Math 535 - General Topology
Glossary

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1 Separation axioms

Definition 1.1. A topological space X is called:

- $\mathbf{T_0}$ or **Kolmogorov** if any distinct points are topologically distinguishable: For $x, y \in X$ with $x \neq y$, there is an open subset $U \subset X$ containing one of the two points but not the other.

- $\mathbf{T_1}$ if any distinct points are separated (i.e. not in the closure of the other): For $x, y \in X$ with $x \neq y$, there are open subsets $U_x, U_y \subset X$ satisfying $x \in U_x$ but $y \notin U_x$, whereas $y \in U_y$ but $x \notin U_y$.

- $\mathbf{T_2}$ or **Hausdorff** if any distinct points can be separated by neighborhoods: For $x, y \in X$ with $x \neq y$, there are open subsets $U_x, U_y \subset X$ satisfying $x \in U_x$, $y \in U_y$, and $U_x \cap U_y = \emptyset$.

- **regular** if points and closed sets can be separated by neighborhoods: For $x \in X$ and $C \subset X$ closed with $x \notin C$, there are open subsets $U_x, U_C \subset X$ satisfying $x \in U_x$, $C \subset U_C$, and $U_x \cap U_C = \emptyset$.

- $\mathbf{T_3}$ if it is T_1 and regular.

- **completely regular** if points and closed sets can be separated by functions: For $x \in X$ and $C \subset X$ closed with $x \notin C$, there is a continuous function $f \colon X \to [0, 1]$ satisfying $f(x) = 0$ and $f|_C \equiv 1$.

- $\mathbf{T_{3\frac{1}{2}}}$ or **Tychonoff** if it is T_1 and completely regular.

- **normal** if closed sets can be separated by neighborhoods: For $A, B \subset X$ closed and disjoint, there are open subsets $U, V \subset X$ satisfying $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

- $\mathbf{T_4}$ if it is T_1 and normal.

- **perfectly normal** if closed sets can be precisely separated by functions: For $A, B \subset X$ closed and disjoint, there is a continuous function $f \colon X \to [0, 1]$ satisfying $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

- $\mathbf{T_6}$ if it is T_1 and perfectly normal.

2 Compactness

Definition 2.1. A topological space X is called:

- **compact** if every open cover of X admits a finite subcover.
- **countably compact** if every countable open cover of X admits a finite subcover.
- **sequentially compact** if every sequence in X has a convergent subsequence.
- **Lindelöf** if every open cover of X admits a countable subcover.
- **locally compact** if every point $x \in X$ has a compact neighborhood.
- **σ-compact** if X is a countable union of compact subspaces.
- **paracompact** if every open cover of X admits a locally finite refinement.
- **hemicompact** if there is a countable collection of compact subspaces $K_n \subseteq X$ such that for any compact subspace $K \subseteq X$, there is an $n \in \mathbb{N}$ satisfying $K \subseteq K_n$.

3 Countability axioms

Definition 3.1. A topological space X is called:

- **first-countable** if every point $x \in X$ has a countable neighborhood basis.
- **second-countable** if the topology on X has a countable basis.
- **separable** if X has a countable dense subset.

4 Connectedness

Definition 4.1. A topological space X is called:

- **connected** if X is not a disjoint union of non-empty open subsets.
- **locally connected** if for all $x \in X$ and neighborhood U of x, there is a connected neighborhood V of x satisfying $V \subseteq U$.
- **path-connected** if any two points of X can be joined by a path.
- **locally path-connected** if for all $x \in X$ and neighborhood U of x, there is a path-connected neighborhood V of x satisfying $V \subseteq U$.

5 Properties of maps

Definition 5.1. A function $f\colon X \to Y$ between topological spaces is called:

- **continuous** if for any open $U \subseteq Y$, the preimage $f^{-1}(U) \subseteq X$ is open in X.
- **open** if for any open $U \subseteq X$, the image $f(U) \subseteq Y$ is open in Y.
- **closed** if for any closed $C \subseteq X$, the image $f(C) \subseteq Y$ is closed in Y.
- a **homeomorphism** if it is bijective and its inverse $f^{-1}\colon Y \to X$ is continuous.
- an **embedding** if it is injective and a homeomorphism onto its image $f(X)$.
- a **quotient map** or **identification map** if it is surjective and Y has the quotient topology induced by f.
- **proper** if for any compact subspace $K \subseteq Y$, the preimage $f^{-1}(K) \subseteq X$ is compact.

Definition 5.2. A function $f\colon X \to Y$ between *metric* spaces is called:

- **uniformly continuous** if for any $\epsilon > 0$, there is a $\delta > 0$ satisfying $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$ for all $x \in X$.
- **Lipschitz continuous** with Lipschitz constant $K \geq 0$ it satisfies the inequality

  $$d\left(f(x), f(x')\right) \leq K d(x, x')$$

  for all $x, x' \in X$.