THE NUMBER OF EQUATIONS DEFINING A DETERMINANTAL VARIETY

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The aim of this note is the following theorem.

THEOREM 1. *Let K be an algebraically closed field, and $L = \mathrm{Hom}_K(K^m, K^n)$. Suppose that $1 \leqslant t \leqslant \min(m,n)$. Then*
(a) *the subvariety $V = \{\phi \in L : \mathrm{rk}\,\phi < t\}$ can be defined set-theoretically by $mn - t^2 + 1$ equations, but not by fewer equations;*
(b) *the same holds for the corresponding projective variety $\mathbf{P}(V) \subset \mathbf{P}(L)$.*

In order to give the most general version of Theorem 1, we introduce the following notation: B is a commutative ring, X an $m \times n$ matrix of indeterminates, and $\mathrm{I}_t(X)$ the ideal generated by the t-minors of X in the polynomial ring $B[X]$. (We use Bruns and Vetter [3] as a reference for the theory of determinantal ideals.) For an ideal J in a commutative ring R, we call

$$\mathrm{ara}\, J = \min\{k: \text{there exist } f_1, \ldots, f_k \in R \text{ such that } \mathrm{Rad}\, J = \mathrm{Rad}(f_1, \ldots, f_k)\}$$

the *arithmetical rank* of J.

Then Theorem 1 clearly is a consequence of the following.

THEOREM 2. *With the notation just introduced,*

$$\mathrm{ara}\, \mathrm{I}_t(X) = mn - t^2 + 1$$

for all t such that $1 \leqslant t \leqslant \min(m,n)$. Furthermore, the $mn - t^2 + 1$ elements generating $\mathrm{I}_t(X)$ up to radical can be chosen homogeneous.

The inequality $\leqslant$ in Theorem 2 is a result of the first author; it will be used in proving the converse. Therefore we restate [3, (5.21)]. It can be seen easily from its proof that the $mn - t^2 + 1$ elements generating $\mathrm{I}_t(X)$ up to radical may be chosen homogeneous; furthermore, this has been noted explicitly in Bruns [2, (2.1)].

LEMMA 1. *Let B be a commutative ring. Then $\mathrm{ara}\, \mathrm{I}_t(X) \leqslant mn - t^2 + 1$, and there are $mn - t^2 + 1$ homogeneous elements generating $\mathrm{I}_t(X)$ up to radical for all t such that $1 \leqslant t \leqslant \min(m,n)$.*

The precise value of $\mathrm{ara}\, \mathrm{I}_t(X)$ has been found only in the trivial case $t = 1$ and the cases (i) $t = \min(m,n)$ and (ii) $t = 2$ in characteristic zero. Case (i) was proved by Hochster (compare the remark following the corollary below). The inequality $\geqslant$ in (ii)

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is a result of Newstead [12, p. 180, Example (i)(a)]. Generalizing Newstead's arguments, we derive the inequality $\geqslant$ from the computation of a topological invariant. (All the information on the topology of algebraic varieties tacitly used below can be found in Łojasiewicz [10].)

LEMMA 2. *With the notation of Theorem* 1, *let* $K = \mathbf{C}$, $N = mn$ *and* $k = mn - t^2$. *Then for every abelian group* G,

$$H^{N+k}(L \setminus V, G) \cong G.$$

(*Here* H *denotes singular cohomology.*) *The higher cohomology groups vanish.*

We draw a corollary regarding the algebraic cohomological dimension of $I = \mathrm{I}_t(X)$ in $R = B[X]$,

$$\operatorname{cd} I = \max\{i : H_I^i(M) \neq 0 \text{ for some finitely generated } R\text{-module } M\},$$

H_I denoting cohomology with support in I. The conclusion strengthens Theorem 2 for torsionfree $\mathbf{Z}$-algebras.

COROLLARY. *Let* B *be a torsionfree* $\mathbf{Z}$*-algebra. Then* $\operatorname{cd} \mathrm{I}_t(X) = mn - t^2 + 1$ *for all* t *such that* $1 \leqslant t \leqslant \min(m, n)$.

Proof. Since $\operatorname{cd} \mathrm{I}_t(X) \leqslant \operatorname{ara} \mathrm{I}_t(X)$ (Hartshorne [7, p. 414, Example 2]), Lemma 1 implies $\operatorname{cd} \mathrm{I}_t(X) \leqslant mn - t^2 + 1$. For $B = \mathbf{C}$, the case $G = \mathbf{C}$ of Lemma 2 yields $\operatorname{cd} L \setminus V \geqslant mn - t^2$ by virtue of [8, p. 146, Proposition 7.2 and p. 147, Theorem]. The long exact sequence of cohomology then gives $\operatorname{cd} \mathrm{I}_t(X) \geqslant mn - t^2 + 1$. One now derives the general case from the flat base change property of local cohomology via $B = \mathbf{Z}$.

For $t = \min(m, n)$ the corollary has been proved by Hochster. His proof exploits the fact that the subalgebra S generated by the maximal minors is a direct S-summand of $K[X]$ if K is a field of characteristic zero (compare [3, (7.12)] or [9, 4.11]).

In contrast, if B has characteristic $p > 0$, with p prime, then

$$\operatorname{cd} \mathrm{I}_t(X) = \operatorname{ht} \mathrm{I}_t(X) = (m - t + 1)(n - t + 1)$$

by Peskine and Szpiro [13, p. 110, Proposition (4.1)] since $\mathrm{I}_t(X)$ is a perfect ideal by the theorem of Hochster and Eagon [3, (5.18)].

In order to prove Theorems 1 and 2 in arbitrary characteristic, one uses the 'étale' analogue of Lemma 2.

LEMMA 2′. *With the notation of Theorem* 1, *let* $N = mn$ *and* $k = mn - t^2$. *Then for all integers* $q \neq 0$ *prime to* $\operatorname{char} K$,

$$H_{\mathrm{et}}^{N+k}(L \setminus V, \mathbf{Z}/q\mathbf{Z}) \cong \mathbf{Z}/q\mathbf{Z}.$$

(*Here* H_{et} *denotes étale cohomology.*) *The higher cohomology groups vanish.*

REMARKS. (a) The proof of Lemma 1 is based on the structure of $B[X]$ as an algebra with straightening law on the poset of minors of X. It should be noted that this structure gives the precise value of $\operatorname{ara} J$ in the closely related situation where J is the defining ideal of a Schubert subvariety of a Grassmannian (compare [3, (5.22)] or [2, (3.4)]).

(b) P. Schenzel pointed out that in the case of a field $B = K$ an upper bound for the arithmetical rank is supplied by the 'symbolic analytic spread'

$$\dim S/\mathrm{I}_1(X)S, \quad S = \bigoplus_{i=0}^{\infty} I^{(i)}/I^{(i+1)}, \quad I = \mathrm{I}_t(X),$$

with S considered as a $K[X]$-algebra via the epimorphism $K[X] \to K[X]/\mathrm{I}_t(X)$. This gives in fact the same bound (compare [**3**, (10.8)]). Note, however, that the ordinary analytic spread is an efficient bound only for $t = \min(m, n)$ (or $t = 1$):

$$\dim G/\mathrm{I}_1(X)G = \begin{cases} mn - t^2 + 1 & \text{if } t = \min(m,n), \\ mn & \text{otherwise,} \end{cases} \quad G = \bigoplus_{i=0}^{\infty} I^{(i)}/I^{(i+1)}$$

(compare [**3**, (9.22), (10.16)]).

(c) Since Lemma 2′ holds in every characteristic, one does not need Lemma 2 for the proof of the theorems. One can even derive Lemma 2 from Lemma 2′. Applying the comparison theorem [**11**, p. 117, Theorem 3.12], one first obtains

$$H^{N+k}(L\backslash V, \mathbf{Z}/q\mathbf{Z}) \cong \mathbf{Z}/q\mathbf{Z}$$

for $q \neq 0$. Since the higher cohomology groups vanish, the universal coefficient theorem then implies

$$H^{N+k}(L\backslash V, \mathbf{Z}) \cong \mathbf{Z}$$

(note that the cohomology groups $H^i(L\backslash V, \mathbf{Z})$ are finitely generated). Another application of the universal coefficient theorem yields Lemma 2. However, for the convenience of readers who feel more comfortable in a topological environment, we give an independent proof of Lemma 2. Of course, the proofs of the lemmas are completely parallel.

(d) With N replaced by $N-1$, Lemmas 2 and 2′ also hold for the complement $\mathbf{P}(L)\backslash\mathbf{P}(V)$ of the corresponding projective variety. This can be seen either by repeating the whole calculation (the fibre $\mathrm{GL}_t(\mathbf{C})$ appearing in the fibration of Lemma 6 changes to $\mathrm{PGL}_t(\mathbf{C})$), or more directly by an application of the Gysin sequence associated with the $\mathbf{C}^*$-fibration $L\backslash V \to \mathbf{P}(L)\backslash\mathbf{P}(V)$.

(e) Our methods should also yield the number of equations defining varieties similar to determinantal ones, for example, varieties defined by pfaffians of alternating matrices or minors of symmetric ones.

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1. *Proof of the main result*

In this section we derive Theorem 2 from the lemmas above, whereas the proofs of the crucial Lemmas 2 and 2′ will be given in the next sections. Because of Lemma 1 we may replace B by a field $B/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of B. Under specialization, the arithmetical rank cannot increase. Thus let $B = K$ be a field.

We first discuss the case $\operatorname{char} K = 0$. The quickest argument now is a reference to the corollary of Lemma 2: $\operatorname{cd} \mathrm{I}_t(X) = mn - t^2 + 1$, and $\operatorname{cd} \mathrm{I}_t(X) \leqslant \operatorname{ara} \mathrm{I}_t(X)$, as noticed in its proof.

A more direct topological argument seems appropriate, however; it will be needed below anyway. Let $g_1, \ldots, g_s \in K[X]$ such that $\mathrm{I}_t(X) = \operatorname{Rad}(g_1, \ldots, g_s)$. (Note that $\mathrm{I}_t(X)$

is a prime ideal by the theorem of Hochster and Eagon.) This is equivalent to the existence of equations which for each t-minor δ express that a power of δ is contained in $(g_1, \dots, g_s)$. There are only finitely many equations, and each of them involves only finitely many polynomials. Thus $I_t(X) = \operatorname{Rad}(g_1, \dots, g_s)$ holds already over a finitely generated field extension K_0 of $\mathbf{Q}$. Embedding K_0 into $\mathbf{C}$ and applying the argument above once more, one finally reduces this case to $K = \mathbf{C}$. (In the following, dimension always denotes complex dimension, more generally K-dimension, unless explicitly indicated otherwise.)

LEMMA 3. (a) *Let W and $\tilde{W}$, with $W \subset \tilde{W}$, be projective varieties over* $\mathbf{C}$ *such that $\tilde{W} \setminus W$ is non-singular and of pure dimension d. If there are k homogeneous equations $f_1, \dots, f_k$ such that $W = \tilde{W} \cap V(f_1, \dots, f_k)$, then*

$$H^{d+i}(\tilde{W} \setminus W, G) = 0 \quad \text{for all } i \geqslant k$$

and all abelian groups G.

(b) *The analogous statement holds for affine varieties.*

Part (a) is [**12**, Theorem 2′], or rather the statement from which that theorem is derived via Lefschetz duality. Part (b) is proved in exactly the same way. If $k = 1$, then $H^{d+i}(\tilde{W} \setminus W, \mathbf{Z}) = 0$ for all $i \geqslant 1$, by the theorem of Andreotti and Frankel [**1**, Theorem 1], and the general case follows by induction based on the Mayer–Vietoris sequence. (The universal coefficient theorem enables one to replace $\mathbf{Z}$ by an arbitrary group.)

Lemmas 1 and 3 finish the argument in characteristic zero once more. Since $H^{N+k}(L \setminus V, G) \neq 0$, $N = mn$, $k = mn - t^2$, the arithmetical rank of $I_t(X)$ is at least $mn - t^2 + 1$.

Let K be an arbitrary field now. For the same reasons as above, we may enlarge K and therefore assume that K is algebraically closed. Then one argues by the 'étale' analogue of Lemma 3, which is as follows.

LEMMA 3′. (a) *Let K be an algebraically closed field, and $W \subset \tilde{W}$ projective varieties. Let $d = \dim \tilde{W} \setminus W$. If there are k homogeneous equations $f_1, \dots, f_k$ such that $W = \tilde{W} \cap V(f_1, \dots, f_k)$, then*

$$H^{d+i}_{\text{et}}(\tilde{W} \setminus W, \mathbf{Z}/q\mathbf{Z}) = 0 \quad \text{for all } i \geqslant k$$

and all $q \in \mathbf{Z}$, with q prime to $\operatorname{char} K$.

(b) *The analogous statement holds for affine varieties.*

This is proved in the same way as Lemma 3: Milne [**11**, p. 25, Theorem 7.2] gives the case $k = 1$, in which $\tilde{W} \setminus W$ is affine, and the general case follows by induction via the Mayer–Vietoris sequence [**11**, Ex. 2.24, p. 110].

2. *The crucial singular cohomology group*

This section is devoted to the proof of Lemma 2. All the cohomology and homology groups are taken with respect to a fixed group G of coefficients. In order to simplify notation we omit the group of coefficients henceforth.

One first notes that the cohomology groups with indices greater than $N+k$ $(N = mn, k = mn - t^2)$ vanish because of Lemmas 1 and 3.

In the simplest case, $m = n = t$, the set $L \setminus V$ is just $\mathrm{GL}_t(\mathbf{C})$.

LEMMA 4. *$H^{t^2}(\mathrm{GL}_t(\mathbf{C})) \cong G$ and the higher cohomology groups vanish.*

This follows immediately from the fact that $\mathrm{GL}_t(\mathbf{C})$ is homotopy-equivalent with $U(t)$, the group of unitary $t \times t$ matrices. We now turn to the general case, and first provide a preparatory result.

LEMMA 5. *Let $\tilde{V} = V(\mathrm{I}_{t+1}(X)) \subset L$ ($\tilde{V} = L$ if $t = \min(m,n)$) and $W = \tilde{V} \backslash V$. Then*

$$H^{N+k}(L \backslash V) \cong H^{2d-t^2}(W), \quad d = \dim W.$$

Proof. In order to apply Lefschetz duality, we first compactify by embedding $L = \mathbf{C}^N$ into $\mathbf{P}^N$ in the usual way. Then $L \backslash V = \mathbf{P}^N \backslash U$, $U = V \cup H_\infty$, where H_∞ denotes the hyperplane at infinity. Since $N+k = 2N-t^2$, Lefschetz duality yields

$$H^{N+k}(L \backslash V) \cong H_{t^2}(\mathbf{P}^N, U).$$

Let $\tilde{U} = \tilde{V} \cup H_\infty$. As part of the exact sequence for homology, one has an exact sequence

$$H_{t^2+1}(\mathbf{P}^N, \tilde{U}) \longrightarrow H_{t^2}(\tilde{U}, U) \longrightarrow H_{t^2}(\mathbf{P}^N, U) \longrightarrow H_{t^2}(\mathbf{P}^N, \tilde{U}).$$

We claim that the first and last terms in this sequence are zero. Obviously, $\tilde{U}$ can be defined set-theoretically by the same number of equations as can $\tilde{V}$. Thus Lemmas 1 and 3, in connection with Lefschetz duality, force these homology groups to vanish.

Since $V = \operatorname{Sing} \tilde{V}$ if $t < \min(m,n)$ (compare [3, (2.6)]), $\tilde{U} \backslash U = \tilde{V} \backslash V = W$ is non-singular. Applying Lefschetz duality once more, one concludes that $H_{t^2}(\tilde{U}, U) \cong H^{2d-t^2}(W)$.

After this reduction it remains to compute $H^{2d-t^2}(W)$.

LEMMA 6. *Let $G_{d,e}$ denote the Grassmannian of d-dimensional vector subspaces of $\mathbf{C}^e$. Then the map $\pi: W \to G_{m-t,m} \times G_{t,n}$ which assigns to each $\phi \in W$ the pair $(\operatorname{Ker}\phi, \operatorname{Im}\phi)$, is a locally trivial fibre bundle (in the Zariski topology) with fibre $\mathrm{GL}_t(\mathbf{C})$.*

Proof. It may suffice to indicate the open subsets of $G_{m-t,m} \times G_{t,n}$ over which the fibration is trivial. One chooses bases $e_1, \ldots, e_m$ of $\mathbf{C}^m$ and $f_1, \ldots, f_n$ of $\mathbf{C}^n$. Then the open subvarieties

$$A = \{(C,D): C \cap \mathbf{C}e_{i_1} + \ldots + \mathbf{C}e_{i_t} = 0 \text{ and } D \cap \mathbf{C}f_{j_1} + \ldots + \mathbf{C}f_{j_{n-t}} = 0\},$$
$$i_1 < \ldots < i_t, \quad j_1 < \ldots < j_{n-t},$$

cover $G_{m-t,m} \times G_{t,n}$, and over each of them the fibration is trivial. ($A \cong \mathbf{C}^{(m-t)t} \times \mathbf{C}^{(n-t)t}$ as a variety.)

Now we can complete the proof of Lemma 2. Since $G_{m-t,m} \times G_{t,n}$ is simply-connected, the Leray spectral sequence for the fibration of Lemma 6 takes the simple form

$$E_2^{u,v} = H^u(G_{m-t,m} \times G_{t,n}, H^v(\mathrm{GL}_t)) \Longrightarrow H^{u+v}(W).$$

Since $H^{2D}(G_{m-t,m}\times G_{t,n})\cong G$, $D=\dim G_{m-t,m}\times G_{t,n}$, $H^{t^2}(\mathrm{GL}_t)\cong G$ and the higher cohomology groups vanish, this yields immediately that

$$H^{2D+t^2}(W)\cong E_\infty^{2D,t^2}=E_2^{2D,t^2}\cong H^{2D}(G_{m-t,m}\times G_{t,n},H^{t^2}(\mathrm{GL}_t))\cong G.$$

Finally, $2D+t^2=2d-t^2$.

3. *The crucial étale cohomology group*

Since a direct comparison does not seem to be possible, we repeat the computation of Section 2 for étale cohomology, step by step. Let K be an algebraically closed field of arbitrary characteristic, and $q\neq 0$ an integer prime to $\mathrm{char}\,K$. All the étale cohomology groups below are taken with respect to the constant sheaf defined by $\mathbf{Z}/q\mathbf{Z}$.

(1) For the case $m=n=t$, there is indeed a comparison theorem. By the theorem of Friedlander and Parshall [**5**, Theorem 1], Lemma 4 implies the corresponding statement for $\mathrm{GL}_t(K)$: $H^{t^2}(\mathrm{GL}_t(K))\cong\mathbf{Z}/q\mathbf{Z}$, and the higher cohomology groups vanish.

(2) The reduction provided by Lemma 5 works in almost the same way. Poincaré duality [**11**, p. 276, Corollary 11.2] shows that $H_{\mathrm{et}}^{2N-i}(L\setminus V)$ and $H_c^i(L\setminus V)$ are duals of each other (non-canonically) with respect to linear maps to $\mathbf{Z}/q\mathbf{Z}$. The corresponding statement holds for $L\setminus\tilde{V}$ and $\tilde{V}\setminus V$, since all these varieties are separated and smooth over K. The exact sequence of homology is replaced by the exact sequence of cohomology with proper support [**11**, p. 94, Remark 1.30]:

$$\cdots\longrightarrow H_c^i(L\setminus\tilde{V})\longrightarrow H_c^i(L\setminus V)\longrightarrow H_c^i(\tilde{V}\setminus V)\longrightarrow\cdots.$$

(3) The fibration given in Lemma 6 is valid for every algebraically closed field.

(4) The properties of the Grassmannians exploited in the proof of Lemma 2 hold over every algebraically closed field: $G_{m-t,m}\times G_{t,n}$ is irreducible, and in particular it is connected. Since it is a proper non-singular rational variety, the algebraic fundamental group $\pi_1(G_{m-t,m}\times G_{t,n})$ vanishes by [**6**, p. 285, Corollary 1.2]. Furthermore, $H_{\mathrm{et}}^{2D}(G_{m-t,m}\times G_{t,n},\mathbf{Z}/q\mathbf{Z})\cong\mathbf{Z}/q\mathbf{Z}$, and the higher cohomology groups vanish. ($G_{m-t,m}\times G_{t,n}$ being proper, this may be considered a consequence of Poincaré duality.)

(5) The only fact which seems to need proof here is that the Leray spectral sequence [**11**, p. 89, Theorem 1.18]

$$H_{\mathrm{et}}^u(G_{m-t,m}\times G_{t,n},R^v\pi_*\mathbf{Z}/q\mathbf{Z})\Longrightarrow H_{\mathrm{et}}^{u+v}(W,\mathbf{Z}/q\mathbf{Z})$$

for the morphism $\pi: W\to G_{m-t,m}\times G_{t,n}$ takes the same simple form as in the topological situation. This is guaranteed by the following proposition, a rather basic fact for which the authors fruitlessly searched the literature.

PROPOSITION. *Let X, F and B be schemes of finite type over an algebraically closed field K, and suppose one has a morphism $\pi: X\to B$ inducing a locally trivial fibration of X over B with fibre F. Assume that q is prime to* $\mathrm{char}\,K$. *If B is simply-connected (that is, B is connected and $\pi_1(B)=0$), then $R^v\pi_*\mathbf{Z}/q\mathbf{Z}$ is the constant sheaf defined by $H_{\mathrm{et}}^v(F,\mathbf{Z}/q\mathbf{Z})$ on B.*

Proof. Let $\mathscr{G}=R^v\pi_*\mathbf{Z}/q\mathbf{Z}$ and let $\mathscr{F}$ denote the constant sheaf defined by $H_{\mathrm{et}}^v(F,\mathbf{Z}/q\mathbf{Z})$ on B. Let (U_i) be an étale covering of X such that π induces the trivial fibration $\pi^{-1}(U_i) \cong B \times_K U_i \to U_i$. We claim that $\mathscr{G}|U_i \cong \mathscr{F}|U_i$. Then $\mathscr{G}$ is a locally constant sheaf, thus constant since X is simply-connected (an easy consequence of [**11**, p. 155, Proposition 1.1]), and necessarily $\mathscr{G} \cong \mathscr{F}$.

In order to prove the claim, one considers the following Cartesian diagram.

$$\begin{array}{ccc} F \times_K U_i & \xrightarrow{\ \phi\ } & F \\ \tilde{\pi} \downarrow & & \downarrow \rho \\ U_i & \xrightarrow{\ \psi\ } & \operatorname{Spec} K \end{array}$$

The constant sheaf $\mathbf{Z}/q\mathbf{Z}$ on $F \times_K U_i$ is the pullback of the constant sheaf $\mathbf{Z}/q\mathbf{Z}$ on F along the base extension ψ. Furthermore, the formation of higher direct images commutes with base extension for schemes of finite type over $\operatorname{Spec} K$ [**4**, p. 236, Theorem 1.9]. Therefore

$$R^v \tilde{\pi}_* \mathbf{Z}/q\mathbf{Z} \cong \psi^*(R^v \rho_* \mathbf{Z}/q\mathbf{Z}).$$

This argument reduces the proposition to the case in which $X = F$ and π is the structure morphism $X \to \operatorname{Spec} K$. For this case it is easy to prove: for every étale morphism $Y \to \operatorname{Spec} K$ of finite type, Y is just the union of finitely many copies of $\operatorname{Spec} K$.

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