Let \((R, \mathfrak{m}, k)\) be a Noetherian local ring, and \(x\) a minimal system of generators of \(\mathfrak{m}\). The Koszul complex \(K_\bullet(x)\) is essentially independent of the choice of \(x\), and thus an invariant of \(R\) (as an alternating algebra equipped with an anti-derivation of degree \(-1\)). Therefore one may write \(H_\bullet(R)\) for its homology; it carries the structure of an alternating \(k\)-algebra and is called the \textit{Koszul algebra} of \(R\). By the universal property of the exterior algebra \(\wedge H_\bullet(R)\), there is always a natural map \(\lambda \colon \wedge H_1(R) \to H_\bullet(R)\) which extends the identity on \(H_1(R)\). (We refer to Bourbaki [2], Ch. X for notation and results related to the Koszul complex, to [2], Ch. III for exterior algebra, and to Matsumura [5] for commutative algebra.)

Using the methods of Tate [8], Assmus [1] gave the following beautiful characterization of complete intersections.

\textbf{THEOREM 1.} Let \((R, \mathfrak{m}, k)\) be a Noetherian local ring. Then the following are equivalent:
\begin{enumerate}
\item \(R\) is a complete intersection;
\item \(H_\bullet(R)\) is (isomorphic with) the exterior algebra of \(H_1(R)\);
\item \(H_\bullet(R)\) is generated by \(H_1(R)\);
\item \(H_2(R) = H_1(R)^2\).
\end{enumerate}

In particular, \(R\) is a complete intersection if (and only if) \(\lambda\) is surjective. In this note we want to describe complete intersections by the injectivity of \(\lambda\). More precisely, we shall prove the following theorem:

\textbf{THEOREM 2.} Let \((R, \mathfrak{m}, k)\) be a Noetherian local ring containing a field. Then:
\begin{enumerate}
\item \(H_i(R) = 0\) for \(i > \text{emb dim } R - \text{dim } R\);
\item in particular, \(R\) is a complete intersection if (and only if) the natural map
\[\lambda \colon \wedge H_1(R) \to H_\bullet(R)\]
\end{enumerate}
is injective.
It is easy to see that part (a) of Theorem 2 implies part (b). In fact, if \( \lambda \) is injective, then (a) yields \( \dim_k H_i(R) \leq \text{emb dim } R - \dim R \), and this holds if and only if \( R \) is a complete intersection (and \( \dim_k H_i(R) = \text{emb dim } R - \dim R \)); see [5], §21.

The crucial argument in proving part (a) of Theorem 2 will be the theorem of Evans-Griffith [3] on order ideals of minimal generators of syzygies. This explains the restriction to rings containing a field: the theorem of Evans-Griffith has not yet been proved in general. (Even if it should fail, Theorem 2 holds 'almost' for arbitrary local rings; cf. Remarks, (a).)

Since the Koszul algebra, the property of being a complete intersection, and the numerical invariants in Theorem 2 are stable under completion, we may assume that \( R \) is complete. Then \( R \) has a presentation \( R = \mathcal{S}/I \) in which \((S, n, k)\) is a regular local ring, and \( I \subset n^2 \) is an ideal of \( S \). We choose a regular system of parameters \( y \) in \( S \).

For the moment, let us consider more generally a (Noetherian) ring \( S \), and ideals \( I \subset n \) of \( S \). Let \( y = y_1, \ldots, y_n \) generate \( n \), and \( a = a_1, \ldots, a_m \) generate \( I \). We write \( a_i \in \sum a_{ji}y_j \) with \( a_{ji} \in S \).

Denote the canonical bases of \( S^n \) and \( S^m \) by \( f_1, \ldots, f_n \) and \( e_1, \ldots, e_m \) resp., and let \( \varphi: S^m \to S^n \) be the map given by the matrix \( (a_{ji}) \). Setting \( u_i = \varphi(e_i) \in S^n \) we have \( d_a(e_i) = a_i = d_y(u_i) \). Here \( d_a \) and \( d_y \) are the differentials in the Koszul complexes \( K(a) \) and \( K(y) \). Furthermore,

\[
\Lambda: K(a) \to K(y).
\]

is a chain map. The induced map \( H(a, S/I) \to H(y, S/I) \) actually yields a homomorphism

\[
\Lambda: (S/n)^m \to H(a, S/n) \to H(y, S/I)
\]

of \( S/n \)-algebras: note that \( H(a, S/I) \cong K(a) \otimes S/I \cong \Lambda(S/I)^m \) and that \( H(y, M) \) is annihilated by \( n \) for an arbitrary \( S \)-module \( M \).

One has natural homomorphisms

\[
\rho: H(a, S/n) \to \text{Tor}(S/I, S/n), \\
\sigma: H(y, S/I) \to \text{Tor}(S/n, S/I).
\]

By a standard argument of homological algebra, \( \text{Tor}(S/I, S/n) = \text{Tor}(S/n, S/I) \). So we have two maps from \( H(a, S/n) \) to \( \text{Tor}(S/I, S/n) \), namely \( \rho \) and \( \sigma \circ \Lambda \). The proof of Theorem 2 hinges on the fact that these maps are essentially equal—under the proper identification of \( \text{Tor}(S/I, S/n) \) and \( \text{Tor}(S/n, S/I) \). This may be a well-known fact, but we do not have a reference, and the argument is short.

We choose free resolutions \( F \) and \( G \) of \( S/I \) and \( S/n \) resp. Then there are chain maps \( K(a) \to F \to S/I \) and \( K(y) \to G \to S/n \). Taking tensor
products yields a commutative diagram

\[
\begin{array}{c}
K.(a) \otimes S/\pi & \xleftarrow{\alpha} & K.(a) \otimes K.(y) \\
\downarrow & & \downarrow \\
F. \otimes S/\pi & \leftarrow & F. \otimes G. \\
\end{array}
\rightarrow S/I \otimes K.(y) \\
\downarrow \\
S/I \otimes G.
\]

The standard argument referred to above is that the bottom row induces an isomorphism

\[
H.(F. \otimes S/\pi) \xrightarrow{\cong} H.(F. \otimes S/\pi) \xrightarrow{\cong} H.(S/I \otimes G.).
\]

This is the identification of

\[
\text{Tor}^S(S/I, S/\pi) \cong H.(F. \otimes S/\pi) \quad \text{and} \quad \text{Tor}^S(S/\pi, S/I) \cong H.(S/I \otimes G.).
\]

which we will use in the following.

**Lemma 1.** One has \(\rho_s = (-1)^s \sigma_s \circ \Lambda_s.\)

**Proof.** Let \(e_1, \ldots, e_m\) and \(f_1, \ldots, f_n\) be bases of \(S^n\) and \(S^m\) and choose elements \(u \in S\) with \(d_u(u) = d_a(e_i)\). It is enough to show that

\[
\rho_s(\tilde{e}_{i_1} \wedge \ldots \wedge \tilde{e}_{i_s}) = (-1)^s \sigma_s(u_{i_1} \wedge \ldots \wedge u_{i_s}),
\]

and in view of the diagram above it suffices to exhibit a cycle \(z \in K.(a) \otimes K.(y)\) such that \(\alpha(z) = \tilde{e}_{i_1} \wedge \ldots \wedge \tilde{e}_{i_s}\) and \(\beta(z) = (-1)^s(u_{i_1} \wedge \ldots \wedge u_{i_s})\). We choose

\[
z = (e_{i_1} \otimes 1 - 1 \otimes u_{i_1}) \cdots (e_{i_s} \otimes 1 - 1 \otimes u_{i_s}).
\]

In order to see that \(z\) is a cycle one uses that the product of cycles in \(K.(a) \otimes K.(y)\) is again a cycle. Thus it is enough to show that \(e_i \otimes 1 - 1 \otimes u_i\) is a cycle, and this is immediate if one uses the definition of the differentiation on a tensor product of complexes. That \(\alpha(z) = \tilde{e}_{i_1} \wedge \ldots \wedge \tilde{e}_{i_s}\) and \(\beta(z) = (-1)^s(u_{i_1} \wedge \ldots \wedge u_{i_s})\) follows from the fact that \(\alpha\) and \(\beta\) are algebra homomorphisms.\[\square\]

Let us return to the special situation above in which \(S\) is a regular local ring, and \(y\) a regular system of parameters. Let \(x\) denote the sequence of residue classes of \(y = y_1, \ldots, y_n\) in \(R = S/I\). One has \(H.(R) \cong H.(y, R)\), and it is well known that the residue classes of the cycles \(u_i\) introduced above are a \(k\)-basis of \(H.(R)\), provided \(a\) is a minimal system of generators of \(I\) (cf. for example Scheja [6]). Therefore the maps \(\lambda\) and \(\Lambda\) differ only by an
automorphism of $\wedge k^m$; both $\lambda_1$ and $\Lambda_1$ are isomorphisms $k^m \to H_1(R)$. Theorem 2 claims that $\lambda_i = 0$ for $i > \text{emb dim } R - \text{dim } R$. Since $y$ is a regular sequence, $K.(y)$ is a free resolution of $k \cong S/n$, and so $\sigma_i$ is an isomorphism. Summarizing our arguments, we have reduced the theorem to the fact that $\rho_i = 0$ for $i > \text{emb dim } R - \text{dim } R$. This follows from the next lemma since $S/I$ has finite projective dimension over $S$. Moreover, one has

$$\text{emb dim } R - \text{dim } R = \text{dim } S - \text{dim } R = \text{height } I.$$ 

**Lemma 2.** Let $(S, n, k)$ be a Noetherian local ring containing a field, and $I \subset n$ an ideal generated by a sequence $a$. If $\text{proj dim } S/I < \infty$, then the natural homomorphism

$$H_i(a, k) = K.(a) \otimes k \to \text{Tor}_i^S(S/I, k)$$

is zero for $i > \text{height } I$.

**Proof.** The natural homomorphism $H_i(a, k) \to \text{Tor}_i^S(S/I, k)$ is induced by a chain map $\gamma_i$ from $K.(a)$ to a free resolution $F.$ of $S/I$. It only depends on $I$ and $a$, so that we may assume that

$$F : 0 \to F_0 \xrightarrow{\varphi_0} F_{s-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to 0$$

is a minimal free resolution. That $H_i(a, k) = K.(a) \otimes k$ and $\text{Tor}_i^S(S/I, k) \cong F. \otimes k$, follows from the minimality of the complexes $K.(a)$ and $F.$ Thus the map

$$H_i(a, k) \to \text{Tor}_i^S(S/I, k)$$

is just $\gamma_i \otimes k$.

For an $S$-module $M$ and $x \in M$ let $\mathcal{O}_M(x) = \{f(x): f \in \text{Hom}_S(M, S)\}$ denote its order ideal. We choose $M = \text{Im } \varphi_i$. The theorem of Evans-Griffith says that

$$\text{height } \mathcal{O}_M(\varphi_i(e)) \geq i \text{ for every element } e \in F_i, e \notin nF_i;$$

cf. [3], Proposition 1.6. We need the stronger assertion that $\text{height } \mathcal{O}_F(\varphi_i(e)) \geq i$ where $F = F_{i-1}$. (Of course, if $g_1, \ldots, g_w$ is a basis of $F$ and $\varphi_i(e) = s_1g_1 + \cdots + s WG_w$ with $s_i \in S$, then $\mathcal{O}_F(\varphi_i(e))$ is the ideal generated by $s_1, \ldots, s_w$.)

In order to prove $\text{height } \mathcal{O}_F(\varphi_i(e)) \geq i$, we show that $\mathcal{O}_F(\varphi_i(e)) = S$ for every prime ideal $\mathfrak{p}$ with height $\mathfrak{p} \leq i - 1$. Since $\text{proj dim } (S/I)_{\mathfrak{p}} \leq i - 1$, the embedding $M_{\mathfrak{p}} \to F_{\mathfrak{p}}$ splits for such a prime ideal; furthermore the formation
of order ideals commutes with localization. Therefore one has $\mathcal{O}_F(\varphi_i(e))_p = \mathcal{O}_M(\varphi_i(e))_p$, and that $\mathcal{O}_M(\varphi_i(e))_p = S_p$ is the result of Evans-Griffith.

The assertion of the lemma amounts to $\gamma_i(K_i(a)) \subseteq nF_i$ for $i > \text{height } I$. Let $z \in K_i(a)$. If $\gamma_i(z) \notin nF_i$, then $\text{height } \mathcal{O}_F(\gamma_i(z)) = \text{height } \mathcal{O}_F(\varphi_i(\gamma_i(z))) \geq i$ as just explained. On the other hand, $\mathcal{O}_F(\gamma_{i-1}(d_a(z))) \subseteq I$ since $\text{Im } d_a \subseteq IK_i(a)$. □

Remarks. (a) Suppose that $(S, n, k)$ is a regular local ring not containing a field. Let $p = \text{char } k$, and $\overline{S} = S/(p)$. Then $S$ is a Cohen-Macaulay local ring containing a field. Let $I$ be an ideal of $S$, and $F$ a minimal free resolution of $S/I$. As in the proof of Lemma 2 we have a comparison map $K_i(a) \to F_i$. Let $F'$ be the truncation

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to 0$$

of $F_i$. Then $F' \otimes \overline{S}$ is a minimal free resolution of $I \otimes \overline{S}$ over $\overline{S}$, and we can apply the theorem of Evans-Griffith to $F' \otimes \overline{S}$ over $\overline{S}$. With the notation of the proof of Lemma 2 it yields $\text{height } (\mathcal{O}_F(\varphi_i(e)) + (p))/(p) \geq i - 1$, and it follows easily that

$$\text{height } \mathcal{O}_F(\varphi_i(e)) \geq i - 1.$$ 

This argument shows that Lemma 2 holds for regular rings not containing a field if we replace $\text{height } I$ by $\text{height } I - 1$. Thus Theorem 2, (a) is valid without the hypothesis that $R$ contains a field if $\text{emb dim } R - \text{dim } R$ is replaced by $\text{emb dim } R - \text{dim } R + 1$.

(b) The method we used to prove Theorem 2 also yields a quick proof of Theorem 1. Again one may assume that $R$ is complete. If $I$ is generated by an $S$-sequence $a$, then $K_i(a)$ resolves $R$, and therefore $\rho_i$ is an isomorphism; it follows that $\lambda_i$ is an isomorphism, proving (a) ⇒ (b). While (b) ⇒ (c) ⇒ (d) is trivial, the implication (d) ⇒ (a) results from the fact that $\rho_2$ must be surjective if $\lambda_2$ is surjective. In order to conclude that (d) ⇒ (a) choose $F_\cdot$ as a minimal free resolution of $S/I$. Then we have a commutative diagram

$$\begin{array}{cccc}
\wedge^2 S^m & \to & S^m & \to S & \to 0 \\
\downarrow \gamma_2 & & \cong & & \\
F_2 & \to & F_1 & \to S & \to 0.
\end{array}$$

The map $\rho_2$ is just $\gamma_2 \otimes k$, and $\gamma_2 \otimes k$ being surjective, $\gamma$ is surjective itself. It follows immediately that $H_i(K_i(a)) = 0$, and this implies that $a$ is an $S$-sequence ([5], Theorem 16.5).
(c) Lemma 2 is false without the hypothesis that proj dim $S/I < \infty$. In fact, Serre [7] showed that the map $H_\ell(a, k) \to \text{Tor}_\ell(S/I, k)$ is injective if $a$ generates $I = n$. If $S$ is not regular, this yields a counterexample.

(d) The reader may have noticed that Theorem 2 is trivial if $R$ is a Cohen-Macaulay ring. Then dim $R = \text{depth } R$, and one always has $H_i(R) = 0$ for $i > \text{emb dim } R - \text{depth } R$ by the grade-sensitivity of the Koszul complex ([5], Theorem 16.8). On the other hand, if $H_\ell(R)^p \neq 0$ for $p = \text{emb dim } R - \text{depth } R$, then it follows easily from a theorem of Wiebe [9] that $R$ is a complete intersection. Cf. Gulliksen-Levin [4], 3.5.3. (There the number $n$ must be replaced by $\text{emb dim } R - \text{depth } R$; one first reduces to the case depth $R = 0$, and then applies Wiebe’s theorem.)

(e) It is easy to find rings $R$ which are not complete intersections, but for which $\lambda_1: H_\ell(R)^p \to H_\ell^p(R)$ is injective for $p = \text{emb dim } R - \text{dim } R$. This shows that Theorem 2 is optimal. □

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