Realization problems in algebraic topology

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Outline

1. Background
2. Obstruction theory
3. Quillen cohomology
4. Classification results
5. Realizability results
6. Related work
Let $X$ be a space.

- $H^*(X; \mathbb{F}_p)$ is an unstable algebra over the Steenrod algebra $\mathcal{A}$.
- $H_*(X; \mathbb{F}_p)$ is an unstable coalgebra over $\mathcal{A}$.
- $H^*(X; \mathbb{Q})$ is graded commutative $\mathbb{Q}$-algebra.
- $\pi_*X$ is a $\Pi$-algebra, i.e., graded group with action of primary homotopy operations.

Let $X$ be a spectrum and $E$ a ring spectrum, e.g., $E = H\mathbb{F}_p$ or $KU$.

- $E^*X$ is an $E^*E$-module.
- $E_*X$ is an $E_*E$-comodule.
- $\pi_*X$ is a $\pi^S_*$-module, where $\pi^S_* = \pi_*(S)$ is the stable homotopy ring.
Π-algebras

Π-algebra ≈ graded group with additional structure which looks like the homotopy groups of a space.

**Definition**

- Π := full subcategory of the homotopy category of pointed spaces consisting of finite wedges of spheres $\bigvee S^{n_i}$, $n_i \geq 1$.
- Π-algebra := product-preserving functor $A: \Pi^{\text{op}} \to \text{Set}_*$.

**Example**

$\pi_* X = [-, X]$ for a pointed space $X$.

**Notation**

Write $A_n := A(S^n)$.
Primary operations

Example

\[ S^n \xrightarrow{\text{pinch}} S^n \vee S^n \]

induces the group structure

\[ A_n \times A_n \xrightarrow{A(\text{pinch})} A_n. \]

Example

\[ S^{p+q-1} \xrightarrow{w} S^p \vee S^q \]

induces the Whitehead product

\[ A_p \times A_q \xrightarrow{A(w)} A_{p+q-1}. \]
Realizations

Realization Problem
Given a \( \Pi \)-algebra \( A \), is there a space \( X \) satisfying \( \pi_* X \cong A \) as \( \Pi \)-algebras?

Classification Problem
If \( A \) is realizable, can we classify all realizations?
Some examples

- Simplest $\Pi$-algebras: Only one non-trivial group $A_n$.
  - Answer: Always realizable (uniquely), by an Eilenberg–MacLane space $K(A_n, n)$.
- Next simplest case: Only 2 non-trivial groups $A_n, A_{n+k}$. Assume $n \geq 2$.
  - Answer: **Not** always realizable...

Warm-up

Case $k = 1$: Always realizable (classic).
Case $k = 2$: Always realizable (a bit of work).
Simply connected rational \( \Pi \)-algebra, i.e., \( A_1 = 0 \) and \( A_n \) is a \( \mathbb{Q} \)-vector space (for every \( n \geq 2 \)).

Same as a reduced graded Lie algebra \( L_* := A_{*+1} \) over \( \mathbb{Q} \), with respect to Whitehead products.

Answer: Always realizable as the homotopy Lie algebra \( L_* \cong \pi_{*+1}X \) of a rational space \( X \), by Quillen’s theorem.

A realization may not be unique, e.g., if \( X \) is not formal.
Classify?

- Naive: List of realizations $= \pi_0 \mathcal{T} \mathcal{M}(A)$.
- Better: **Moduli space** $\mathcal{T} \mathcal{M}(A)$ of realizations.

**Remark**

Relative moduli space $\mathcal{T} \mathcal{M}'(A)$: Realizations $X$ with identification $\pi_* X \simeq A$. Have fiber sequence:

\[ \mathcal{T} \mathcal{M}'(A) \xrightarrow{\text{forget}} \mathcal{T} \mathcal{M}(A) \to B \text{Aut}(A) \]

and $\mathcal{T} \mathcal{M}(A) \simeq \mathcal{T} \mathcal{M}'(A)_{h \text{Aut}(A)}$. 

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Realization problems

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\( \mathcal{M}(A) = \text{nerve of the category with} \)

- Objects: Realizations \( X \).
- Morphisms: Weak equivalences \( X \to X' \).

\[
\mathcal{M}(A) \simeq \bigsqcup_{\langle X \rangle} B \text{Aut}^h(X).
\]
Building $\mathcal{T}M(A)$

- Blanc–Dwyer–Goerss (2004): Obstruction theory for building $\mathcal{T}M(A)$.
- $\infty$-categorical reinterpretation by Pstrągowski (2017).
- Successive approximations $\mathcal{T}M_n(A)$, $0 \leq n \leq \infty$. 
Building $\mathcal{T}M(A)$

\[
\begin{array}{c}
\mathcal{T}M \\
\sim \\
\mathcal{T}M_\infty \longrightarrow \text{holim}_n \mathcal{T}M_n \\
\Downarrow \\
\vdots \\
\Downarrow \\
\mathcal{T}M_1 \\
\Downarrow \\
\mathcal{T}M_0
\end{array}
\]
Building $\mathcal{T}M(A)$

- $\mathcal{T}M_0(A) \simeq B\text{Aut}(A)$.
- $\mathcal{T}M_n(A) \to \mathcal{T}M_{n-1}(A)$ related by a fiber square.
- For $Y$ in $\mathcal{T}M_{n-1}$ and $M(Y) \subseteq \mathcal{T}M_{n-1}$ its component, we have:
  \[
  \mathcal{H}^{n+1}(A; \Omega^n A) \to \mathcal{T}M_n(A)_Y \to M(Y)
  \]
  where fiber $=$ Quillen cohomology “space”.

- Obstruction to lifting $\in HQ^{n+2}(A; \Omega^n A)$
- Lifts classified by $\pi_0(\text{fiber}) = HQ^{n+1}(A; \Omega^n A)$.

**Problem**

Can we compute the obstruction groups?
Beck modules

Definition

Let \( C \) be an algebraic category and \( X \) an object in \( C \). A (Beck) module over \( X \) is an abelian group object in the slice category over \( X \):

\[(C/X)_{ab}.\]

Example

\( C = \text{Groups} \). A Beck module over \( G \) is a split extension:

\[G \ltimes M \twoheadrightarrow G.\]

Note: \((g, m)(g', m') = (gg', m + gm')\).
Example

$C = \text{Commutative rings. A Beck module over } R \text{ is a square-zero extension:}$

$$R \oplus M \rightarrow R.$$ 

Note: $(r, m)(r', m') = (rr', rm' + mr').$
**Definition**

**Quillen cohomology** of $X$ with coefficients in a module $M$ is:

$$
HQ^*(X; M) := \pi^* \text{Hom}(C\ldots, M)
$$

where $C\ldots \sim X$ is a cofibrant replacement in $sC$, the category of simplicial objects in $C$.

**Example**

For $C =$ Commutative rings, this is the classic André–Quillen cohomology.
Truncated $\Pi$-algebras

**Definition**

A $\Pi$-algebra $A$ is $n$-truncated if it satisfies $A_i = \ast$ for all $i > n$.

- Postnikov truncation $P_n : \Pi\text{Alg} \rightarrow \Pi\text{Alg}^n_1$.
- $P_n$ is left adjoint to inclusion $\iota : \Pi\text{Alg}^n_1 \rightarrow \Pi\text{Alg}$.
- Unit map $\eta_A : A \rightarrow P_nA$. 
Let $A$ be a $\Pi$-algebra and $N$ a module over $A$ which is $n$-truncated. Then the natural comparison map

$$\text{HQ}_{\Pi\text{Alg}}^n(P_nA; N) \xrightarrow{\approx} \text{HQ}_{\Pi\text{Alg}}(A; N).$$

induced by the Postnikov truncation functor $P_n$ is an isomorphism.
Highly connected $\Pi$-algebras

**Definition**

A $\Pi$-algebra $A$ is **$n$-connected** if it satisfies $A_i = \ast$ for all $i \leq n$.

- $n$-connected cover $C_n : \Pi\text{Alg} \to \Pi\text{Alg}_n$.  
- $C_n$ is *right* adjoint to inclusion $\iota : \Pi\text{Alg}_n \to \Pi\text{Alg}$.
- Counit map $\epsilon_A : C_nA \to A$.  

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Theorem (F.)

Let $B$ be an $n$-connected $\Pi$-algebra and $M$ a module over $\iota B$. Then the natural comparison map

$$\text{HQ}^*_{\Pi \text{Alg}}(\iota B; M) \xrightarrow{\text{R}} \text{HQ}^*_{\Pi \text{Alg}}^{\infty}(B; C_n M)$$

induced by the connected cover functor $C_n$ is an isomorphism.

Remark

More general comparison theorem for adjunctions $F : C \rightleftarrows D : G$ between algebraic categories.
2-stage Example

- Take $A_i = 0$ for $i \neq 1, n$.
- $A$ is realizable, e.g., Borel construction

$$BA_1(A_n, n) := EA_1 \times_{A_1} K(A_n, n) \to BA_1.$$  

Theorem

$$\mathcal{TM}(A) \simeq \text{Map}_{BA_1}(BA_1, BA_1(A_n, n + 1))_{h \text{Aut}(A)}.$$

Upshot

Classification by a $k$-invariant is promoted to a moduli statement: The moduli space of realizations is the mapping space where the $k$-invariant lives.
Corollary

- \( \pi_0 \mathcal{TM}(A) \simeq H^{n+1}(A_1; A_n)/Aut(A) \)

- For any choice of basepoint in \( \mathcal{TM}(A) \), we have:

\[
\pi_i \mathcal{TM}(A) \simeq \begin{cases} 
0, & i > n \\
\text{Der}(A_1, A_n), & i = n \\
H^{n+1-i}(A_1; A_n), & 2 \leq i < n 
\end{cases}
\]

and \( \pi_1 \mathcal{TM}(A) \) is an extension by \( H^n(A_1; A_n) \) of a subgroup of \( Aut(A) \) corresponding to realizable automorphisms.
Stable 2-types

- Take $A_i = 0$ for $i \neq n, n + 1$, for some $n \geq 2$.
- $A$ is realizable.

**Theorem**

$\mathcal{T} \mathcal{M}'(A)$ is connected and its homotopy groups are:

$$
\pi_i \mathcal{T} \mathcal{M}'(A) \simeq \begin{cases} 
0, & i \geq 3 \\
\text{Hom}_\mathbb{Z}(A_n, A_{n+1}), & i = 2 \\
\text{Ext}_\mathbb{Z}(A_n, A_{n+1}), & i = 1.
\end{cases}
$$
Corollary

\[ TM(A) \cong TM' (A)_{h \text{Aut}(A)} \text{ is connected; its homotopy groups are:} \]

\[ \pi_i TM(A) \cong \begin{cases} 0, & i \geq 3 \\ \text{Hom}_\mathbb{Z}(A_n, A_{n+1}) i = 2 & \end{cases} \]

and \( \pi_1 TM(A) \) is an extension of \( \text{Aut}(A) \) by \( \text{Ext}_\mathbb{Z}(A_n, A_{n+1}) \). In particular, all automorphisms of \( A \) are realizable.

Remark

Few higher automorphisms.
Homotopy operation functors

A \( \Pi \)-algebra \( A \) concentrated in degrees \( n, n + 1, \ldots, n + k \) can be described inductively by abelian groups and structure maps:

\[
\begin{align*}
A_n \\
\eta_1 : \Gamma_n^1(A_n) &\rightarrow A_{n+1} \\
\eta_2 : \Gamma_n^2(A_n, \eta_1) &\rightarrow A_{n+2} \\
&\ldots \\
\eta_k : \Gamma_n^k(\pi_n, \eta_1, \ldots, \eta_{k-1}) &\rightarrow A_{n+k}.
\end{align*}
\]

Example

\[
\Gamma_n^1(A_n) = \begin{cases} 
\Gamma(A_n) & \text{for } n = 2 \\
A_n \otimes_{\mathbb{Z}} \mathbb{Z}/2 & \text{for } n \geq 3.
\end{cases}
\]

and \( \eta_1 : \Gamma_n^1(A_n) \rightarrow A_{n+1} \) is precomposition by the Hopf map \( \eta : S^{n+1} \rightarrow S^n \).
A 2-stage $\Pi$-algebra $A$ consists of the data

$$\eta_k : \widetilde{\Gamma}_n^k(A_n) := \Gamma_n^k(A_n, 0, \ldots, 0) \to A_{n+k}.$$

Example

$$\widetilde{\Gamma}_3^2(A_3) = \Lambda(A_3) = A_3 \otimes A_3/(a \otimes a),$$

the exterior square, and

$$\eta_2 : \Lambda(A_3) \to A_5$$

encodes the Whitehead product.
2-stage case (cont’d)

Notation

\( Q_{k,n} \) := indecomposables of \( \pi_{n+k}(S^n) \)

In the stable range \( k \leq n - 2 \), we have \( Q_{k,n} = Q_k^S \), where \( Q_k^S := \) indecomposables of the graded ring \( \pi_*^S \).

Proposition

Assuming \( k \neq n - 1 \), we have

\[
\widetilde{\Gamma}^k_n(A_n) = A_n \otimes_{\mathbb{Z}} Q_{k,n}.
\]

In particular, in the stable range we have \( \widetilde{\Gamma}^k_n(A_n) = A_n \otimes_{\mathbb{Z}} Q_k^S \).
Criterion for realizability

Theorem (Baues,F.)

The 2-stage $\Pi$-algebra given by $\eta_k : \tilde{\Gamma}_n^k(A_n) \to A_{n+k}$ is realizable if and only if the map $\eta_k$ factors through the map $\gamma_K(A_n,n)$:

$$
\begin{array}{cc}
\tilde{\Gamma}_n^k(A_n) & A_{n+k} \\
\gamma_K(A_n,n) & H_{n+k+1}(K(A_n,n)) \\
\eta_k & \downarrow \\
& \gamma
\end{array}
$$
Corollary

Fix $n \geq 2$ and $k \geq 1$. Then an abelian group $A_n$ has the property that “every $\Pi$-algebra concentrated in degrees $n, n + k$ with prescribed group $A_n$ is realizable” if and only if the map

$$\gamma_{K(A_n, n)} : \Gamma_n^k(A_n) \to H_{n+k+1}K(A_n, n)$$

is split injective.
First few stable homotopy groups of spheres $\pi_*^S$ and their indecomposables $Q_*^S$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\pi^S_k$</th>
<th>$Q^S_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>1</td>
<td>$\mathbb{Z}/2\langle \eta \rangle$</td>
<td>$\mathbb{Z}/2\langle \eta \rangle$</td>
</tr>
<tr>
<td>2</td>
<td>$\mathbb{Z}/2\langle \eta^2 \rangle$</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>$\mathbb{Z}/24 \simeq \mathbb{Z}/8\langle \nu \rangle \oplus \mathbb{Z}/3\langle \alpha \rangle$</td>
<td>$\mathbb{Z}/12 \simeq \mathbb{Z}/4\langle \nu \rangle \oplus \mathbb{Z}/3\langle \alpha \rangle$</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>$\mathbb{Z}/2\langle \nu^2 \rangle$</td>
<td>0</td>
</tr>
</tbody>
</table>
Look at stem $k = 3$.  

**Proposition**  

Let $n \geq 5$. The (stable) $\Pi$-algebra concentrated in degrees $n, n + 3$ given by $A_n = \mathbb{Z}$ and $A_{n+3} = \mathbb{Z}/4$ with structure map

$$\eta_3 : A_n \otimes_\mathbb{Z} Q^S_3 \cong \mathbb{Z}/4 \langle \nu \rangle \oplus \mathbb{Z}/3 \langle \alpha \rangle \to \mathbb{Z}/4$$

sending $\nu$ to 1 is not realizable.

**Proof.**

$HZ_4HZ \cong \mathbb{Z}/6$

$$\gamma : Q^S_3 \cong \mathbb{Z}/4 \langle \nu \rangle \oplus \mathbb{Z}/3 \langle \alpha \rangle \to HZ_4HZ \text{ sends } 2\nu \text{ to } 0.$$
Infinite families

Look at Greek letter elements in the stable homotopy groups of spheres $\pi^S_*$.

**Proposition**

Assume $p \geq 3$.

1. The first alpha element $\alpha_1 \in Q^{S}_{2(p-1)-1}$ is **not** in the kernel of $\gamma$.

2. Higher alpha elements $\alpha_i \in Q^{S}_{2i(p-1)-1}$ for $i > 1$ are in the kernel of $\gamma$.

3. Generalized alpha elements $\alpha_{i/j} \in Q^{S}_{*}$ for $j > 1$ satisfy $p\alpha_{i/j} \neq 0$ but $\gamma(p\alpha_{i/j}) = 0$.

**Proof.**

(3) $\alpha_{i/j}$ has order $p^i$ in $\pi^S_*$.

The $p$-torsion in $HZ_*HZ$ is all of order $p$ (and not $p^2$, $p^3$, etc.).
Infinite families (cont’d)

Upshot
This provides infinite families of non-realizable 2-stage (stable) $\Pi$-algebras.
Let $E$ be a homotopy commutative ring spectrum.

$X$ an $E_\infty$ ring spectrum $\rightsquigarrow E_*X$ is an $E_*$-algebra in $E_*E$-comodules.

Realizations of $E_*E$ correspond to $E_\infty$ ring structures on $E$.

Applications to chromatic homotopy theory. Morava $E$-theory $E_n$ admits a unique $E_\infty$ ring structure.
Realizing unstable algebras over the Steenrod algebra as $H^*(X; \mathbb{F}_p)$ for some space $X$.

Classifying realizations via higher order cohomology operations [Blanc–Sen (2017)].

Realizing unstable coalgebras over the Steenrod algebra as $H_*(X; \mathbb{F}_p)$ for some space $X$. [Blanc (2001), Biedermann–Raptis–Stelzer (2015)]

Stable analogues.
Let $E$ be an $H_\infty$ ring spectrum.

- $X$ an $H_\infty$ $E$-algebra $\sim \pi_* X$ is an $E_*$-algebra with power operations.
- $E = H_{\mathbb{F}_p}$: Dyer-Lashof operations, e.g., acting on the mod $p$ homology of an infinite loop space.
- $E = K^\wedge_p$: $\theta$-algebras over the $p$-adic integers $\mathbb{Z}_p$.
- $E = \text{Morava } E$-theory $E_n$: power operations have been studied.
Higher order operations

$X$ a space or spectrum $\sim H^*(X; \mathbb{F}_p)$ a module over the Steenrod algebra (primary cohomology operations)
+ secondary operations
+ tertiary operations
+ etc.

With all higher order cohomology operations, we can recover the $p$-type of $X$. 
Thank you!
