ON THE NUMBER OF ELEMENTS INDEPENDENT
WITH RESPECT TO AN IDEAL

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Let $R$ denote a commutative noetherian ring, and $a$ an ideal of $R$. Elements $x_1, \ldots, x_n \in R$ are called independent with respect to $a$, or simply $a$-independent, if every form $F \in R[X_1, \ldots, X_n]$ such that $F(x_1, \ldots, x_n) = 0$ has all its coefficients in $a$. Valla [10] introduced the notation $\text{sup } a$ for the maximal number of $a$-independent elements in $a$. He found that $\text{sup } a$ is bounded above by the height of $a$ (which we denote by $\text{ht } a$) whereas the lower bound grade $a$ was established by Rees:

**Proposition 1.** grade $a \leq \text{sup } a \leq \text{ht } a$.

(The grade of $a$ is the maximal length of a $R$-sequence contained in $a$.) More precisely Rees provided in [9] the following proposition.

**Proposition 2.** Let $x_1, \ldots, x_n$ be a $R$-sequence. Then $x_1, \ldots, x_n$ are independent with respect to the ideal they generate in $R$.

The relationship between $R$-sequences and sequences of elements which are independent with respect to an ideal can best be explained in terms of the Rees ring of an ideal $a$. It is the graded ring

$$R(a) := \bigoplus_{i \geq 0} a^i.$$ Let $x_1, \ldots, x_n$ generate $a$. Then, as was shown in the proof of Theorem 2.1 of [9], $x_1, \ldots, x_n$ are independent with respect to an ideal $b \supset a$ if and only if the homomorphism $R[X_1, \ldots, X_n] \rightarrow R(a)$, which sends $X_i$ to $x_i$, induces an isomorphism

$$\phi : (R/b)[X_1, \ldots, X_n] \rightarrow R(a)/bR(a),$$

and exactly when $x_1, \ldots, x_n$ form a $R$-sequence, $\phi$ is already an isomorphism for $b = a$ (cf. [6; Theorem p. 202]), and observe that $R(a)/aR(a)$ is just the associated graded ring of $R$ with respect to $a$.

A system of parameters in a local ring $R$ is independent with respect to the maximal ideal $m_R$ of $R$; the most important consequence of this theorem is the analytic independence of a system of parameters in a complete equicharacteristic local ring [13; Theorem 2.1 and Corollary 2]. By an elementary localization argument one can further conclude that $\text{sup } b = \text{ht } b$ for radical ideals $b$ [10; Theorem 5.4].

We do not believe that $\text{sup } a$ can always be expressed in terms of better

Received 4 September, 1979.

J. LONDON MATH. SOC. (2), 22 (1980), 57–62]
understood invariants of \( a \). There are simple examples of ideals \( a \) with \( \sup a^n < \sup a \) for some \( n > 1 \). They demonstrate that it is in general impossible to compute \( \sup a \) from invariants which do not change if \( a \) is replaced by a power of itself. Thus one is led to study the “asymptotic stabilization”

\[
\sup^\infty a := \inf \{ \sup a^n : n \geq 1 \},
\]

and this is the main object of our note. The final theorem is a bit too technical to be given here. However, two special cases suffice to indicate the nature of our results:

(1) Let \( R \) be a local ring, and \( m_R \) its maximal ideal; then \( \sup^\infty m_R \) is the minimal Krull dimension of an associated prime ideal in the \( m_R \)-adic completion \( \hat{R} \) of \( R \):

\[
\sup^\infty m_R = \min \{ \dim \hat{R}/p : p \in \text{Ass } \hat{R} \}.
\]

(2) If \( R \) is an excellent domain, then \( \sup^\infty a = \sup a = \text{ht } a \) for every ideal \( a \) of \( R \).

We start our investigation by giving an upper bound for \( \sup^\infty a \). It is a consequence of Proposition 1 and the lemma of Artin–Rees.

**Proposition 3.** Let \( R \) be a noetherian ring, \( a \) an ideal of \( R \), and \( S \) a flat \( R \)-algebra. Then \( \sup^\infty a \leq \text{ht } (aS + p)/p \) for every associated prime ideal \( p \) of \( S \).

**Proof.** According to Proposition 1 it is enough to prove that \( \sup^\infty a \leq \sup^\infty (aS + p)/p \). An associated prime ideal \( p \) is the annihilator of an element \( x \in S \). As a consequence of the lemma of Artin–Rees (cf. [8, (3.12)]), there exists an integer \( j \) such that

\[
(a^jS) : xS \subset p + a^{i-j}S
\]

for \( i > j \). Let \( x_1, \ldots, x_n \) be elements of \( a^{m+j} \), where \( m \) is chosen such that \( \sup (a^mS + p)/p = \sup (a^{m}S + p)/p \). In case \( n > \sup (a^{m}S + p)/p \), one can find a form \( F \in S[X_1, \ldots, X_n] \), which is not contained in \( (a^mS + p)S[X_1, \ldots, X_n] \) and vanishes at \( (x_1, \ldots, x_n) \) modulo \( p \). Then \( xF(x_1, \ldots, x_n) = 0 \), and at least one coefficient of \( xF \) lies outside \( a^{m+j}S \) because of \((*)\). Since the extension from \( R \) to \( S \) is flat by hypothesis, every \( S \)-relation of elements of \( R \) is a linear combination of \( R \)-relations. Therefore \( x_1, \ldots, x_n \) can not be \( a^{m+j} \)-independent.

As an immediate consequence of Proposition 3 one obtains \( \sup^\infty m_R \leq \min \{ \dim \hat{R}/p : p \in \text{Ass } \hat{R} \} \) for the maximal ideal of a local ring \( R \). The lemma below is the first step towards the proof of the opposite inequality.

**Lemma.** Let \( R \) be a complete local ring with a single associated prime ideal \( p \), and \( x_1, \ldots, x_r \) a system of parameters of \( R \). Then there is a ring \( S \subset R \) with the following properties:

(a) \( S \) is a complete local ring. \( S \) is regular or a residue class ring of a regular local ring modulo a power of a prime element;
(b) \( R \) is a finitely generated torsionfree \( S \)-module;
(c) the elements \( x_1, \ldots, x_r \) form a system of parameters in \( S \);
(d) for every non-zero \( x \in R \) there exists a \( f \in \text{Hom}_S(R, S) \) with \( f(x) \neq 0 \).
Proof. We use the structure theorems for complete local rings [5]. If \( R \) is equicharacteristic, then we choose \( S = K[[x_1, \ldots, x_r]] \) where \( K \) is a coefficient field of \( R \). Otherwise there exist a complete discrete valuation domain \( V \) and a local homomorphism \( \phi : V[[X_1, \ldots, X_r]] \to R \), which sends \( X_i \) to \( x_i \) and induces an isomorphism \( V/m_v \cong R/m_R \). Then we take \( S = V[[X_1, \ldots, X_r]]/\text{Ker } \phi \). In both cases \( S \) is complete, \( R \) is a finitely generated \( S \)-module by [8; (30.6)], and \( S \) is regular in the equicharacteristic case. The fundamental theorems on integral extensions imply that the dimensions of \( R \) and \( S \) coincide, that \( x_1, \ldots, x_r \) is a system of parameters in \( S \), and that \( p \cap S \) is the only associated prime ideal of \( S \). Therefore every non-zero divisor of \( S \) is a non-zero divisor of \( R \): the \( S \)-module \( R \) is torsionfree. Since

\[
\dim V[[X_1, \ldots, X_r]] = \dim R + 1 = \dim S + 1,
\]

\( \text{Ker } \phi \) is an ideal of height 1, and has a single prime divisor, as was just shown. Hence \( \text{Ker } \phi \) is generated by a power of a prime element in the factorial ring \( V[[X_1, \ldots, X_r]] \). This completes the proof of (a), (b) and (c).

Property (d) follows from (a) and (b): Every finitely generated torsionfree module \( M \) over \( S \) is torsionless, i.e. the natural homomorphism

\[
M \to \text{Hom}_S(\text{Hom}_S(M, S), S)
\]

is injective. In fact, it suffices that the localization of \( S \) with respect to its associated prime ideal is a Gorenstein ring [12; Theorem (A.1)].

PROPOSITION 4. Let \( R \) be a complete local ring with a single associated prime ideal, and \( x_1, \ldots, x_r \) a system of parameters in \( R \). Then, given an integer \( m \), the elements \( x_1^i, \ldots, x_r^i \) are independent with respect to \( m^i \) for \( i \) sufficiently large.

Proof. We choose \( S \) as in the preceding lemma. Let \( a_i \) denote the ideal generated by \( x_1^i, \ldots, x_r^i \) in \( S \), and put

\[
N_i := \bigcap \{ f^{-1}(a_i) : f \in \text{Hom}_S(R, S) \}.
\]

A theorem of Chevalley [8; (30.1)] tells us: If \( (b_i) \) is a descending chain of ideals in a complete local ring such that \( \bigcap b_i = 0 \), then \( b_i \subseteq m_i^d \) for all sufficiently large \( i \). This theorem remains valid if one replaces \( (b_i) \) by a descending chain \( (M_i) \) of submodules of a finitely generated \( T \)-module \( M \) and, correspondingly, \( m_i^d \) by \( m_i^d M \).

The \( S \)-modules \( N_i \) of \( R \) form a descending chain, Since \( \bigcap a_i = 0 \), property (d) of \( S \), as given by the lemma, guarantees that \( \bigcap N_i = 0 \), and thus \( N_i \subseteq m_i^d R \subseteq m_i^d \) as soon as \( i \) is large enough. Let now

\[
F = \sum_{j_1 + \ldots + j_r = d} a_{j_1 \ldots j_r} X_1^{j_1} \ldots X_r^{j_r}
\]

be a form in \( R[[X_1, \ldots, X_r]] \) such that \( F(x_1^i, \ldots, x_r^i) = 0 \). Then for all \( f \in \text{Hom}_S(R, S) \) we obtain

\[
f(F(x_1^i, \ldots, x_r^i)) = \sum_{j_1 + \ldots + j_r = d} f(a_{j_1 \ldots j_r}) x_1^{j_1 i} \ldots x_r^{j_r i} = 0.
\]
By Proposition 2, all the elements \( f(a_1, \ldots, a_r) \) are contained in \( a_i \), hence \( a_1 \cdot \ldots \cdot a_r \in N_i \subseteq m_R^n \) if \( i \) is sufficiently large.

The idea, by which we shall now derive the first main result, has already been used in the proof of [4; Satz 6].

**Theorem 1.** Let \( R \) be a local ring, and \( n_1 \cap \ldots \cap n_s = 0 \) a reduced primary decomposition of the zero ideal in \( \hat{R} \). Let the images of \( x_1, \ldots, x_s \in R \) under the natural homomorphism \( R \to \hat{R}/n_t \) form part of a system of parameters in \( \hat{R}/n_t \) for \( t = 1, \ldots, s \). Then, given \( m \), the elements \( x'_1, \ldots, x'_s \) are independent with respect to \( m_R^n \) for \( i \) sufficiently large.

**Proof.** Since \( m_R^n \cap R = m_R^n \), we may assume that \( R = \hat{R} \). By the preceding proposition, the elements \( x'_1, \ldots, x'_s \) modulo \( n_t \) are independent with respect to the \( k \)-th power of the maximal ideal of \( \hat{R}/n_t \) if \( i \) is chosen large enough. Hence \( x'_1, \ldots, x'_s \) themselves are independent with respect to

\[
a_k := \bigcap_{i=1}^s (m_R^k + n_t).
\]

Now we only need to prove that the inclusion \( a_k \subseteq m_R^n \) holds for large \( k \). This, however, is again a consequence of Chevalley's theorem since we have

\[
\bigcap_k a_k = \bigcap_{i=1}^s \left( \bigcap_k (m_R^k + n_t) \right) = \bigcap_{i=1}^s n_t = 0.
\]

**Corollary.** Let \( R \) be a local ring. Then

\[
\sup^\infty m_R = \min \{ \dim \hat{R}/p : p \in \text{Ass } \hat{R} \}.
\]

**Proof.** With the notation of Theorem 1, we can find elements \( x_1, \ldots, x_s \in R \), satisfying the hypothesis of that theorem, by elementary prime avoidance arguments, as long as \( r \leq \min \{ \dim \hat{R}/p : p \in \text{Ass } \hat{R} \} \).

The minimal dimension of an associated prime divisor of \( \hat{R} \) was thoroughly investigated in [2], and was (under mild restrictions) characterized as the maximal length of so-called quasi-\( R \)-sequences [2; (5.5)]. The preceding corollary provides another characterization by intrinsic properties of \( R \).

Now we can compute \( \sup^\infty a \) for ideals \( a \) in arbitrary noetherian rings, and simultaneously improve the lower bound of Proposition 1. Brodmann [3] recently showed that the set of prime ideals in \( R \), which are associated to \( R/a^nR \), is independent of \( n \) for \( n \) sufficiently large. We denote this collection of prime ideals by \( \text{Ass}^\infty a \). (To avoid any ambiguity: \( \hat{R}_p \) is the \( \mathbb{P}R_p \)-adic completion of \( R_p \).)

**Theorem 2.** Let \( R \) be a noetherian ring, and \( a \) an ideal in \( R \). Then

\[
\sup a \geq \min \{ \text{ht} (a\hat{R}_p + q)/q : p \in \text{Ass } R/a, \quad q \in \text{Ass } \hat{R}_p \},
\]

and

\[
\sup^\infty a = \min \{ \text{ht} (a\hat{R}_p + q)/q : p \in \text{Ass}^\infty a, \quad q \in \text{Ass } \hat{R}_p \}.
\]
Proof. In view of Proposition 3 it is enough to prove the inequality. We use an obvious localization argument. The ideal $a$ has a reduced primary decomposition:

$$a = \bigcap_{i=1}^{s} b_i, \quad b_i \text{ being primary to a prime ideal } p_i, \text{ and } \{ p_1, \ldots, p_s \} = \text{Ass } R/a.$$ 

Let $r$ denote the number on the right hand side of the inequality. Using Krull's Principal Ideal Theorem and elementary prime avoidance arguments one constructs elements $x_1, \ldots, x_r \in a$, which have the following property: their images under each of the natural homomorphism $R \rightarrow \hat{R}_p/q$ form part of a system of parameters in $\hat{R}_p/q$, $p \in \text{Ass } R/a$, $q \in \text{Ass } \hat{R}_p$. Then $x_1, \ldots, x_r$ satisfy the requirements of Theorem 1 in every localization $\hat{R}_p$, $p \in \text{Ass } R/a$. Hence $x_1', \ldots, x_r'$ are $b_i R_p$-independent, $b_i R_p$ containing a power of $p_i R_p$, as soon as $i$ is large enough. This suffices to render $x_1', \ldots, x_r'$ independent with respect to $a$ since $a = \bigcap_{i=1}^{s} b_i$ and $b_i = b_i R_p \cap R$.

Theorem 2 answers many questions raised in [1] and [11]. In particular, we can describe the rings in which $\sup a = \text{ht } a$ for all ideals.

**Corollary 1.** Let $R$ be a noetherian ring. Then $\sup a = \text{ht } a$ for all ideals $a$ of $R$ if and only if $\dim \hat{R}_p/q = \text{ht } p$ for all prime ideals $p$ of $R$ and all associated prime ideals $q$ of $\hat{R}_p$.

**Proof.** If $\sup a = \text{ht } a$ for all ideals $a$ of $R$, then in particular $\sup^\infty p = \text{ht } p$, and

$$\sup^\infty p \leq \text{ht } (p \hat{R}_p + q)/q \leq \text{ht } p \hat{R}_p = \text{ht } p$$

by Proposition 3. On the other hand, if $\dim \hat{R}_p/q = \text{ht } p$ for all prime ideals $p$ and all associated prime ideals $q$ of $\hat{R}_p$, then

$$\text{ht } a = \min \{ \dim R_p : p \in \text{Ass } R/a \}$$

$$= \min \{ \dim \hat{R}_p/q : p \in \text{Ass } R/a, \quad q \in \text{Ass } \hat{R}_p \}$$

$$\leq \sup a \leq \text{ht } a$$

by Proposition 1 and Theorem 2.

Nagata gave an example of a two dimensional local domain $R$ [8; pp. 204, 205] such that $\hat{R}$ has an associated prime ideal $q$ with $\dim \hat{R}/q = 1$. Thus, even in local domains $\sup a$ does not always equal $\text{ht } a$. However, in a class of rings, which behave well under completion, we can simplify our formula considerably:

**Corollary 2.** Let $R$ be an excellent integral domain. Then for every ideal $a$ in $R$ one has $\sup a = \text{ht } a$.

**Proof.** The localizations of $R$ are excellent again, and the completions of excellent local domains are (reduced and) equi-dimensional (cf. [7]).
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References


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