0$. By Lemma 2.1, J is unmixed. Dualizing we obtain an exact sequence

$$0 \to J^* \to M^* \to R^* \xrightarrow{\delta} \operatorname{Ext}^1(J, R) \to \operatorname{Ext}^1(M, R) \to 0.$$

Identifying R^* with R, one has $J \subset \ker \delta$. Consequently δ induces a homomorphism $\bar{\delta}: R/J \to \Omega^0_{R/J} \cong \operatorname{Ext}^1(J, R)$. We want to show that $\bar{\delta}$ is an isomorphism. The first hypothesis of Lemma 2.4 is satisfied because J is unmixed. Let P be a prime ideal of R such that $P \supset J$ and $\operatorname{depth}(R/J)_Q \leqslant 1$, where $Q = P/J$. Then $\operatorname{depth} J_P \leqslant 2$ and $\operatorname{depth} R_P \leqslant 3$ since $\operatorname{depth} M_P \geqslant \min(3, \operatorname{depth} R_P)$ and $\operatorname{depth} M_P = \operatorname{depth} J_P$ as soon as $\operatorname{depth} R_P \geqslant 4$. For these prime ideals P the R_P-module M_P is free; hence $\operatorname{Ext}^1(M, R)_P = 0$, and $\bar{\delta}$ is an epimorphism locally at P or, equivalently, at Q. Now Lemma 2.4 applies and shows that $\bar{\delta}$ is an isomorphism. Immediate consequences are that $\operatorname{Ext}^1(M, R) = 0$, that the exact sequence $0 \to R \to M \to J \to 0$ represents a generator of $\operatorname{Ext}^1(J, R)$, and that $\Omega^0_{R/J} \cong R/J$. By the last statement of Corollary 2.3, R/J is not a Cohen–Macaulay ring. This concludes the proof of part (b).

Let us now finish the proof of part (a). We have seen than $\bar{\delta}$ is an isomorphism, in particular $\ker \delta = J$. Thus we have an exact sequence $0 \to R \to M \to J \to 0$. Since M is a third syzygy, $\operatorname{Ext}^1(M^*, R) = 0$ by part (c) of the Auslander–Bridger criteria mentioned at the start of Section 1. Then the exact sequence $0 \to R \to M^* \to J \to 0$ also represents a generator of $\operatorname{Ext}^1(J, R)$, and necessarily $M \cong M^*$.

Part (c), finally, follows immediately from part (a) and Lemma 2.3.

The self-duality exhibited by third syzygies of rank two has been shown by Miller [30] to be a common trait of reflexive rank two modules.

Remark 2.6. If the ring R in Theorem 2.5 is chosen such that it has minimal dimension among all regular local rings with nonfree third syzygies of rank two, then one can say more about R/J: $\operatorname{depth} R/J = 2$, and R/J is locally a complete intersection on its punctured spectrum, because in this case M is locally free on the punctured spectrum of R and $\operatorname{depth} M = 3$. (If there should be a nonmaximal prime ideal P in R with M_P nonfree, one could reduce the dimension by localizing at P, and in case M is locally free on the punctured spectrum of R, but $\operatorname{depth} M \geqslant 4$, the dimension can be decreased by replacing R by R/xR and M by M/xM, where x is a nonunit outside the square of the maximal ideal.)

Remark 2.7. Part (a) of Theorem 2.5 shows that it is very difficult to find counterexamples to the syzygy problem in rank two (if there are any). The known constructions of reflexive modules which are locally free on the punctured spectrum of a local ring R [23, 24, 39] start with a module of rank higher than desired, and then decrease the rank by two methods (which are dual to each other):

(i) Replace M by the kernel of a homomorphism $f \in M^*$ such that im f is primary to the maximal ideal, and

(ii) replace M by M/Rx, where $x \in M$ generates a free direct summand on the punctured spectrum.

The first method fails because ker f is never a third syzygy (one has $\operatorname{pd} \ker f = \operatorname{depth} R - 2$), the second one as a consequence of part (a) of Theorem 2.5.

The self-duality of rank two third syzygies enables us to reprove Theorem 1.1(c) for $r = 2$ (without any restriction on the characteristic of R) as a special case of a more general theorem. Let M be a self-dual module, let

(A) $\quad 0 \to F_k \to \cdots \to F_1 \to F_0 \to M \to 0$

be a free resolution, and

(B) $\quad 0 \to M \to F_0^* - 1 \to \cdots \to F_k^* \to 0$

its dual.

Then one can splice (A) and (B) by an isomorphism $M \to M^*$ and obtains the free complex

(C) $\quad 0 \to F_k \to \cdots \to F_1 \to F_0 \to F_0^* \to F_1^* \to \cdots \to F_k^* \to 0.$

We want to show that $\operatorname{syz} M \leqslant (\dim R + 1)/2$ if M is not free. Assume the contrary. Since $\operatorname{syz} M$ does not decrease under localization, we may assume that M is locally free on the punctured spectrum of R. Then (A) and (B) are split exact on the punctured spectrum of R, and (C) shares this property.

The assumption $\operatorname{syz} M > (\dim R + 1)/2$ implies $k \leqslant \dim R - \operatorname{syz} M < (\dim R - 1)/2$; hence $2k + 1 < \dim R$. Now the lemme d'acyclicité of Peskine–Szpiro [33] shows that (C) is actually exact, and so M is free. We have proved part (a) of the following proposition. Part (b) results from part (a) and Theorem 2.5(a) (after a separate discussion of the elementary cases $\operatorname{syz} M = 0, 1, 2$).

PROPOSITION 2.8. *Let R be a regular local ring.*

(a) *If M is a nonfree self-dual R-module, then* $\operatorname{syz} M \leqslant (\dim R + 1)/2$.

(b) *If M is a nonfree R-module of rank two, then* $\operatorname{syz} M \leqslant (\dim R + 1)/2$.

Our final observation concerning rank two third syzygies deals with graded syzygies M over a polynomial ring $k[X_0, X_1,..., X_n]$, with k a field, such that M corresponds to a vector bundle on $\mathbb{P}^n$. In regard to extending vector bundles from $\mathbb{P}^{n-1}$ to $\mathbb{P}^n$, a negative answer to the sygyzy problem with respect to graded second syzygies, which are locally free away from the irrelevant maximal ideal, would say that one can have an indecomposable rank two vector bundle E on $\mathbb{P}^{n-1}$ with associated graded *reflexive* module M/X_nM where M is the graded reflexive module associated to the extension of E on $\mathbb{P}^n$. Normally one must take the bidual of M/X_nM with respect to the polynomial ring $k[X_0, X_1,..., X_{n-1}]$. The problem of extending rank two vector bundles has been considered by Schwarzenberger [36] and by Barth and Van de Ven [2]. They show that a vector bundle of rank two which can be indefinitely extended to $\mathbb{P}^n$ for arbitrarily large n is necessarily a direct sum of two line bundles. Theorem 1.1 implies a weaker assertion. Associated modules of vector bundles cannot be extended indefinitely to associated modules of vector bundles.

3. RANK TWO THIRD SYZYGIES AND FACTORIAL RINGS

In this section we examine a connection between non–Cohen–Macaulay factorial rings and rank two third syzygies. Samuel [35] asked if every factorial ring is Cohen–Macaulay. In [3] Bertin constructed an example of a non–Cohen–Macaulay factorial ring in characteristic p. Later Freitag and Kiehl [18] gave an example in characteristic 0. Both of these rings are images of regular local rings. If a local ring S is the image of a local ring R, then we say that the codimension of S in R is the dimension of R minus the dimension of S. The codimension of S is the minimum of such codimensions where R varies over all regular local rings which map onto S. If m is the maximal ideal of S, then the codimension of S is easily seen to be the minimal number of generators of m minus the dimension of S. The codimension of the examples of Bertin and of Freitag and Kiehl are all greater than 4. Furthermore it is known [17] that Bertin's example can be completed to give an example of a complete local factorial ring which is not Cohen–Macaulay. In this section we will show that in the equicharacteristic case the existence of nonfree rank two third syzygies is equivalent to the existence of non–Cohen–Macaulay factorial rings of codimension two.

Let S be such a ring, $S = R/J$, R regular, and J a prime ideal of grade two. Ω_S^0 is a divisorial ideal of S, hence principal. Therefore J satisfies the hypothesis of Theorem 2.5(c) which tells us how a nonfree third syzygy of rank two can be produced from J.

Surprisingly, one can also construct a non–Cohen–Macaulay factorial ring of codimension two from a nonfree third syzygy of rank two:

THEOREM 3.1. *Let R be an equicharacteristic regular local ring, and M a nonfree third sygyzy of rank two over R. Then there exists a non–Cohen–Macaulay factorial local ring T of codimension two and with an algebraically closed residue class field.*

Proof. Localizing if needed, we can assume that M is locally free on the punctured spectrum of R. Completing R and M at the maximal ideal, we can suppose that R is a power series ring in n variables $X_1,..., X_n$ over a field K. After an extension of K we may further assume that K is algebraically closed. We now follow the procedure in [14] to obtain the "generic" ideal P which is an "image" of M, and use it to produce a non–Cohen–Macaulay factorial ring.

We will briefly sketch the construction of P here. The reader should refer to [14] for details. Let M be generated by $m_1,..., m_k$ and let S be $R[Y_1,..., Y_k]$. We form the exact sequence

$$0 \to S \to M \otimes S \to P \to 0,$$

where $f(1) = \sum_{i=1}^{k} m_i \otimes Y_i$. By depth considerations there is a one-form $g = \sum_{i=1}^{n} k_i X_i$ of S such that g is not a zero-divisor on S/P. Then P will be a prime ideal if P_g is a prime ideal of S_g. By Theorem 4.1 (cf. the next section) M_g is a free S_g-module. Thus we can proceed as in [14] and conclude that P_g is generated by two new polynomial indeterminates of $R_g[Y_1,..., Y_k]$ over R_g. Hence P_g and P are prime ideals. By Flenner's version of Bertini's theorem [15] there is a one-form g which generates a prime ideal in S/P. Nagata's well-known theorem on factorial rings [32] shows that S/P will be factorial if $(S/P)_g$ is. But, as above, P_g is generated by two polynomial indeterminates. Thus $(S/P)_g$ is isomorphic to $R_g[Z_1,..., Z_{k-2}]$, where the Z_i are indeterminates. Hence $(S/P)_g$ and S/P are factorial as desired. Note that P is contained in the extension Q of the maximal ideal of

R to S. Otherwise $M \otimes S_Q$ would be a free S_Q-module, and this is impossible since S_Q is a faithfully flat extension of R. We finally choose T as $(S/P)_{Q'}$ where Q' is the maximal ideal of S generated by Q and $Y_1,..., Y_k$. T has the same residue class field as R. If T were a Cohen–Macaulay ring, then S_Q/PS_Q, being a localization of T, would also be a Cohen–Macaulay ring. This again would force $M \otimes S_Q$ to be free.

Remark 3.2. If the ring R in Theorem 3.1 was chosen minimal in the sense of Remark 2.6, then depth $M = 3$; hence depth $M \otimes S_Q = 3$ and depth $S_Q/PS_Q = 2$, but $\dim S_Q = \dim R \geqslant 5$ and therefore $\dim S_Q/PS_Q \geqslant 3$: T does not satisfy Serre's condition (S_3). In case char $T = 0$ this implies that neither the completion nor the power series extension $T[\![X]\!]$ are factorial [9, 10, 28].

Remark 3.3. For each field k there exist nonfree third syzygies M of depth three over $k[\![X_1,..., X_n]\!]$, $n \geqslant 4$, which are locally free on the punctured spectrum of this ring. (Simply choose the third syzygy in a resolution of k.) Applying the construction of [14] to this module, one obtains normal local domains T with residue class field k which do not satisfy (S_3) for the same reason as above. If char $k = 0$, then T does not have discrete divisor class group, i.e., the natural homomorphism of the class group of T into the one of $T[\![X]\!]$ is not an isomorphism [9, 10].

4. Projective Modules over Certain Laurent Series Rings are free

In this section we prove that projective modules over certain Laurent series rings are free. This result has already been used in Section 3. It is a variant of a theorem of Swan [38], and extends the theorem of Lindel and Lütkebohmert [27] on projective modules over polynomial algebras of power series rings. The proof follows the one in [27] rather closely, but for the convenience of the reader we will give it in detail.

THEOREM 4.1. *Let K be a field. Then every finitely generated projective module over the ring $K[\![X_1,..., X_n]\!][X^{-1}, Y_1,..., Y_m]$ is free, where X is a one-form in the power series variables $X_1,..., X_n$.*

The main tools of the proof are the Weierstrass division and preparation theorems which will allow us to change a power series variable into a polynomial variable, thereby providing a proof by induction on n. We recall that a Weierstrass polynomial in X_n of degree s is an element

$$w = X_n^s + a_{n-1}X_n^{s-1} + \cdots + a_0 \in K[\![X_1,..., X_{n-1}]\!][X_n],$$

where $a_i \in K[\![X_1,\ldots,X_{n-1}]\!]$. The division theorem is used in the following slightly generalized version given by [27, Lemma]:

LEMMA 4.2. *Let $w \in K[\![X_1,\ldots,X_{n-1}]\!][X_n]$ be a Weierstrass polynomial of degree s. Then every element $h \in K[\![X_1,\ldots,X_n]\!][Y_1,\ldots,Y_m]$ can uniquely be written $h = gw + r$, where $g \in K[\![X_1,\ldots,X_n]\!][Y_1,\ldots,Y_m]$ and $r \in K[\![X_1,\ldots,X_{n-1}]\!][X_n, Y_1,\ldots,Y_m]$ with X_n-*degree *of r at most $s-1$.*

The following lemma essentialy handles the induction step in the proof of Theorem 4.1. It is an extension of [27, Satz 1].

LEMMA 4.3. *Let P be a finitely generated projective $\tilde{R} := K[\![X_1,\ldots,X_n]\!][X_1^{-1}, Y_1,\ldots,Y_m]$-module, $n \geqslant 2$. If there is a Weierstrass polynomial $w \in K[\![X_1,\ldots,X_{n-1}]\!][X_n]$ such that P_w is a free $\tilde{R}_w$-module, then there exists a finitely generated projective $\tilde{S} := K[\![X_1,\ldots,X_{n-1}]\!][X_1^{-1}, X_n, Y_1,\ldots,Y_n]$-module P', from which P is extended: $P \cong P' \otimes_{\tilde{S}} \tilde{R}$.*

Proof. Let

$$R := K[\![X_1,\ldots,X_n]\!][Y_1,\ldots,Y_m]$$

and

$$S := K[\![X_1,\ldots,X_{n-1}]\!][X_n, Y_1,\ldots,Y_m].$$

We choose a $\tilde{R}_w$-basis $e_1,\ldots,e_r$ of P_w and a $\tilde{R}$-system of generators $p'_1,\ldots,p'_s$ of P. After dividing $e_1,\ldots,e_r$ by a sufficiently high power of X_1 and replacing w by a sufficiently high power of itself, we may assume:

$$wp'_i = \sum_{j=1}^{r} a_{ij} e_j \qquad \text{with} \quad a_{ij} \in R.$$

According to Lemma 4.1 we can write $a_{ij} = q_{ij} w + r_{ij}$, and have

$$wp'_i = w\left(\sum_{j=1}^{r} q_{ij} e_j\right) + \sum_{j=1}^{r} r_{ij} e_j.$$

Now let $p_i = p'_i - \sum_{j=1}^{r} q_{ij} e_j$, $i = 1,\ldots,s$. Then

$$wp_i = \sum_{j=1}^{r} r_{ij} e_j \in Se_1 + \cdots + Se_r.$$

Define the S-homomorphism $\Phi_S\colon S^{r+s} \to P$ by sending the first r basis elements $f_1,\ldots,f_r$ of S^{r+s} to $e_1,\ldots,e_r$, respectively, and the last s basis elements $f_{r+1},\ldots,f_{r+s}$ to $p_1,\ldots,p_s$. Consider S^{r+s} as a S-submodule of R^{r+s} in the natural way and simply extend Φ_S to the R-homomorphism

$\Phi_R\colon R^{r+s}\to P$. Let $M=\operatorname{im}\Phi_S$ and $N=\operatorname{im}\Phi_R$. The first assertion we will have to prove is

(A) $N\cong M\otimes_S R$.

Now assume that (A) holds. By the choice of $e_1,\dots,e_r$ and $p_1,\dots,p_s$ one has $P=\tilde R\cdot N$. Since $\tilde R$ is a ring of quotients of R, an elementary argument shows that actually $P\cong\tilde R\otimes_R N$. Another elementary argument, essentially the associativity of the tensor product, then leads to $P\cong(M\otimes_S\tilde S)\otimes_{\tilde S}\tilde R$. So we will finally have to prove:

(B) $M\otimes_S\tilde S$ is a projective $\tilde S$-module.

Proof of (A). We have an exact sequence

$$0\to\ker\varphi_S\to S^{r+s}\to M\to 0.$$

The kernel of $\varphi_S\otimes_S R\colon R^{r+s}\to M\otimes_S R$ is $R\cdot\ker\varphi_S$, the R-submodule of R^{r+s} generated by $\ker\varphi_S$. The sequence

$$0\to R\cdot\ker\varphi_S\to R^{r+s}\to M\otimes_S R\to 0$$

is exact. Because of $R\cdot\ker\varphi_S\subset\ker\varphi_R$ and the exactness of the sequence

$$0\to\ker\varphi_R\to R^{r+s}\to N\to 0$$

it remains to show that $\ker\varphi_R\subset R\cdot\ker\varphi_S$.

Let $v_i:=\sum_{j=1}^r r_{ij}f_j-wf_{r+i}$, $i=1,\dots,s$. Then clearly $v_i\in\ker\varphi_S$, $i=1,\dots,s$. Assume that

$$x'=\sum_{j=1}^r A_j'f_j+\sum_{i=1}^s B_if_{r+i}\in\ker\varphi_R.$$

Decompose $B_i=Q_iw+R_i$ according to Lemma 4.2. This yields

$$\begin{aligned}\sum_{i=1}^s B_if_{r+i}&=\sum_{i=1}^s Q_iwf_{r+i}+\sum_{i=1}^s R_if_{r+i}\\&=\sum_{i=1}^s Q_i\left(\sum_{j=1}^r r_{ij}f_j-v_i\right)+\sum_{i=1}^s R_if_{r+i}\\&=-\sum_{i=1}^s Q_iv_i+\sum_{j=1}^r\left(\sum_{i=1}^s Q_ir_{ij}\right)f_j+\sum_{i=1}^s R_if_{r+i}.\end{aligned}$$

Since $v_i\in\ker\varphi_S$, $i=1,\dots,s$, it suffices to prove that the elements $x\in\ker\varphi_R$ such that

$$x=\sum_{j=1}^r A_jf_j+\sum_{i=1}^s R_if_{r+i},\qquad\text{with}\quad A_j\in R\quad\text{and}\quad R_i\in S,$$

are in $R \cdot \ker \varphi_S$. We have

$$\sum_{j=1}^{r} A_j w e_j = -\sum_{i=1}^{s} R_i w p_i = -\sum_{i=1}^{s} R_i\left(\sum_{j=1}^{r} r_{ij} e_j\right)$$
$$= \sum_{j=1}^{r} S_j e_j \qquad \text{with} \quad S_j \in S$$

because $\varphi_R(wx) = 0$, $R_i \in S$, $r_{ij} \in S$. By the linear independence of $e_1, \dots, e_r$: $A_j w = S_j \in S$, $j = 1, \dots, r$. A simple argument based on the uniqueness of the Weierstrass division shows that this is only possible if $A_j \in S$, $j = 1, \dots, r$. Therefore $\ker \varphi_R \subset R \cdot \ker \varphi_S$. This concludes the proof of (A).

Proof of (B). Let $\tilde{M} := M \otimes_S \tilde{S}$, ad Q be a maximal ideal of $\tilde{S}$. First assume that $w \notin Q$. It is not hard to check that the $\tilde{S}_Q$-homomorphism

$$\psi\colon \tilde{S}_Q^{r+s} \to \ker \varphi_S \otimes_S \tilde{S}_Q$$

given by

$$\psi(f_j) = 0, \qquad j = 1, \dots, r$$

and

$$\psi(f_{r+i}) = f_{r+i} - w^{-1} \sum_{j=1}^{r} r_{ij} f_j$$

is surjective and a section to the natural inclusion $\ker \varphi_S \otimes_S \tilde{S}_Q \subset \tilde{S}_Q^{r+s}$. Therefore the exact sequence

$$0 \to \ker \varphi_S \otimes_S \tilde{S}_Q \to \tilde{S}_Q^{r+s} \to \tilde{M}_Q \to 0$$

splits as required.

Now assume $w \in Q$. We already know that $P = \tilde{M} \otimes_{\tilde{S}} \tilde{R}$. Therefore it is enough to prove the existence of a maximal ideal in R, which lies over Q, because the extension $\tilde{S} \to \tilde{R}$ is flat. By the Weierstrass division $R/Rw = S/Sw$; hence $\tilde{R}/\tilde{R}w = \tilde{S}/\tilde{S}w$, and the maximal ideals of $\tilde{S}$ containing w. This concludes the proof of (B).

Proof of Theorem 4.1. We may assume $X = X_1$ without restriction. We proceed by induction on the number n of power series variables. Since $K[\![X_1]\!][X_1^{-1}]$ is a field, the case $n = 1$ is covered by the theorem of Quillen–Suslin [34] on projective modules over polynomial rings over fields.

Now assume $n \geqslant 2$. Let T be the set of nonzero elements of $K[\![X_1,..., X_n]\!]$. Again the theorem of Quillen–Suslin shows that projective modules over

$$\begin{aligned} &K[\![X_1,..., X_n]\!][X_1^{-1}, Y_1,..., Y_n]_T \\ &\quad \cong K[\![X_1,..., X_n]\!]_T\,[Y_1,..., Y_n] \end{aligned}$$

are free. Therefore one can find an element $w' \in K[\![X_1,..., X_n]\!]$ such that $P_{w'}$ is a free $K[\![X_1,..., X_n]\!][X_1^{-1}, Y_1,..., Y_m]_{w'}$-module. The element X_1 is a unit in $K[\![X_1,..., X_n]\!][X^{-1}, Y_1,..., Y_m]$. Hence we may assume that w' as an element of $K[\![X_1,..., X_n]\!]$ is not divisible by X_1: $w' = f + X_1 g$ where $f \neq 0$ is a power series in the variables $X_2,..., X_n$ only. After a linear substitution involving these variables only, we may assume that $w'(0,..., 0, X_n) \neq 0$. Now the Weierstrass preparation theorem implies that $w' = ew$, where e is a unit in $K[\![X_1,..., X_n]\!]$ and w a Weierstrass polynomial in $K[\![X_1,..., X_{n-1}]\!][X_n]$. Lemma 4.3 shows that P is the extension (by induction hypothesis) of a free $K[\![X_1,..., X_{n-1}]\!][X_1^{-1}, X_n, Y_1,..., Y_m]$-module, and therefore is free itself.

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