

MODULES DEFINED BY GENERIC SYMMETRIC AND ALTERNATING MAPS

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The object of this note is to prove a conjecture of the first author ([Br], p. 192) concerning the perfection of modules defined by generic symmetric or alternating maps of a given rank.

Let B be a commutative noetherian ring, X_{ij} , $1 \leq i \leq j \leq n$, a family of indeterminates, $X_{ji} = X_{ij}$, and X the symmetric matrix (X_{ij}) . For an integer r , $0 \leq r \leq n$, put $R_r = B[X]/I_{r+1}(X)$, $I_{r+1}(X)$ denoting the ideal generated by the $(r+1)$ -minors of X . Then the residue class x of the matrix X defines an R_r -homomorphism $x: F \rightarrow F^*$, $F = (R_r)^n$, which for obvious reasons may be called the *generic symmetric map of rank r* over B . The modules we are interested in are $M = \text{Coker } x$, $M^* = \text{Ker } x$, and $\text{Im } x$.

Analogously a *generic alternating map* is defined with respect to an alternating matrix X of indeterminates, $X_{ij} = -X_{ji}$, $X_{ii} = 0$. The necessarily even rank r is now fixed by the vanishing of the $(r+2)$ -pfaffians of X : one considers $R_r = B[X]/\text{Pf}_{r+2}(X)$ and the map $x: F \rightarrow F^*$ as above.

The results of [Br] for generic $m \times n$ matrices were proved by the method of principal radical systems and a duality argument. In [BV], Section 13 the former has been replaced by filtrations and depth bounds based on the straightening law of the underlying ring. In the alternating case one again has an algebra with straightening law, cf. [DEP], and, mutatis mutandis, the arguments of [BV] go through. In the symmetric case we will draw upon Kutz's paper [Ku]. In both cases local duality will be needed, too. Its application here is easier than in [Br] and [BV], since $B[X]/\text{Pf}_{r+2}(X)$ is Gorenstein (over a Gorenstein B), cf. [KL], and $B[X]/I_{r+1}(X)$, X symmetric, has a very simple canonical module, if it is not Gorenstein [Go].

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LEMMA 2. Let $B = \mathbf{Z}$ (or any Cohen–Macaulay ring), and $\mathfrak{p} \subset R = R_r$, a prime ideal, $\mathfrak{p} \supset I_1(x)$. Then

$$\text{depth } M_{\mathfrak{p}} \geq \text{depth } R_{\mathfrak{p}} - r \geq \frac{1}{2} \text{depth } R_{\mathfrak{p}}.$$

Proof. By [Ku]

$$\text{depth } R_{\mathfrak{p}} \geq nr - r(r-1)/2,$$

implying the second inequality.

In order to prove the first inequality we put $M_i = \sum_{j=i+1}^n R\bar{e}_j$, \bar{e}_j denoting the residue class in M of the j th canonical basis element of F^* . One has a filtration

$$M = M_0 \supset M_1 \supset \dots \supset M_r.$$

We claim: (i) M_r is a free R -module.

(ii) The annihilator J_i of M/M_i is the ideal generated by the i -minors of the first i columns of x .

(iii) The generator \bar{e}_i of M_{i-1}/M_i is linearly independent over R/J_i .

It follows from these claims that M has a filtration with quotients R and R/J_i , $i = 1, \dots, r$. By [Ku] all these rings are Cohen–Macaulay, and $\dim R/J_i = \dim R + i - r - 1$.

Claim (i) is clear: $\text{rank } x = r$, and the first r columns are linearly independent, hence $\text{rank } M/M_r = 0 = \text{rank } M - (n - r)$. Since M/M_i is represented by the matrix $(x|i)$ consisting of the first i columns of x , $\text{Ann } M/M_i \supset J_i$. On the other hand the first $i - 1$ columns of $(x|i)$ are linearly independent over R/J_i (again by [Ku]), and by the same argument as used for (i) one concludes (iii) and (ii). ■

Now we can already prove part (a) of the proposition under whose hypotheses $R = R_r$ is a Gorenstein ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Arguing inductively one may suppose that $M_{\mathfrak{q}}$ is a maximal Cohen–Macaulay module for all primes \mathfrak{q} strictly contained in \mathfrak{p} , and, by Lemma 1, that $\mathfrak{p} \supset I_1(x)$. Let $D = \text{Coker } x^*$ be the Auslander–Bridger dual of M . Of course $D \cong M$. Lemma 2 and the assumptions so far imply that $M_{\mathfrak{p}}$ is a d th syzygy module, $d = \text{depth } M_{\mathfrak{p}}$, hence

$$\text{Ext}_{R_{\mathfrak{p}}}^i(M_{\mathfrak{p}}, R_{\mathfrak{p}}) = \text{Ext}_{R_{\mathfrak{p}}}^i(D_{\mathfrak{p}}, R_{\mathfrak{p}}) = 0 \quad \text{for } i = 1, \dots, d,$$

(cf. [BV], Section 16 for example). On the other hand $\text{depth } M_{\mathfrak{p}} \geq d$ is equivalent to

$$\text{Ext}_{R_{\mathfrak{p}}}^i(M_{\mathfrak{p}}, R_{\mathfrak{p}}) = 0 \quad \text{for } i = \text{depth } R_{\mathfrak{p}} - d + 1, \dots, \text{depth } R_{\mathfrak{p}}$$

by local duality. Hence $\text{Ext}_{R_{\mathfrak{p}}}^i(M_{\mathfrak{p}}, R_{\mathfrak{p}}) = 0$ for all $i > 0$, and $M_{\mathfrak{p}}$ is a maximal Cohen–Macaulay module.

Part (b) finally follows by induction, the numerical information supplied above, and:

LEMMA 3. Let $B = \mathbf{Z}$.

- (a) As an (R/J_r) -module M/M_r is reflexive.
- (b) Its dual over R/J_r is isomorphic to J_{r-1}/J_r .
- (c) M/M_r is a maximal Cohen–Macaulay module over R/J_r .

(In order to include the case $r = 1$: A 0-minor has the value 1.)

Proof. To simplify the notation write \bar{R} for R/J_r and \bar{M} for M/M_r . Let us first observe that (c) holds in case $n \equiv r+1$ (2) since, as has just been proved, M is a maximal Cohen–Macaulay module over R .

Next one notices that the case $r = 1$ is indeed trivial, M/M_1 being free of rank 1 over R/J_1 . Suppose that $r > 1$ and proceed by induction. Then, via M and Lemma 1, it follows that \bar{M}_p is a maximal Cohen–Macaulay module over \bar{R}_p for all $p \in \text{Spec } \bar{R}$, $p \not\supseteq I_1(x)/J_r$.

For (a) it is enough to show that (i) \bar{M}_p is free for all primes p such that $\text{depth } R_p \leq 1$, and (ii) $\text{depth } \bar{M}_p \geq 2$ for the remaining ones. (i) is clear: $\text{grade } I_{r-1}(x|r)/J_r \geq 2$, and \bar{M}_p is free if $p \not\supseteq I_{r-1}(x|r)/J_r$. In order to verify (ii) one may now assume that $n \geq r+2$, $r > 1$, and $p \supseteq I_1(x)/J_r$. Then Lemma 2 implies (ii).

The dual of \bar{M} is isomorphic to the kernel of the map $\bar{R}^r \rightarrow \bar{R}^n$ defined by the transpose y of $(x|r)$. Taking the determinantal relations of the rows of y , one sees that J_{r-1}/J_r is embedded in $\text{Ker } y$ such that this embedding splits at all prime ideals not containing $I_{r-1}(x|r)/J_r$, in particular at all primes p such that $\text{depth } R_p \leq 1$. Since J_{r-1}/J_r is a maximal Cohen–Macaulay module over \bar{R} , (b) follows easily.

It remains to prove (c) for $n \not\equiv r+1$ (2). In this case J_r is the canonical module of R , so $\bar{R} = R/J_r$ is a Gorenstein ring, cf. [HK], 6.13. By (b), the dual of \bar{M} is Cohen–Macaulay, so is \bar{M} by (a). ■

Once proved over \mathbf{Z} , Lemma 3 holds over any Cohen–Macaulay ring B , in fact over arbitrary B if one replaces the Cohen–Macaulay property by perfection.

2. Generic alternating maps

Let X be an alternating $n \times n$ matrix of indeterminates over B and $R_r = B[X]/\text{Pf}_{r+2}(X)$, $r \geq 0$ an even integer.

THEOREM 2. Let $x: F \rightarrow F^*$, $\text{rank } F = n$, be the generic alternating map of rank r over B . Suppose that $r < n-1$ if n is odd. Then $M = \text{Coker } x$, $M^* = \text{Ker } x$, and the self-dual module $\text{Im } x$ are perfect $B[X]$ -modules.

The case $r = n-1$, n odd, is a true exception: $\text{Coker } x \cong \text{Pf}_r(X)$ is an ideal of projective dimension 2 [BE].

The proof of Theorem 2 is reduced to the case $B = \mathbf{Z}$ and $r+2 \leq n$ by the same arguments as above. It is enough to show that $\text{depth } M_p \geq \text{depth } R_p - 1$

for all $\mathfrak{p} \in \text{Spec } R_r$, $r = \text{rank } x$. Together with the fact that $M_{\mathfrak{p}}$ is free for $\mathfrak{p} \neq \text{Pf}_r(x)$, $\text{grade Pf}_r(x) = 2(n-r) + 1 \geq 5$, this implies that M is reflexive and M^* , the second syzygy of M , is Cohen–Macaulay, whence $M = M^{**}$ is Cohen–Macaulay, too. After all, R_r is a Gorenstein ring (over any Gorenstein B), cf. [KL].

For indices i_1, \dots, i_u we denote (the residue class of) the pfaffian of the $u \times u$ matrix $(X_{i_v i_w}: 1 \leq v, w \leq u)$ by $[i_1, \dots, i_u]$. Note that $[i_1, \dots, i_u] = 0$ if $i_v = i_w$ for some $v, w, v \neq w$. One has the expansions

$$\begin{aligned} [i_1, \dots, i_u] &= \sum_{j=1}^u (-1)^{v+j} \sigma(j, v) [i_1, \dots, \hat{i}_j, \dots, \hat{i}_v, \dots, i_u] [i_j, i_v] \\ &= \sum_{j=1}^u (-1)^{v+j} \sigma(v, j) [i_1, \dots, \hat{i}_j, \dots, \hat{i}_v, \dots, i_u] [i_v, i_j] \end{aligned}$$

along a row or column resp., $\sigma(j, v)$ denoting the sign of $v-j$.

As above we put $M_r = \sum_{j=r+1}^n R_r \bar{e}_j$, $\bar{M} = M/M_r$, and, now, $\bar{R} = R_r/R_r[1, \dots, r]$. By what has just been said, it is enough to show the following lemma:

LEMMA 4. *Let $r < n$.*

(a) *M_r is a free R_r -module.*

(b) *$[1, \dots, r] \bar{M} = 0$.*

(c) *As an \bar{R} -module \bar{M} is isomorphic to the ideal $I \subset \bar{R}$ generated by the pfaffians $[1, \dots, \hat{i}, \dots, r+1]$, $1 \leq i \leq r$.*

(d) *Suppose that $r+2 \leq n$. Then I is a maximal Cohen–Macaulay module over \bar{R}_r .*

Proof. (a) follows as the corresponding statement in Lemma 1: $\text{rank } M = n-r$ and $[1, \dots, r | 1, \dots, r] = [1, \dots, r]^2$ is not a zero-divisor.

(b) follows from arguments analogous to those detailed below in the proof of (c).

For the proof of (c) and (d) we use the structure of R_r as an algebra with straightening law over B , cf. [DEP], Section 12. The underlying poset is formed by the s -pfaffians of x , $s \leq r$, ordered by $[a_1, \dots, a_u] \leq [b_1, \dots, b_v] \Leftrightarrow u \geq v$, $a_i \leq b_i$, $i = 1, \dots, v$. This poset is obviously a distributive lattice, in particular it is wonderful. As one sees easily

$$\bar{R}/I \cong R_r/J,$$

J being generated by all the pfaffians in the set $\Psi = \{\pi: \pi \not\leq [1, \dots, r-1, r+2]\}$. Ψ is a poset ideal, and by [DEP], Lemma 8.2 or [BV], (5.13) \bar{R}/I is an ASL on a wonderful poset again. This implies that \bar{R}/I is a Cohen–Macaulay ring. Using [DEP], Section 6 or [BV], (5.10) to determine dimensions one sees that (under the hypothesis of (d)) $\dim \bar{R}/I = \dim R_r - 2 = \dim \bar{R} - 1$. Together with the Cohen–Macaulay property of \bar{R} and \bar{R}/I the equation $\dim \bar{R}/I = \dim \bar{R} - 1$ implies (d).

In order to prove (c) one starts with the natural presentation

$$\bar{R}^n \xrightarrow{(x|r)} \bar{R}^r \rightarrow M \rightarrow 0.$$

Then one defines an epimorphism $\varphi: \bar{R}^r \rightarrow I$ by $\varphi(e_i) = (-1)^i \times [1, \dots, \hat{i}, \dots, r+1]$. The expansion of the pfaffian $[1, \dots, r+1, k]$ along its $(r+2)$ -th row,

$$[1, \dots, r+1, k] = - \sum_{i=1}^{r+1} (-1)^i X_{ki} [1, \dots, \hat{i}, \dots, r+1]$$

together with $[1, \dots, r+1, k] = [1, \dots, r] = 0$ in \bar{R} shows that the rows of $(x|r)$ are relations of the generators of I , the map φ factors through M . It remains to show that the kernel of φ is generated by the rows of $(x|r)$. By virtue of [BV], (5.6) it is enough to find linear combinations of the rows of $(x|r)$ of the form

$$\pi e_i - \sum_{j>i} a_j e_j, \quad a_j \in \bar{R},$$

for all $i = 1, \dots, r$ and all pfaffians $\pi \neq [1, \dots, \hat{i}, \dots, r+1]$. Let π be such a pfaffian. Then

$$\pi = [1, \dots, i, b_1, \dots, b_q], \quad i < b_1 < \dots < b_q, \quad i+q \equiv 0 \pmod{2}.$$

Let x_j be the j th row of $(x|r)$ and $[a_1, \dots, a_{i+q}] = [1, \dots, i, b_1, \dots, b_q]$. The element

$$y = \sum_{j=1}^{i+q} (-1)^{i+j} \sigma(i, j) [a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_{i+q}] x_j$$

has the desired form: Its component with respect to $e_k, k \leq i$, is $[1, \dots, k, \dots, i-1, k, b_1, \dots, b_q]$, hence 0 for $k < i$, and $[1, \dots, i, b_1, \dots, b_q]$ for $k = i$. ■

Remarks. (a) Both Theorems 1 and 2 can be generalized along the lines of [Br], Theorems 1 and 2, and supplemented by statements analogous to [Br], Theorems 4 and 5. Furthermore it is not difficult to show that $\text{Ker } x$ is generated by the natural determinantal resp. pfaffian relations of the rows of x .

(b) Under the hypotheses of Theorem 2 let n be an odd integer and $r+2 = n-1$. As remarked above, $I = \text{Pf}_{r+1}(X)$ is a (Gorenstein) ideal of grade 3, and obviously $M \cong I/I^2$. Thus Theorem 2 may be considered a generalization of Herzog's result [He], Satz 2.8 saying essentially that I/I^2 is a perfect module over $B[X]$.

References

- [Br] W. Bruns, *Generic maps and modules*, Compos. Math. 47 (1982), 171–193.
- [BV] W. Bruns and U. Vetter, *Determinantal Rings*, Lecture Notes in Math. 1327, Springer, Berlin–Heidelberg–New York 1988.

- [BE] D. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3*, Amer. J. Math. 99 (1977), 447–485.
 - [DEP] C. De Concini, D. Eisenbud and C. Procesi, *Hodge Algebras*, Astérisque 91, 1982.
 - [Go] S. Goto, *On the Gorensteinness of determinantal loci*, J. Math. Kyoto Univ. 19 (1979), 371–374.
 - [He] J. Herzog, *Ein Cohen–Macaulay-Kriterium mit Anwendungen auf den Konormalenmodul und den Differentialmodul*, Math. Z. 163 (1978), 149–162.
 - [HK] J. Herzog und E. Kunz, *Der kanonische Modul eines Cohen–Macaulay-Rings*, Lecture Notes in Math. 238, Springer, Berlin–Heidelberg–New York 1971.
 - [Jo] T. Józefiak, *Ideals generated by minors of a symmetric matrix*, Comment. Math. Helv. 53 (1978), 595–607.
 - [KL] H. Kleppe and D. Laksov, *The algebraic structure and deformation of Pfaffian schemes*, J. Algebra 64 (1980), 167–189.
 - [Ku] R. E. Kutz, *Cohen–Macaulay rings and ideal theory of invariants of algebraic groups*, Trans. Amer. Math. Soc. 194 (1974), 115–129.
 - [Mt] H. Matsumura, *Commutative Algebra*, Second Ed., Benjamin/Cummings, Reading 1980.
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