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THE EISENBUD-EVANS GENERALIZED PRINCIPAL IDEAL THEOREM AND DETERMINANTAL IDEALS

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ABSTRACT. In [2] Eisenbud and Evans gave an important generalization of Krull's Principal Ideal Theorem. However, their proof, using maximal Cohen-Macaulay modules, may have limited the validity of their theorem to a proper subclass of all local rings. (Hochster proved the existence of maximal Cohen-Macaulay modules for local rings which contain a field, cf. [4]). In the first section we present a proof which is simpler and guarantees the Generalized Principal Ideal Theorem for all local rings. The main result of the second section was conjectured in [2]. Under a hypothesis typically being satisfied for the most important fitting invariant of a module, it improves the Eagon-Northcott bound [1] on the height of a determinantal ideal considerably. Finally we will discuss the implications of a recent theorem of Faltings [3] on determinantal ideals.

1. The Generalized Principal Ideal Theorem. We recall some notations from [2]. Let R be a commutative noetherian ring, and M a finitely generated R-module. The *order ideal* $M^*(x)$ of an element $x \in M$ is given by

$$M^*(x) := \{f(x): f \in M^*\},$$

where M^* denotes the dual $\operatorname{Hom}_R(M, R)$ of M. Since M is finitely presented, the formation of $M^*(x)$ commutes with flat ring extensions, in particular with localizations, completions, and the adjunction of indeterminates. The *rank* of M is the maximum of $\dim_{R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}} M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$, $\mathfrak{p}$ ranging over the minimal primes of R. For all unexplained notations and terminology we refer the reader to [7].

Theorem 1 below extends Theorem 1.1 of [2] to all (local) rings R. It was named "Generalized Principal Ideal Theorem" because one recovers Krull's Principal Ideal Theorem for elements $x_1, \ldots, x_m \in R$ from it by specializing M to R^m and x to $(x_1, \ldots, x_m) \in R^m$. (Theorem 1 was called the "Eisenbud-Evans Principal Ideal Conjecture" in [5].)

THEOREM 1. *Let R be a noetherian ring, M a finitely generated R-module, and $x \in M$. If there is a prime ideal $\mathfrak{p}$ of R with $x \in \mathfrak{p}M_\mathfrak{p}$, then*

$$\operatorname{ht} M^*(x) \leqslant \operatorname{rank} M.$$

PROOF. It is enough to prove $\operatorname{ht} M_\mathfrak{q}^*(x) \leqslant \operatorname{rank} M_\mathfrak{q}$ for a prime ideal $\mathfrak{q}$ of R (with $x \in \mathfrak{q}R_\mathfrak{q}$): By the way rank M was defined, it cannot increase under localization, and $\operatorname{ht} M^*(x) \leqslant \operatorname{ht} M_\mathfrak{q}^*(x)$ simply because $(M^*(x))_\mathfrak{q} = M_\mathfrak{q}^*(x)$.

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Let us first assume that there is a prime ideal $\mathfrak{q}$ of R such that $M_{\mathfrak{q}}$ is a free $R_{\mathfrak{q}}$-module and $x \in \mathfrak{q}M_{\mathfrak{q}}$. Then $\operatorname{ht} M_{\mathfrak{q}}^*(x) \leqslant \operatorname{rank} M_{\mathfrak{q}}$ by Krull's Principal Ideal Theorem since $M_{\mathfrak{q}}^*$ is generated by $\operatorname{rank} M_{\mathfrak{q}}$ elements.

In the general case we may assume that R is local with maximal ideal $\mathfrak{p}$. We may even suppose that R is a complete local ring, height and rank being stable under completion. Finally we can factor out a minimal prime ideal $\mathfrak{q}$ of R for which $\operatorname{ht} M^*(x) = \operatorname{ht}(M^*(x) + \mathfrak{q})/\mathfrak{q}$, cf. [2]. So we only need to prove the theorem for universally catenarian local domains.

There are elements $e_1, \ldots, e_m \in M$ such that $x = a_1e_1 + \cdots + a_me_m$ with $a_i \in \mathfrak{p}$. Let S denote the localization of $R[T_1, \ldots, T_m]$ with respect to the maximal ideal generated by $\mathfrak{p}$ and the indeterminates $T_1, \ldots, T_m$. The ideal

$$\mathfrak{r} := S(a_1 + T_1) + \cdots + S(a_m + T_m)$$

is a prime ideal of S with $\mathfrak{r} \cap R = \{0\}$. Thus

$$(M \otimes S)_{\mathfrak{r}} = M_{(0)} \otimes S_{\mathfrak{r}}$$

is a free $S_{\mathfrak{r}}$-module ($M_{(0)}$ denotes the localization of M with respect to the zero-ideal of R). The element

$$y := (a_1 + T_1)e_1 + \cdots + (a_m + T_m)e_m$$

is contained in $\mathfrak{r}(M \otimes S)$. By what has been shown above, $\operatorname{ht}(M \otimes S)^*(y) \leqslant \operatorname{rank} M \otimes S = \operatorname{rank} M$.

S is a catenarian ring. Consequently, there is a prime ideal $\mathfrak{q}$ of S containing $(M \otimes S)^*(y)$ as well as $T_1, \ldots, T_m$, such that $\operatorname{ht} \mathfrak{q} \leqslant \operatorname{rank} M + m$. Then $\mathfrak{q}$ must also contain a minimal prime ideal $\tilde{\mathfrak{q}}$ of $(M \otimes S)^*(x) = M^*(x)S$. All minimal prime ideals of $M^*(x)S$ are extended from prime ideals of R. Therefore

$$\tilde{\mathfrak{q}} \subset \tilde{\mathfrak{q}} + ST_1 \subset \cdots \subset \tilde{\mathfrak{q}} + ST_1 + \cdots + ST_m$$

is a strictly ascending chain of prime ideals, whence

$$\operatorname{ht} M^*(x) = \operatorname{ht} M^*(x)S \leqslant \operatorname{ht} \tilde{\mathfrak{q}} \leqslant \operatorname{rank} M.$$

The depth (or grade) of an ideal $\mathfrak{a}$ with respect to an (arbitrary) R-module N, i.e., the length of a maximal N-sequence contained in $\mathfrak{a}$, is bounded above by $\operatorname{ht} \mathfrak{a}$. Therefore Theorem 1 implies the corresponding inequality for depth. Being immediate consequences of Theorem 2.1 of [2], Corollaries 1.2 and 1.3 of [2] become valid for all local rings.

2. Determinantal ideals. As above, let R be a commutative noetherian ring. The ideal generated by the determinants of the $t \times t$ submatrices of an $m \times n$ matrix φ over R is denoted by $I_t(\varphi)$ (with the usual conventions, $I_t(\varphi) = R$ for $t \leqslant 0$ and $I_t(\varphi) = 0$ for $t > \min(m, n)$). We define the *kth fitting invariant* $F_k(M)$ of a finitely generated R-module M [6] to be the ideal $I_{n-k}(\varphi)$ where φ represents a homomorphism $R^m \to R^n$ such that $M = \operatorname{Coker} \varphi$. The fitting invariants determine the level sets of the (locally constant) function assigning to each prime ideal $\mathfrak{p}$ the minimal number of generators of the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$: $\mathfrak{p} \supset F_k(M)$ whenever $M_{\mathfrak{p}}$ cannot be spanned by fewer than $k + 1$ elements.

Let us say that M has f-*rank* r if $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module of constant rank r for all associated primes $\mathfrak{p}$ of R. The f-rank of M is denoted by frk M. (This is the definition of rank proposed in [**8**].) In general, not every module has an f-rank. However, when R is an integral domain or M has a finite free resolution, then frk M is defined, and, in the latter case, given by the Euler characteristic of a finite free resolution. The reader will check that frk $M = r$ if and only if $\mathrm{F}_r(M)$ contains a nonzero divisor and $\mathrm{F}_{r-1}(M) = 0$. Furthermore, in case frk $M = r$, a localization $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module if and only if $\mathfrak{p} \not\supseteq \mathrm{F}_r(M)$. This property renders $\mathrm{F}_r(M)$ the most important of all fitting invariants and explains a great deal of our interest in a bound on ht $\mathrm{I}_t(\varphi)$ under the condition $\mathrm{I}_{t+1}(\varphi) = 0$.

The classical bound on the height of determinantal ideals was given by Eagon and Northcott in [**1**, Theorem 3]:

$$\mathrm{ht}\, \mathrm{I}_t(\varphi) \leqslant \mathrm{EN}(m, n, t) := (m - t + 1)(n - t + 1)$$

for $t = 1, \ldots, \min(m, n)$, regardless of any hypothesis on φ (except, of course, $\mathrm{I}_t(\varphi) \neq R$). The "generic" case, in which φ is a matrix of indeterminates over the integers, demonstrates that the Eagon-Northcott bound is optimal in general. One then has $\mathrm{ht}\, \mathrm{I}_t(\varphi) = \mathrm{EN}(m, n, t)$ and, hence, $\mathrm{ht}\, \mathrm{I}_t(\varphi)/\mathrm{I}_{t+1}(\varphi) = m + n - 2t + 1$. The last equation presumably led Eisenbud and Evans to conjecture the following theorem [**2**, Conjecture 2.6]:

THEOREM 2. *Let R be a commutative noetherian ring, and φ an $m \times n$ matrix over R. If $\mathrm{I}_t(\varphi) \neq R$ and $\mathrm{I}_{t+1}(\varphi) = 0$, then*

$$\mathrm{ht}\, \mathrm{I}_t(\varphi) \leqslant m + n - 2t + 1.$$

PROOF. We use induction on t, and may restrict ourselves to complete local integral domains R and ideals $\mathrm{I}_t(\varphi)$ primary to the maximal ideal $\mathfrak{m}$ of R. Let

$$\varphi = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix}.$$

If there is an $x_{ij} \notin \mathfrak{m}$, then one reduces the assertion to the case $t - 1$ by applying elementary row and column operations to φ. So we may assume that all $x_{ij} \in \mathfrak{m}$, and, by induction on n, that there is a prime ideal $\mathfrak{p} \neq \mathfrak{m}$ containing $\mathrm{I}_t(\varphi')$, where

$$\varphi' = \begin{bmatrix} x_{11} & \cdots & x_{1n-1} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mn-1} \end{bmatrix}.$$

We claim $\mathrm{ht}\, \mathrm{I}_t(\varphi') \leqslant n - t$. Consider φ as a map $R^m \to R^n$ and, correspondingly, φ' as a map $R^m \to R^{n-1}$. Let $M := \mathrm{Coker}\, \varphi$ and $M' := \mathrm{Coker}\, \varphi'$. M' is isomorphic to $M/R\bar{e}_n$, $e_1, \ldots, e_n$ denoting the elements of the canonical basis of R^n. Since $\mathrm{I}_t(\varphi) \not\subset \mathfrak{p}$, $M_{\mathfrak{p}}$ needs exactly $n - t$ generators. So does $M'_{\mathfrak{p}}$ because $\mathrm{I}_t(\varphi') \subset \mathfrak{p}$. Necessarily $\bar{e}_n \in \mathfrak{p}M_{\mathfrak{p}}$, and $\mathrm{ht}\, M^*(\bar{e}_n) \leqslant \mathrm{rank}\, M = n - t$ by Theorem 1. Regarding the determinantal relations of the columns of φ as elements of R^{n*} which vanish on $\operatorname{Im}\varphi$ (the submodule of R^n generated by the rows of φ) we conclude $I_t(\varphi') \subset M^*(\bar{e}_n)$, and obtain the claim.

In complete local domains the equation $\operatorname{ht}\mathfrak{a} + \dim R/\mathfrak{a} = \dim R$ holds for all ideals $\mathfrak{a}$. Consequently Theorem 2 is settled once we have shown that $\dim R/I_t(\varphi') = \operatorname{ht} I_t(\varphi)/I_t(\varphi') \leqslant m - t + 1$.

LEMMA. *Let R be a local ring, and φ an $m \times n$ matrix over R, whose last column consists of elements in the maximal ideal $\mathfrak{m}$ of R. Let φ' be the matrix formed by the first $n-1$ columns of φ. If $I_t(\varphi') = 0$, then*

$$\operatorname{ht} I_t(\varphi) \leqslant m - t + 1.$$

The lemma just extends Theorem 2.1 of [2] to all local rings. The following hint will enable the reader to prove it. Consider the transpose of φ and adjoin a column to it:

$$\tilde{\varphi} := \begin{bmatrix} x_{11} & \cdots & x_{m1} & 0 \\ \vdots & & & \vdots \\ x_{1n-1} & \cdots & x_{mn-1} & 0 \\ x_{1n} & \cdots & x_{mn} & -1 \end{bmatrix}.$$

Now $\tilde{\varphi}$ and φ are related in the same way as φ and φ' in the proof of Theorem 2, and $\bar{e}_{m+1} = x_{1n}\bar{e}_1 + \cdots + x_{mn}\bar{e}_m \in \mathfrak{m}M$, the notations corresponding to those above.

COROLLARY 1. *Let R be as in Theorem 2, φ an $m \times n$ matrix over R, and ψ a $u \times v$ submatrix of φ such that all coefficients of φ outside ψ generate a proper ideal of R. If $I_t(\varphi) \neq R$, then*

$$\operatorname{ht} I_t(\varphi)/I_{t+k}(\psi) \leqslant \operatorname{EN}(m, n, t) - \operatorname{EN}(u, v, t + k)$$

for all $k = 0, \ldots, \min(u, v) - t + 1$.

PROOF. After the by now usual reduction to the case of a complete local domain, one applies Theorem 2 inductively to obtain the assertion in the case $\varphi = \psi$. Then one uses the lemma to complete the proof by induction on $(m + n) - (u + v)$.

Corollary 1, essentially predicted in [2], generalizes the Eagon-Northcott bound, to which it specializes for $\varphi = \psi$, $t + k = \min(m, n) + 1$. It does not say (in general): $\operatorname{ht} I_t(\varphi) \geqslant \operatorname{EN}(m, n, t)$ implies $\operatorname{ht} I_{t+k}(\psi) \geqslant \operatorname{EN}(u, v, t + k)$. The corresponding statement for $\dim R - \dim R/I_t(\varphi)$ and $\dim R - \dim R/I_{t+k}(\psi)$, however, is always true (cf. [2, proof of Corollary 2.4]). Again the reader should observe that the inequalities for height imply the corresponding inequalities for depth.

We now return to the interpretation of determinantal ideals as fitting invariants. For a closed subset A of $\operatorname{Spec} R$ we put

$$\operatorname{codim} A := \min\{\operatorname{ht}\mathfrak{p} : \mathfrak{p} \in A\}.$$

For every finitely generated R-module M

$$\operatorname{Nf} M := \{\mathfrak{p} \in \operatorname{Spec} R : M_{\mathfrak{p}} \text{ is not a free } R_{\mathfrak{p}}\text{-module}\}$$

is a closed subset of Spec R and consists of the prime ideals $\mathfrak{p} \supset \mathrm{F}_r(M)$ in case frk $M = r$, as was noted above.

COROLLARY 2. *Let R be as in Theorem 2, and M a finitely generated R-module with an* f-*rank. Let N be a second syzygy of M. If M is not free, then*

$$\operatorname{codim} \operatorname{Nf} M \leqslant \operatorname{frk} M + \operatorname{frk} N + 1.$$

PROOF. Consider an exact sequence

$$0 \to N \to R^m \xrightarrow{\varphi} R^n \to M \to 0$$

and put $t := n - \operatorname{frk} M$. Then $\mathrm{I}_t(\varphi) \neq R$, $\mathrm{I}_{t+1}(\varphi) = 0$, $\operatorname{frk} N = m - t$, and the conclusion follows from Theorem 2.

It would be extremely interesting to construct modules over regular local rings for which the bound in Corollary 2 is attained. It easy to write down examples with rank $N = 0$ (equivalently, proj dim $M = 1$), and rather nontrivial ones with rank $N = 1$ can be found in [9], but we know of no such modules with rank $M > 1$ and rank $N > 1$.

In our last corollary $\mu(N)$ shall denote the minimal number of generators of an R-module N.

COROLLARY 3. *Let R be as in Theorem 2, M a torsion-free R-module with an* f-*rank. Then*

$$\operatorname{codim} \operatorname{Nf} M \leqslant \mu(M) + \mu(M^*) - 2(\operatorname{frk} M) + 1.$$

PROOF. Let $m := \mu(M)$, $n := \mu(M^*)$, and choose generators $x_1, \ldots, x_m$ of M and $f_1, \ldots, f_n$ of M^*. Let φ be the $m \times n$ matrix $(f_j(x_i))$. Then $\operatorname{Im} \varphi = M/U$, where U is the kernel of the natural homomorphism $M \to M^{**}$. Since M has an f-rank, U is a torsion module and thus $U = 0$. It is easy to check that $\operatorname{Nf} M = \operatorname{Nf} \operatorname{Coker} \varphi$, $\operatorname{frk} \operatorname{Coker} \varphi = \mu(M^*) - \operatorname{frk} M$, and $\operatorname{frk} \ker \varphi = \mu(M) - \operatorname{frk} M$. Now the conclusion follows at once from Corollary 2.

Theorem 3, which is a consequence of a theorem of Faltings [3], gives a better bound on ht $\mathrm{I}_t(\varphi)$, provided R is regular and t is small compared to m or n.

THEOREM 3. *Let R be a regular local ring, and φ an $m \times n$ matrix over R. If $\mathrm{I}_t(\varphi) \neq R$ and $\mathrm{I}_{t+1}(\varphi) = 0$, then*

$$\operatorname{ht} \mathrm{I}_t(\varphi) \leqslant \max(n, m - t + 1).$$

PROOF. Localizing with respect to a minimal prime ideal of $\mathrm{I}_t(\varphi)$, we may suppose $\mathrm{I}_t(\varphi)$ primary to the maximal ideal of R. Regard φ as a map of $R^m \to R^n$, and put $M := \operatorname{Coker} \varphi$. If $\dim R > n$, then by Satz 1 of [3], $n - t$ among the residues $\bar{e}_1, \ldots, \bar{e}_n$ of the canonical basis of R^n, say, $\bar{e}_{t+1}, \ldots, \bar{e}_n$, generate a free direct summand of rank $n - t$ in every localization $M_\mathfrak{p}$, $\mathfrak{p}$ nonmaximal. Therefore $M' := M/R\bar{e}_{t+1} + \cdots + R\bar{e}_n$ has finite length. Now M' is isomorphic to $\operatorname{Coker} \varphi'$, φ' consisting of the first t columns of φ. $\mathrm{I}_t(\varphi')$ is again primary to the maximal ideal of R, hence $\dim R \leqslant m - t + 1$ by Theorem 2 (for the classical case of maximal minors).

Faltings gives his theorem in a more general setting. For complete local domains the inequality of Theorem 3 becomes

$$\operatorname{ht} I_t(\varphi) \leqslant \max(m + \operatorname{embdim} R - \dim R, n - t + 1),$$

embdim R denoting the embedding dimension of R, i.e., the minimal number of generators of the maximal ideal of R. For the most general case cf. [**3**].

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