

# LIFTABLE INTEGRAL CLOSURE

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ABSTRACT. In this paper, we introduce a new closure operation on submodules, the *liftable integral closure* of a submodule. We develop its basic properties, including its relationships with the two prevailing notions of integral closure of submodules. We show that for a quite general class of local rings, every finite length module may be represented as a quotient of the form  $T/L$ , where  $T$  is torsionless and integrally dependent on  $L$ . In the final section, we provide an approach toward solving the direct summand conjecture.

## INTRODUCTION

The classical notion of integral closure of ideals (developed in the 1940s) became so useful that in 1987, Rees [Ree87] defined integral closure of modules. Since then, other definitions were proposed, not all of them equivalent, but each having its advantages. On the other hand, in the words of Huneke and Swanson [HS06, in the middle of page 303], “Every choice of definition has its own problems.”

Some definitions (most notably the one in [EHU03]) place a value on independence of embedding. That is, if  $L \subseteq M \subseteq N$ , then the integral closure of  $L$  in  $M$  should be the intersection of  $M$  with the integral closure of  $L$  in  $N$ . According to this approach also, if  $N, T$  are “big enough” modules containing  $L$ , then the integral closure of  $L$  in  $N$  should agree with the integral closure of  $L$  in  $T$ .

In the present work, we take the opposite approach. That is, if  $L \subseteq M$ , our ‘liftable integral closure’ of  $L$  in  $M$ , denoted  $L_M^-$ , depends *only* on the embedding, in the sense that it only depends on the module structure of the quotient module  $M/L$ . The idea came to the first named author because he was trained in tight closure theory, and this is the approach taken when defining tight closure of submodules. As it happens, this approach has multiple surprising applications, not least a theorem about representing finite length modules in terms of integral extensions of torsionless modules.

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We wish to stress here that we are not claiming this new notion to be the “right” definition of the integral closure of a submodule. We agree, of course, with the Huneke-Swanson quote above. Instead, the new operation should be seen as a different and useful perspective in the theory of integral closure of modules.

The structure of the paper is as follows: In §1, we discuss some pre-existing notions of integral closure and their interrelationships, paying special attention to the case of not necessarily finitely generated modules. In §2, we define the *liftable integral closure*  $L_M^-$  relative to a submodule inclusion  $L \subseteq M$  and develop its basic properties (see especially Lemma 2.4 and Proposition 2.5). In §3 we give notions of ‘test ideals’ relative to liftable integral closure, and we show that there are usually only one or two such ideals per ring (denoted  $\tau_{\mathcal{I}}$  and  $\tau_{\mathcal{M}}$ ), despite a plethora of definitions (see Definition/Proposition 3.1). In §4, we characterize these ideals based mostly on the dimension of the ring. For 0-dimensional rings,  $\tau_{\mathcal{I}}$  is the socle (Proposition 4.2); for 1-dimensional Cohen-Macaulay rings,  $\tau_{\mathcal{I}}$  is the conductor (Theorem 4.4), thus generalizing a theorem of J. Vassilev; for equidimensional unmixed rings of dimension at least two,  $\tau_{\mathcal{M}} = \tau_{\mathcal{I}} = 0$  (Theorem 4.5). In §5, we provide some surprising consequences of this latest result. For instance (Theorem 5.2), given a finite length module  $M$  over a Noetherian equidimensional unmixed local ring of dimension at least two, there exists a finite free module  $F$  and submodules  $L \subseteq T \subseteq F$  such that  $T$  is integral over  $L$  in  $F$ . This raises the question: what other sorts of finitely generated modules may be represented as a quotient of finite torsionless modules sharing a common integral closure? Finally, in §6, we show how liftable integral closure may be combined with a theorem of G. Dietz to attempt a solution of the direct summand conjecture, and we provide a first step (Proposition 6.2) toward realizing this goal.

## 1. SOME HISTORY OF INTEGRAL CLOSURES OF MODULES

Relative to an embedding, there is a commonly held notion of the integral closure of a submodule of a free module, especially in the case where the submodule has a rank. More generally, we make the following definition:

**Definition 1.1** (Integral closure in a free module). Let  $L \subseteq F$  be  $R$ -modules, where  $F$  is free. Let  $S$  be the (naturally graded) symmetric algebra over  $R$  defined by  $F$ , and let  $T$  be the subring of  $S$  induced by the inclusion  $L \subseteq F$ . Note that  $S$  is  $\mathbb{N}$ -graded and generated in degree 1 over  $R$ , and  $T$  is an  $\mathbb{N}$ -graded subring of  $S$ , also generated over  $R$

in degree 1. Then the *integral closure of  $L$  in  $F$* , denoted  $L_F^-$ , is the degree 1 part of the integral closure of the subring  $T$  of  $S$ .

This definition agrees with all the usual definitions of the integral closure relative to an embedding in any situation where  $R$  is Noetherian and  $F$  is finitely generated and free. See [HS06, Chapter 16] for details. As for integral closure of a submodule of an arbitrary (finitely generated) module, there seem to be two prevailing notions – one by Rees [Ree87], and one by Eisenbud, Huneke, and the second author of this work [EHU03]. We give them both below:

**Definition 1.2** (Rees integral closure). Let  $L \subseteq M$  be modules<sup>1</sup> over a Noetherian ring  $R$ . First suppose  $R$  is a domain, and let  $Q$  be its fraction field. Then for  $x \in M$ , we say that  $x$  is in the *Rees integral closure of  $L$  in  $M$* , written  $x \in L_M^{-\text{Rs}}$ , if for every valuation ring  $V$  between  $R$  and  $Q$ ,  $x$  is in the image  $LV$  of  $L$  in  $M \otimes_R V$ .

In the general case, we write  $x \in L_M^{-\text{Rs}}$  if for every minimal prime  $\mathfrak{p}$  of  $R$ , we have  $x + \mathfrak{p}M \in \left(\frac{L + \mathfrak{p}M}{\mathfrak{p}M}\right)_{M/\mathfrak{p}M}^{-\text{Rs}}$ , as modules over the ring  $R/\mathfrak{p}$ .

**Definition 1.3** (EHU integral closure). Let  $L \subseteq M$  be modules over a Noetherian ring  $R$ . Then for  $x \in M$ , we say that  $x$  is in the *EHU integral closure of  $L$  in  $M$* , written  $x \in L_M^{-\text{EHU}}$ , if for every  $R$ -homomorphism of the form  $g : M \rightarrow F$  for free modules  $F$ , we have  $g(x) \in g(L)_F^-$ .

These would seem to have nothing to do with each other. But we have the following:

**Proposition 1.4** (Valuative Criterion). *Let  $L \subseteq M$  be modules over a Noetherian ring  $R$  and let  $x \in M$ . Then  $x \in L_M^{-\text{EHU}}$  if and only if for every map  $M \rightarrow F$  to a free module and every homomorphism  $R \rightarrow V$  to a valuation ring  $V$  with kernel a minimal prime of  $R$ , we have  $(L + Rx)' = L'$ , where  $(-)'$  denotes tensoring with  $V$  and taking images in  $V \otimes_R F$ .*

Under the assumption that  $M$  is finitely generated (and using only Noetherian valuation rings), this was proved as [EHU03, the “*Valuative Criterion*” part of Theorem 2.2]. We postpone the proof of the general case until page 6, after we have developed some tools for dealing with non-finitely generated modules.

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<sup>1</sup>Rees only defined this notion for finitely generated modules, and he required that the valuation rings  $V$  be Noetherian. However, one may employ the usual tricks to show that for finitely generated modules, the definition given here is equivalent to his.

**Corollary 1.5.** *Let  $R$  be a Noetherian ring and  $L \subseteq M$   $R$ -modules. Then  $L_M^{-\text{Rs}} \subseteq L_M^{-\text{EHU}}$ , with equality if  $M$  is free (in which case both coincide with the first definition of this paper) or if  $R$  is a domain.*

In general, however, Rees integral closure may be strictly smaller than EHU integral closure:

**Example 1.6.** [EHU03, immediately after the proof of Theorem 2.2] Let  $k$  be a field,  $R := k[x]/(x^2)$ , and  $M := Rx$ . Then  $0_M^{-\text{EHU}} = M$ , but  $0_M^{-\text{Rs}} = 0$ .

## 2. LIFTABLE INTEGRAL CLOSURE - BASIC PROPERTIES

Next we come to the primary definition of the current paper:

**Definition 2.1** (Liftable integral closure). Let  $L \subseteq M$  be  $R$ -modules. Let  $\pi : F \twoheadrightarrow M$  be a surjection of a free  $R$ -module  $F$  onto  $M$ . Let  $K := \pi^{-1}(L)$ . Then the *liftable integral closure* of  $L$  in  $M$  is

$$L_M^- := \pi(K_F^-).$$

Given such a surjection,  $L_M^-$  is obviously a submodule of  $M$ . We need to show this is well-defined (i.e. independent of  $\pi$ ). First we need the following.

**Proposition 2.2.** *Let  $F, G$  be free  $R$ -modules, let  $L \subseteq F$  and  $M \subseteq G$  be submodules. Then  $(L \oplus M)_{F \oplus G}^- = L_F^- \oplus M_G^-$ .*

*In particular,  $(L \oplus G)_{F \oplus G}^- = L_F^- \oplus G$ .*

*Proof.* Say  $F$  is freely generated over  $R$  by the (possibly finite, possibly uncountable) list  $X_1, X_2, \dots$ , and  $G$  is freely generated by the (possibly finite, possibly uncountable) list  $Y_1, Y_2, \dots$ .

Let  $S := \text{Sym}(F) = R[X_1, X_2, \dots]$ ,  $T := \text{Sym}(G) = R[Y_1, Y_2, \dots]$ , and  $U := S \otimes_R T = R[X_1, X_2, \dots, Y_1, Y_2, \dots]$ .

We note that  $U$  has two important gradings as an  $R$ -algebra: First, it has an  $(\mathbb{N} \times \mathbb{N})$ -grading (a *bigrading*). In this grading, we assign to each  $r \in R$  degree  $(0,0)$ , we let each  $X_i$  have degree  $(1,0)$ , and we let each  $Y_j$  have degree  $(0,1)$ . Secondly, it has an  $\mathbb{N}$ -grading (a *standard* grading) by giving each monomial in the  $X$ s and  $Y$ s its total degree. In particular, this means that if a monomial  $m$  has  $(\mathbb{N} \times \mathbb{N})$ -degree  $(i, j)$ , then the standard (total) degree of  $m$  is  $i + j$ . For example, the bidegree of  $X_1 X_3^2 Y_4^7$  is  $(3, 7)$ , but its total degree is 10.

First take any  $z \in L_F^-$ . Then there is an integer  $n$ , and homogeneous polynomials  $f_1, \dots, f_n \in S$  such that  $\deg f_j = j$  and an equation  $z^n + f_1 z^{n-1} + \dots + f_n = 0$  in  $S$ . But this equation holds in  $U$  as well, and each  $f_j$  has bidegree  $(j, 0)$ , and thus has total degree  $j$ . So the given

equation shows that  $(z, 0) \in (L \oplus 0)_{F \oplus G}^- \subseteq (L \oplus M)_{F \oplus G}^-$ . Similarly for any  $w \in M_G^-$ ,  $(0, w) \in (0 \oplus M)_{F \oplus G}^- \subseteq (L \oplus M)_{F \oplus G}^-$ . So  $L_F^- \oplus M_G^- \subseteq (L \oplus M)_{F \oplus G}^-$ .

Conversely, let  $(z, w) \in (L \oplus M)_{F \oplus G}^-$ . Then in  $U$ ,  $z$  has bidegree  $(1, 0)$ ,  $w$  has bidegree  $(0, 1)$  and there is a homogeneous equation of integrality for  $z+w$  over the  $R$ -subalgebra  $V$  of  $U$  generated by  $L \oplus M$ . That is, we have an equation

$$(1) \quad (z+w)^n + f_1(z+w)^{n-1} + \cdots + f_{n-1}(z+w) + f_n = 0,$$

where each  $f_k \in V$  is homogeneous and has *total* degree  $k$ . So we get a decomposition

$$f_k = \sum_{i=0}^k f_{i, k-i},$$

such that each  $f_{i,j}$  has bidegree  $(i, j)$ . By the bigrading of both  $V$  and  $U$ , the graded part of the left-hand side of Equation 1 of each bidegree  $(k, l)$  ( $k+l=n$ ) is also 0, and each  $f_{i,j} \in V$ . In particular, expanding everything out, the part of Equation 1 of bidegree  $(n, 0)$  yields the following:

$$z^n + f_{1,0}z^{n-1} + f_{2,0}z^{n-2} + \cdots + f_{n-1,0}z + f_{n,0} = 0.$$

The latest displayed equation holds in  $S$  itself, and is an equation of integrality of  $z$  over the subalgebra generated by  $L$ , which shows that  $z \in L_F^-$ . Proceeding similarly with the  $(0, n)$ -bigraded piece of Equation 1, we see that  $w \in M_G^-$ . Hence  $(z, w) \in L_F^- \oplus M_G^-$ .

For the final statement, we need only observe that  $G = G_G^-$ .  $\square$

**Proposition 2.3.** *Let  $L \subseteq M$  be a submodule inclusion. Then  $L_M^-$  is independent of choice of the free module surjection onto  $M$ .*

*Proof.* Let  $F \xrightarrow{\pi} M$  and  $G \xrightarrow{\sigma} M$  be surjections from free modules. Let  $H := F \oplus G$ . Then we construct the following commutative diagram:

$$\begin{array}{ccc} & H & \\ p \swarrow & & \searrow q \\ F & & G \\ \pi \searrow & & \swarrow \sigma \\ & M & \end{array}$$

where  $p$  and  $q$  are the projection maps. Then  $(\pi \circ p)((\pi \circ p)^{-1}(L)_H^-) = (\pi \circ p)((\pi^{-1}(L) \oplus G)_H^-) = (\pi \circ p)(\pi^{-1}(L)_F^- \oplus G) = \pi(\pi^{-1}(L)_F^-)$ , by Proposition 2.2. But the same trick works on the other side of the diagram. So  $\pi(\pi^{-1}(L)_F^-) = (\pi \circ p)((\pi \circ p)^{-1}(L)_H^-) = (\sigma \circ q)((\sigma \circ q)^{-1}(L)_H^-) = \sigma(\sigma^{-1}(L)_G^-)$ , as required.  $\square$

Note that if  $M$  is free, then  $L_M^- = L_M^-$ . In particular, for any ideal  $I$ ,  $I_R^- = \bar{I}$ .

The following lemma allows us frequently to “reduce” to the finitely generated case.

**Lemma 2.4.** *Let  $\Lambda$  be any index set. Suppose  $L \subseteq M$  are modules, and we have submodules  $\{L_\lambda\}_{\lambda \in \Lambda}$ ,  $\{M_\lambda\}_{\lambda \in \Lambda}$  of  $M$  such that  $M = \sum_{\lambda \in \Lambda} M_\lambda$ ,  $L = \sum_{\lambda \in \Lambda} L_\lambda$ , and  $L_\lambda \subseteq L \cap M_\lambda$  for each  $\lambda \in \Lambda$ . Then*

$$L_M^- = \bigcup_{\text{finite subsets } \sigma \subseteq \Lambda} \left( \sum_{\lambda \in \sigma} L_\lambda \right)_{\sum_{\lambda \in \sigma} M_\lambda}^-$$

*Proof.* First, by definition of liftable integral closure, we may immediately quotient out by  $L$  and assume that every  $L_\lambda = 0$ .

Next, for each  $\lambda \in \Lambda$ , let  $\pi_\lambda : F_\lambda \twoheadrightarrow M_\lambda$  be a surjection from a free  $R$ -module. For each  $\lambda$ , let  $X_\lambda$  be a set of free module generators of  $F_\lambda$  over  $R$ , defined in such a way that all the sets  $\{X_\lambda \mid \lambda \in \Lambda\}$  are disjoint from one another. Then with  $F := \bigoplus_{\lambda \in \Lambda} F_\lambda$ , we have a short exact sequence

$$0 \rightarrow U \rightarrow F \xrightarrow{\pi} M \rightarrow 0,$$

where  $\pi$  is induced from the maps  $\pi_\lambda$ .

Now let  $z \in 0_M^-$ . Let  $y \in F$  such that  $\pi(y) = z$ . Then in  $S := \text{Sym}(F)$ , there is an equation of integrality of the form

$$(2) \quad y^n + a_1 y^{n-1} + \cdots + a_{n-1} y + a_n = 0,$$

where each  $a_j \in (US)^j$ . That is each  $a_j$  is an  $R$ -linear combination of  $j$ -fold formal products of elements of  $U$ , where *each* such element is an element of a direct sum of only finitely many of the  $F_\lambda$ s. For the same reason,  $y$  itself can be expressed in terms of only finitely many of the  $F_\lambda$ s. Putting this together, it follows that there is some finite list  $\lambda_1, \dots, \lambda_t$  of elements of  $\Lambda$  such that Equation 2 is an equation of integrality in  $F' := \bigoplus_{i=1}^t F_{\lambda_i}$  of  $y$  over the submodule  $U' := U \cap F'$ .

Combine this with the fact that  $\sum_{i=1}^t M_{\lambda_i} \cong F'/U'$  canonically, and the statement follows.  $\square$

At this point, we possess the tools to prove the Valuative Criterion in general:

*Proof of Proposition 1.4.* Throughout, we may assume that  $M$  is a free  $R$ -module (albeit for different reasons from when we are dealing with liftable integral closure).

For the first inclusion, take any  $x \in L_M^{-\text{EHU}} = L_M^-$ . Then by Lemma 2.4, there is some finitely generated free submodule  $G$  of  $M$  such that, letting  $K := L \cap G$ , we have  $x \in K_G^-$ . Then by the finitely generated case of the Valuative Criterion,  $x \in K_G^{-\text{Rs}} \subseteq L_M^{-\text{Rs}}$ .

For the opposite inclusion, we first prove the result when  $R$  is a domain. Let  $Q$  be its field of fractions. Let  $T$  be the subring of  $\text{Sym}_R(M)$  generated by  $L$ . Let  $W$  be a valuation domain between  $T$  and  $\text{Frac}(T)$ . Then  $V := W \cap Q$  is a  $Q$ -valuation domain containing  $R$ . So for any  $x$  satisfying the Valuative Criterion, we have  $x \in LV$ , the image of the map  $L \otimes_R V \rightarrow F \otimes_R V$ . But  $LV \subseteq W$ , so  $x \in W$ . Since  $W$  was arbitrary, it follows that  $x$  is integral over  $T$ , and hence  $x \in L_M^{-\text{EHU}}$ .

Finally we show the general case, where  $R$  is not necessarily a domain but  $x$  satisfies the Valuative Criterion in  $M$  over  $L$ . Then for each  $\mathfrak{p} \in \text{Min}(R)$ ,  $\bar{x} \in M/\mathfrak{p}M$  satisfies the valuative criterion over  $(L + \mathfrak{p}M)/\mathfrak{p}M$  as  $(R/\mathfrak{p})$ -modules. But  $R/\mathfrak{p}$  is a domain, so by the previous paragraph,  $x \in (L + \mathfrak{p}M)_{M/\mathfrak{p}M}^-$  as  $(R/\mathfrak{p})$ -modules. Now fix a free basis of  $M$  over  $R$ , and let  $\{M_\lambda\}$  be the set of all free submodules of  $M$  generated by finite subsets of that basis. Then by Lemma 2.4 (and since there are only finitely many minimal primes of  $R$ ), there is some  $M' := M_\lambda$  such that if we let  $L' := L \cap M'$ , we have  $x \in M'$  and for each  $\mathfrak{p} \in \text{Min}(R)$ ,  $x \in (L' + \mathfrak{p}M')_{M'/\mathfrak{p}M'}^-$  as  $(R/\mathfrak{p})$ -modules. But then by the usual relationship between integral closure in finitely generated free modules and minimal primes, it follows that  $x \in (L')_{M'}^- \subseteq L_M^- = L_M^{-\text{EHU}}$ , as was to be shown.  $\square$

We next collect several important properties of liftable integral closure:

**Proposition 2.5.** *Let  $R$  be Noetherian and let  $L \subseteq M$  be  $R$ -modules.*

- (1) (Idempotence)  $L_M^- = (L_M^-)^-$ .
- (2) (Functoriality) *If  $h : M \rightarrow N$  is a homomorphism of  $R$ -modules, then  $h(L_M^-) \subseteq h(L)_N^-$ .*
- (3) (Direct sums) *If  $N \subseteq T$  is another inclusion of  $R$ -modules, then*

$$(L \oplus N)_{M \oplus T}^- = L_M^- \oplus N_T^-$$

- (4) (Submodules) *If  $K \subseteq L$  is a submodule, then  $K_L^- \subseteq K_M^- \subseteq L_M^-$ .*
- (5) (Comparison to tight closure) *Suppose that  $M$  is finitely generated and either (a)  $R$  has characteristic  $p$ , or (b)  $R$  is excellent of equal characteristic 0. Then  $L_M^- \subseteq L_M^*$ .*
- (6) (Comparison to Rees integral closure) *We have  $L_M^- \subseteq L_M^{-\text{Rs}}$ .*
- (7) (Colons) *If  $L = L_M^-$ ,  $J$  is an ideal, and  $U \subseteq M$  another submodule, then  $(L :_M J)_M^- = L :_M J$  and  $\overline{(L :_R U)} = L :_R U$ .*

- (8) (Persistence) *Let  $R \rightarrow S$  be a homomorphism of Noetherian rings. Then*

$$\text{im}(L_M^- \otimes_R S \rightarrow M \otimes_R S) \subseteq (\text{im}(L \otimes_R S \rightarrow M \otimes_R S))_{M \otimes_R S}^-$$

where the closure on the right-hand side is taken as  $S$ -submodules.

- (9) (Normal base change) *If in the situation of Property (8), the ring homomorphism is normal (i.e. flat with geometrically normal fibers), then the displayed  $S$ -module containment is an equality.*
- (10) (Semiprime property) *For any ideal  $J$ ,  $(JL)_M^- = (\bar{J}L_M^-)_M^-$ .*
- (11) (Nakayama property) *If  $(R, \mathfrak{m})$  is local and Noetherian, and  $M/L$  is finitely generated, then  $L_M^- \subseteq L + \mathfrak{m}M$ . Thus,  $L_M^- = M$  if and only if  $L = M$ .*

*Proof.* (1): We may assume that  $M$  is free. Then by looking at the symmetric algebra of the free module  $M$ , the property follows from the standard fact that whenever  $A \subseteq B$  is an extension of commutative rings, the integral closure of  $A$  in  $B$  is integrally closed in  $B$ .

(2): First, we may assume  $L = 0$ , so that we want to show that  $h(0_M^-) \subseteq 0_N^-$ .

Next, lift to a map  $\tilde{h} : F \rightarrow G$  of free modules, so that we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & F & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & \downarrow h_0 & & \downarrow \tilde{h} & & \downarrow h & & \\ 0 & \longrightarrow & V & \longrightarrow & G & \xrightarrow{p} & N & \longrightarrow & 0 \end{array}$$

Then  $\tilde{h}$  lifts to a map  $H : \text{Sym}(F) \rightarrow \text{Sym}(G)$  of symmetric algebras. Let  $x \in 0_M^-$ . Then there is some  $y \in F$  with  $\pi(y) = x$  such that  $y \in U_F^-$ , which means that  $y$  is in the integral closure of the subalgebra  $S_F(U)$  generated by  $U$  in  $\text{Sym}(F)$ . By persistence of integral closure of subalgebras along ring maps, it follows that  $\tilde{h}(y) = H(y)$  is integral over the subalgebra  $S_G(V)$  generated by  $V$  in  $\text{Sym}(G)$ , which means that  $\tilde{h}(y) \in V_G^-$ , whence  $h(x) = h(\pi(y)) = p(\tilde{h}(y)) \in 0_N^-$ , as required.

(3): This follows from Property (2), applied to the canonical projection and inclusion maps of the direct sum.

(4): The first inclusion follows from (2) applied to the inclusion map  $L \hookrightarrow M$ . The second inclusion is clear since we may assume  $M$  is a free module.

(5): First, since both tight closure and liftable integral closure are independent of surjection from free modules, we may assume  $M$  is free.



Recall that for a regular Noetherian local ring  $S$ , any submodule of any  $S$ -module is tightly closed.

Now let  $x \in L_M^*$  and let  $R \rightarrow V$  be a map to a Noetherian valuation ring with kernel a minimal prime of  $R$ . Then  $x' \in LV$  (since  $V$  is regular), so  $x \in L_M^{-\text{Rs}} = L_M^-$ .

(6): Let  $\pi : F \twoheadrightarrow M$  be a surjection from a free module, let  $K := \pi^{-1}(L)$ , and let  $x \in L_M^-$ . Let  $y \in F$  with  $\pi(y) = x$ . Going mod a minimal prime, we may assume  $R$  is a domain, with quotient field  $Q$ . Let  $V$  be a valuation ring between  $R$  and  $Q$ . Then by the Valuative Criterion,  $y' \in KV$ . But then  $x' = \pi(y)' = \pi'(y') \in LV$ , so that  $x \in L_M^{-\text{Rs}}$ .

(7): Let  $\pi : F \twoheadrightarrow M$  be a surjection from a free module. Let  $K := \pi^{-1}(L)$  and  $V := \pi^{-1}(U)$ . It follows easily that  $K :_F J = \pi^{-1}(L :_M J)$  and  $L :_R U = K :_R V$ , so it suffices to prove the analogous formulas for  $K$ ,  $V$ , and  $F$ .

First let  $y \in (K :_F J)_F^-$ . Then in  $S := \text{Sym}(F)$ , we have an equation of the form:

$$y^n + \alpha_1 y^{n-1} + \cdots + \alpha_n = 0,$$

where each  $\alpha_i \in ((K :_F J)S)^i$ . Take any  $j \in J$ . Then  $j^i \alpha_i \in (KS)^i$ , so multiplying the displayed equation by  $j^n$ , we get

$$(jy)^n + (j\alpha_1)(jy)^{n-1} + \cdots + (j^{n-1}\alpha_{n-1})(jy) + j^n \alpha_n = 0.$$

This shows that  $jy \in K_F^- = K$ , as required.

Similarly, if  $r \in (K :_R V)$ , then we have the following equation in  $R$ :

$$r^n + c_1 r^{n-1} + \cdots + c_n = 0,$$

where each  $c_j \in (K :_R V)^j$ . Now let  $v \in V$ , and in  $S = \text{Sym}(F)$  it follows that  $c_j v^j \in (KS)^j$ . So multiplying the displayed equation by  $v^n$  in  $S$ , we get:

$$(rv)^n + (c_1 v)(rv)^{n-1} + \cdots + c_n v^n = 0,$$

whence  $rv \in K_F^- = K$ . Since  $v \in V$  was arbitrary, it follows that  $r \in K :_R V$ , as required.

(8): We may assume immediately that  $M$  is free. Let  $\{M_\lambda\}$  be the set of finitely generated free  $R$ -submodules of  $M$ , and let  $L_\lambda := L \cap M_\lambda$  for each  $\lambda$ . Note that  $\{M_\lambda\}$  form a direct limit system under inclusion, whose direct limit is  $M$ , and similarly for the  $\{L_\lambda\}$ . Accordingly, let  $z \in \text{im}(L_M^- \otimes_R S \rightarrow M \otimes_R S)$ . Then there is some  $\lambda$  such that  $z \in \text{im}((L_\lambda)_{M_\lambda}^- \otimes_R S \rightarrow M \otimes_R S)$ . Hence,

$$z = \sum_{i=1}^n z_i \otimes s_i,$$

where  $z_i \in (L_\lambda)_{M_\lambda}^-$  and  $s_i \in S$  for  $1 \leq i \leq n$ . But since Property (8) holds for submodules of finitely generated free modules (which follows,

in turn, from the persistence of integral closure of subrings), we have that

$$z_i \otimes s_i \in \text{im}[(\text{im } L_\lambda \otimes S \rightarrow M_\lambda \otimes S)_{M_\lambda \otimes S}^- \rightarrow M \otimes S].$$

But then by Properties (2) and (4), it follows that

$$z_i \otimes s_i \in (\text{im}(L \otimes_R S \rightarrow M \otimes_R S))_{M \otimes_R S}^-,$$

as was to be shown.

(9): As usual, we may assume  $M$  is free. We need only show that  $(L \otimes_R S)_{M \otimes_R S}^- \subseteq L_M^- \otimes_R S$ . So let  $z \in (L \otimes_R S)_{M \otimes_R S}^-$ . Let  $\{M_\lambda\}$  and  $\{L_\lambda\}$  be as above. By flatness, it follows that the systems  $\{L_\lambda \otimes_R S\}$  and  $\{M_\lambda \otimes_R S\}$ , of  $S$ -submodules of  $L \otimes S$  and  $M \otimes S$  respectively, are also injective direct limit systems, with unions equal to  $L \otimes S$  and  $M \otimes S$  respectively. Accordingly, by Lemma 2.4, there is some  $\lambda$  such that  $z \in (L_\lambda \otimes_R S)_{M_\lambda \otimes_R S}^-$ . But since Property (9) holds for submodules of finitely generated free modules (as the argument in the proof of [HS06, Corollary 19.5.2] works just as well in this case), it follows that  $z \in (L_\lambda)_{M_\lambda}^- \otimes_R S$ . By Property (4) on the  $R$ -module inclusions involved,  $(L_\lambda)_{M_\lambda}^- \subseteq L_M^-$ , so that by flatness,

$$z \in (L_\lambda)_{M_\lambda}^- \otimes_R S \subseteq L_M^- \otimes_R S,$$

as was to be shown.

(10): (sketch) As usual we may assume  $M$  is free. Use Lemma 2.4 to reduce to the case where  $M$  is finitely generated. Then the statement is clear via use of the Valutive Criterion.

(11): Suppose  $L \neq M$ . Let  $G \xrightarrow{\psi} F \rightarrow M/L \rightarrow 0$  be a minimal free presentation of  $M/L$ , and let  $U := \text{im } \psi$ . By minimality,  $U \subseteq \mathfrak{m}F$ . But  $\mathfrak{m}F$  is integrally closed in  $F$ , as is clear by tensoring everything with  $R/\mathfrak{m}$  and noting that subspaces of vector spaces are always integrally closed. Hence  $U_F^- \subseteq (\mathfrak{m}F)_F^- = \mathfrak{m}F$ , so that

$$\frac{L_M^-}{L} = 0_{M/L}^- = \pi(U_F^-) \subseteq \mathfrak{m}\pi(F) = \mathfrak{m}(M/L) = \frac{L + \mathfrak{m}M}{L}.$$

whence  $L_M^- \subseteq L + \mathfrak{m}M$ . The last statement follows from the Nakayama lemma.  $\square$

**Remark.** Property (11) of the above Proposition shows that for finitely generated modules, it makes no sense to say that a module is ‘liftably integral’ over a submodule (though it does for infinite modules; see Proposition 5.1 below). This shows that liftably integral closure is quite a distinct notion from previous notions of integral closure of submodules. For example, let  $R$  be any Noetherian local domain that contains an ideal  $J$  that is not integrally closed. Let  $I = \bar{J}$  be its integral closure. Then  $J_I^{-\text{Rs}} = J_I^{-\text{EHU}} = I$ , but  $J_I^- \neq I$ .

Finally for this section, here is an equivalent way to define liftable integral closure:

**Lemma 2.6.** *Let  $R$  be a Noetherian ring, and let  $L \subseteq M$  be (resp. finite)  $R$ -modules.*

*Consider all surjections  $\pi : Z \twoheadrightarrow M$  from  $R$ -modules (resp. from finite  $R$ -modules)  $Z$ , and introduce the notation  $L^\pi := \pi^{-1}(L)$  and  $M^\pi := Z$ . Then*

$$L_M^- = \bigcap_{\pi} \pi((L^\pi)_{M^\pi}^{-\text{EHU}}) = \bigcap_{\pi} \pi((L^\pi)_{M^\pi}^{-\text{Rs}}).$$

*Proof.* Since for any  $R$ -modules  $A \subseteq B$  and any  $R$ -module map  $g : B \rightarrow C$ , we have  $g(A_B^{-\text{Rs}}) \subseteq g(A)_C^{-\text{Rs}}$  and  $g(A_B^{-\text{EHU}}) \subseteq g(A)_C^{-\text{EHU}}$ , and since every (finite)  $R$ -module is the surjective image of a (finite) free  $R$ -module, we need only consider surjections from (finite) free modules. And then the result follows by definition of liftable integral closure and Proposition 2.3.  $\square$

That is, an element  $z \in M$  is *liftably integral* over  $L$  if for all ‘liftings’  $\pi$  of  $M$ , a preimage of  $z$  is in the (Rees or EHU) integral closure of  $L^\pi$  in  $M^\pi$ , and the *liftable integral closure* of  $L$  in  $M$  is the set of all such elements.

### 3. ‘TEST IDEALS’ FOR LIFTABLE INTEGRAL CLOSURE

A main point of this work is to determine what kills liftable integral closures of zero in some ‘universal’ way, inspired by the tight closure notion of “test elements”.

**Definition/Proposition 3.1.** *Let  $(R, \mathfrak{m}, k, E)$  be a local Noetherian ring. Then*

(1)

$$\begin{aligned} \tau_{\mathcal{M}} &:= \bigcap_{M \text{ artinian}} \text{ann } 0_M^- = \bigcap_{M \text{ fin. length}} \text{ann } 0_M^- \\ &= \bigcap_{M \text{ fin. gen.}} \text{ann } 0_M^- = \text{ann } 0_E^-, \end{aligned}$$

*where the intersections in question are taken over all  $R$ -modules of the specified types.*

(2)

$$\tau_{\mathcal{I}} := \bigcap_{\text{ideals } I} (I : \bar{I}) = \bigcap_{\mathfrak{m}\text{-primary ideals } I} (I : \bar{I}).$$

(3)  $\tau_{\mathcal{I}} \subseteq \bigcap_{\mathfrak{m}\text{-primary param. ideals } J} (J : \bar{J})$ , with equality if  $k$  is infinite.

- (4)  $\tau_{\mathcal{M}} \subseteq \tau_{\mathcal{I}}$  with equality if  $\dim R = 0$  or  $R$  is approximately Gorenstein.

Whenever  $\tau_{\mathcal{M}} = \tau_{\mathcal{I}}$ , we call it the integral test ideal of  $R$ .

Recall [Hoc77] that a local ring  $(R, \mathfrak{m}, k, E)$  is *approximately Gorenstein* if there exists a sequence  $\{I_t\}_{t \in \mathbb{N}}$  of irreducible  $\mathfrak{m}$ -primary ideals such that for all  $N \in \mathbb{N}$ , there exists  $t \in \mathbb{N}$  such that  $I_t \subseteq \mathfrak{m}^N$ . (In other words, in the  $\mathfrak{m}$ -adic topology, the diameter of the sequence approaches 0.). Moreover, we have

**Theorem 3.2.** [Hoc77] *Let  $(R, \mathfrak{m}, k, E)$  be a local Noetherian ring.  $R$  is approximately Gorenstein if and only if  $\hat{R}$  is. If  $R$  is approximately Gorenstein, then there is a sequence  $\{I_t\}_{t \in \mathbb{N}}$  of irreducible  $\mathfrak{m}$ -primary ideals and a sequence  $i_t : R/I_t \hookrightarrow R/I_{t+1}$  of injective  $R$ -linear maps such that  $E \cong \varinjlim (R/I_t)$  under the action of these maps.*

Suppose that one of the following conditions holds:

- (1)  $\hat{R}$  is reduced.
- (2)  $R$  is excellent and reduced.
- (3)  $\text{depth } R \geq 2$ .
- (4)  $R$  is a normal domain.

Then  $R$  is approximately Gorenstein.

Thus, for rings of dimension  $\geq 2$ , the condition is quite general.

*Proof of Proposition 3.1.* We will start with the  $\tau_{\mathcal{M}}$  equalities (that is, part (1)). For the purposes of the proof, let  $\tau_1 := \bigcap_{M \text{ artinian}} \text{ann } 0_M^-$ ,

$$\tau_2 := \bigcap_{M \text{ fin. length}} \text{ann } 0_M^-, \quad \tau_3 := \bigcap_{M \text{ fin. gen.}} \text{ann } 0_M^-, \quad \text{and} \quad \tau_4 := \text{ann } 0_E^-.$$

$\tau_1 \subseteq \tau_4$ : since  $E$  is artinian.

$\tau_4 \subseteq \tau_2$ : Let  $M$  be a finite length  $R$ -module, and  $G := E_R(M)$ , the injective hull of  $M$ . Then  $G$  is isomorphic to a direct sum of finitely many copies of  $E$ , say  $G := E^n$ . Then  $0_M^- \subseteq 0_G^- \cong (0_E^-)^{\oplus n}$  (by Proposition 2.5 (3)), and hence any element of  $R$  that annihilates  $0_E^-$  must annihilate  $0_M^-$  as well.

$\tau_2 \subseteq \tau_1$ : Let  $N$  be an artinian  $R$ -module and  $z \in 0_N^-$ . As  $N$  is the directed union of its finite length submodules, it follows from Lemma 2.4 that  $z \in 0_M^-$  for some finite length submodule  $M \subseteq N$ . Since any  $c \in \tau_2$  annihilates  $0_M^-$ , we have  $cz = 0$ .

At this point we have shown that  $\tau_1 = \tau_2 = \tau_4$ . To complete the proof of the equalities for  $\tau_{\mathcal{M}}$ , we need only show that  $\tau_2 = \tau_3$ .

$\tau_3 \subseteq \tau_2$ : since every finite length module is finitely generated.

$\tau_2 \subseteq \tau_3$ : Let  $N$  be a finitely generated  $R$ -module and let  $c \in \tau_2$ . Let  $z \in 0_N^-$ . Since  $\lrcorner$  respects inclusions (by Proposition 2.5 (4)), we have

$z \in (\mathfrak{m}^t N)_N^-$  for all integers  $t \geq 1$ . Let  $N_t := N/\mathfrak{m}^t N$ . Since  $N_t$  has finite length, it follows that  $c$  annihilates  $0_{N_t}^-$ , so that  $c \cdot \bar{z} = \bar{0}$  in  $N_t$ . In other words,  $cz \in \mathfrak{m}^t N$  for all  $t$ , *i.e.*  $cz \in \bigcap_{t \geq 1} \mathfrak{m}^t N = 0$  by the Krull intersection theorem.

We proceed with the assertions about  $\tau_{\mathcal{I}}$  (that is, parts (2) and (3)): Let  $\tau_5 := \bigcap_{\text{ideals } I} (I : \bar{I})$ ,  $\tau_6 := \bigcap_{\text{m-primary ideals } I} (I : \bar{I})$ , and  $\tau_7 := \bigcap_{\text{m-prim. param. ideals } J} (J : \bar{J})$ . The proof that  $\tau_5 = \tau_6$  is essentially identical to the proof that  $\tau_2 = \tau_3$ .  $\tau_6 \subseteq \tau_7$  for obvious reasons, so it remains only to see that  $\tau_7 \subseteq \tau_6$  when  $k$  is infinite. But it is well known [NR54] that in this case, every  $\mathfrak{m}$ -primary ideal  $I$  has a minimal reduction  $J$  which is generated by a system of parameters. So for any such  $I$ , any  $z \in \bar{I}$ , and any  $c \in \tau_7$ , since  $\bar{I} = \bar{J}$ , we have  $cz \in J \subseteq I$ , so  $c \in (I : \bar{I})$ .

As for (4): First, it is clear that  $\tau_{\mathcal{M}} \subseteq \tau_{\mathcal{I}}$  in all cases, since for any ideal  $I$  (with closure being taken as  $R$ -modules),  $0_{R/I}^- = \bar{I}/I$ , so that  $\text{ann } 0_{R/I}^- = \text{ann}(\bar{I}/I) = (I :_R \bar{I})$ . For the other direction, if  $R$  is approximately Gorenstein, then  $E$  is a directed union of finite length *cyclic*  $R$ -modules, so Lemma 2.4 shows that  $\tau_{\mathcal{I}} \subseteq \tau_{\mathcal{M}}$  in this case. Finally, the case  $\dim R = 0$  is given in Proposition 4.2 below.  $\square$

It is also interesting to consider top local cohomology modules. As one may expect, the results are at their cleanest when the ring is Cohen-Macaulay:

**Proposition 3.3.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring. Let  $d := \dim R$ . Then*

$$\tau_{\mathcal{I}} \subseteq \text{ann } 0_{H_{\mathfrak{m}}^d(R)}^-,$$

*and equality holds whenever  $R$  is Cohen-Macaulay with infinite residue field.*

*Proof.* As is well-known, whenever  $x_1, \dots, x_d$  is a system of parameters, we can express the module  $H_{\mathfrak{m}}^d(R)$  as a direct limit, namely

$$H_{\mathfrak{m}}^d(R) = \lim_{\rightarrow} \frac{R}{(x_1^t, \dots, x_d^t)}$$

where the maps are given by multiplication by the element  $x = \prod_{i=1}^d x_i$ . To be explicit, let  $j_t : R/(x_1^t, \dots, x_d^t) \rightarrow H_{\mathfrak{m}}^d(R)$  and  $\alpha_{t,n} : R/(x_1^t, \dots, x_d^t) \rightarrow R/(x_1^{t+n}, \dots, x_d^{t+n})$  be the corresponding maps, and let  $\pi_t : R \rightarrow R/(x_1^t, \dots, x_d^t)$  be the natural projection map.

To prove the first inclusion, let  $a \in \tau_{\mathcal{I}}$  and  $0 \neq u \in 0_{H_{\mathfrak{m}}^d(R)}^-$ . For each  $t$ , let  $\kappa_t := \ker(j_t \circ \pi_t)$ . By Lemma 2.4, there is some  $t$  such that  $u \in 0_{j_t(R/(x_1^t, \dots, x_d^t))}^-$ . That is, there is some  $y \in R$  such that  $y +$

$(x_1^t, \dots, x_d^t)$  represents  $u$  in the direct limit system and  $y \in \overline{\kappa_t}$ . But for some  $n$ , we have  $x^n \kappa_t \subseteq (x_1^{n+t}, \dots, x_d^{n+t})$ , so that  $x^n y \in \overline{(x_1^{n+t}, \dots, x_d^{n+t})}$  by functoriality of integral closure of ideals. Since  $a$  annihilates the integral closure of ideals, this implies that  $ax^n y \in (x_1^{n+t}, \dots, x_d^{n+t})$ . But  $x^n y + (x_1^{n+t}, \dots, x_d^{n+t})$  represents  $u$  in the direct limit system, so it follows that  $au = 0$ . Thus,  $a \in \text{ann } 0_{H_{\mathfrak{m}}^d(R)}^-$ .

As for the reverse inclusion (assuming  $R$  is Cohen-Macaulay with infinite residue field), choose  $a \in \text{ann } 0_{H_{\mathfrak{m}}^d(R)}^-$  and let  $J$  be an  $\mathfrak{m}$ -primary parameter ideal. Since  $R$  is Cohen-Macaulay, we have an inclusion  $R/J \hookrightarrow H_{\mathfrak{m}}^d(R)$ , which in turn induces an inclusion  $\bar{J}/J \hookrightarrow 0_{H_{\mathfrak{m}}^d(R)}^-$ , since as  $R$ -modules we have  $0_{R/J}^- = \bar{J}/J$ . But then since  $a$  annihilates  $0_{H_{\mathfrak{m}}^d(R)}^-$ , it annihilates the submodule  $\bar{J}/J$ , so that  $a \in (J :_R \bar{J})$ . Since this holds for all  $\mathfrak{m}$ -primary parameter ideals  $J$ , the result follows from Proposition 3.1.  $\square$

#### 4. CHARACTERIZATIONS OF $\tau_{\mathcal{I}}$ AND $\tau_{\mathcal{M}}$

Next, we characterize  $\tau_{\mathcal{I}}$  (and sometimes  $\tau_{\mathcal{M}}$ ) under certain conditions on the local ring  $R$ .

**Dimension zero.** We begin with the following observation.

**Proposition 4.1.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring. The following are equivalent:*

- (1)  $\dim R = 0$ .
- (2) For every finitely generated  $R$ -module  $M$ ,  $0_M^- = \mathfrak{m}M$ .
- (3) For every  $R$ -module inclusion  $L \subseteq M$  such that  $M/L$  is finitely generated,  $L_M^- = L + \mathfrak{m}M$ .

*Proof.* It is clear that (2) and (3) are equivalent.

To see that (2)  $\implies$  (1), suppose  $\dim R \geq 1$ . Then letting  $M = R$ , we see that

$$0_M^- = 0_R^- = \sqrt{0} \neq \mathfrak{m} = \mathfrak{m}M.$$

To see that (1)  $\implies$  (2), let  $\dim R = 0$  and let  $M$  be a finite  $R$ -module. Let  $\pi : F \twoheadrightarrow M$  be a surjection from a finite free module. Say  $F = \bigoplus_{i=1}^t R\mathbf{e}_i$ , where the  $\mathbf{e}_i$  are free module generators. Then

$$\mathfrak{m}F = \bigoplus_{i=1}^t \mathfrak{m}\mathbf{e}_i = \bigoplus_{i=1}^t (0_R^-)\mathbf{e}_i \subseteq 0_F^- \subseteq \mathfrak{m}F,$$

so that all inequalities become equalities and  $\mathfrak{m}F = 0_F^-$ . Then by Proposition 2.5 (11),

$$\mathfrak{m}M = \pi(\mathfrak{m}F) = \pi(0_F^-) \subseteq 0_M^- \subseteq \mathfrak{m}M$$

which finishes the proof.  $\square$

**Proposition 4.2.** *Let  $(R, \mathfrak{m}, k, E)$  be a Noetherian local ring of dimension 0. Then  $\tau_{\mathcal{I}} = \tau_{\mathcal{M}} = \text{Soc } R$ .*

*Proof.* We have  $E = \omega_R$ , the *canonical module*, which is finitely generated and faithful. Thus, by Proposition 4.1,  $\tau_{\mathcal{M}} = \text{ann } 0_E^- = \text{ann } (\mathfrak{m}E)$ . But since  $R$  is complete,  $\text{ann } (\mathfrak{m}E) = \text{ann } \mathfrak{m} = \text{Soc } R$ , so  $\tau_{\mathcal{M}} = \text{Soc } R$ . On the other hand, for any proper ideal  $I$ ,  $(I : \bar{I}) = (I : \mathfrak{m}) \supseteq (0 : \mathfrak{m}) = \text{Soc } R$ , and  $(0 : \bar{0}) = \text{ann } \mathfrak{m} = \text{Soc } R$ , so  $\tau_{\mathcal{I}} = \text{Soc } R$ .  $\square$

**Dimension one.** To deal the dimension 1 case, we define the *conductor*  $\mathfrak{f}_R$  of a Noetherian ring  $R$  to be the ideal  $\mathfrak{f}_R = (R :_R \bar{R})$ , where  $\bar{R}$  is the integral closure of  $R$  in its total ring of fractions. This agrees with the usual definition (e.g. in [HS06, Chapter 12]) when  $R$  is reduced.

Next, we note the following:

**Proposition 4.3.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring. Then<sup>2</sup>  $\text{grade } \mathfrak{f}_R \geq 1$  if and only if  $\bar{R}$  is finitely generated as an  $R$ -module.*

*Proof.* Let  $Q$  be the total quotient ring of  $R$ .

First suppose  $\bar{R}$  is finitely generated as an  $R$ -module. Say  $\bar{R} = \sum_{i=1}^t R z_i$ , with each  $z_i \in Q$ . We have  $z_i = a_i/x_i$ , where  $a_i \in R$  and  $x_i$  is a non-zerodivisor of  $R$ . Let  $x := \prod_i x_i$ . Then  $x$  is a non-zerodivisor of  $R$ , and  $x z_i \in R$  for each  $i$ , whence  $x \bar{R} \subseteq R$ , so that  $x \in \mathfrak{f}_R$ , giving that ideal positive grade.

Conversely, suppose  $\text{grade } \mathfrak{f}_R \geq 1$ . Then  $\mathfrak{f}_R$  contains a non-zerodivisor, say  $x$ . Then  $x \bar{R}$  is an  $R$ -submodule of  $R$ , hence an ideal, hence finitely generated. Say  $x \bar{R} = (a_1, \dots, a_t)R$ . Then

$$\bar{R} = \sum_{i=1}^t R \cdot \frac{a_i}{x}$$

is finitely generated as an  $R$ -module.  $\square$

**Theorem 4.4.** *Let  $(R, \mathfrak{m}, k, E)$  be a Noetherian local ring of dimension one, with infinite residue field. Then  $R$  is Cohen-Macaulay if and only if  $\tau_{\mathcal{I}} = \mathfrak{f}_R$ ; otherwise  $\mathfrak{f}_R = R$  and  $\tau_{\mathcal{I}}$  is proper.*

*Proof.* First suppose  $R$  is not Cohen-Macaulay. Then since  $\dim R = 1$ , we have  $\text{depth } R = 0$ , whence  $\mathfrak{m} \in \text{Ass } R$ . Thus, the total ring of fractions  $Q$  of  $R$  is  $R$  itself, whence  $\bar{R} = R$ . Thus,  $\mathfrak{f}_R = R :_R \bar{R} = R :_R R = R$ . On the other hand,  $R$  is non-reduced, so that in particular  $0 \neq \sqrt{0} = (0)^-$ , whence  $((0) : (0)^-) \neq R$ , so that  $\tau_{\mathcal{I}} = \bigcap_I (I : \bar{I}) \subseteq (0 : (0)^-) \neq R$ .

<sup>2</sup>Recall in particular the convention that  $\text{grade } R = \infty$ .

Conversely, suppose  $R$  is Cohen-Macaulay. By Proposition 3.1, we have

$$\begin{aligned} \tau_{\mathcal{I}} &= \bigcap_I (I : I^-) = \bigcap_{x \text{ parameter}} ((x) : (x)^-) \\ &= \bigcap_{x \text{ nzd}} ((x) : (x)^-) = \bigcap_{x \text{ nzd}} ((x) : (x\bar{R} \cap R)) \\ &\supseteq \bigcap_{x \text{ nzd}} (xR :_R x\bar{R}) = \bigcap_{x \text{ nzd}} (R :_R \bar{R}) = \mathfrak{f}_R. \end{aligned}$$

Now we need only show that  $\tau_{\mathcal{I}} \subseteq \mathfrak{f}_R$ . Let  $Q$  be the total quotient ring of  $R$ . There is some index set  $J$  and some set of elements  $\{z_j\}_{j \in J}$  of  $\bar{R}$  with  $\bar{R} = \sum_{j \in J} Rz_j$  as an  $R$ -submodule of  $Q$ . We have  $\mathfrak{f}_R = (R :_R \sum_j Rz_j) = \bigcap_j (R :_R Rz_j)$ . Moreover, each  $z_j = a_j/x_j$  for some  $a_j \in R$  and some non-zero-divisor  $x_j \in R$ . Then  $x_j \in (R :_R Rz_j)$ , so that  $x_j z_j \in \bar{R}x_j \cap R = (x_j)^-$ . This means that  $((x_j) :_R (x_j)^-) \subseteq ((x_j R) :_R x_j z_j) = (R :_R Rz_j)$ . Putting it all together, we get

$$\tau_{\mathcal{I}} = \bigcap_I (I :_R I^-) \subseteq \bigcap_j ((x_j) :_R (x_j)^-) \subseteq \bigcap_j (R :_R Rz_j) = \mathfrak{f}_R.$$

□

**Remark.** A part of Theorem 4.4 was proved by Janet C. Vassilev in her doctoral thesis [Vas97, Theorem 3.11 and Remark 3.12] (a special case of which is mentioned in [Hun98, Example 3.5]). Namely, she proved that if  $R$  is a one-dimensional local integral domain of characteristic  $p > 0$  with infinite residue field, the ‘test ideal’  $\tau(R)$  (for tight closure) equals the conductor. However, in any equicharacteristic Noetherian ring of dimension 1, tight closure is identical to integral closure, and hence  $\tau(R) = \tau_{\mathcal{I}}(R)$ . So in our terms, Vassilev showed that when  $R$  is a one-dimensional integral domain of prime characteristic,  $\tau_{\mathcal{I}} = \mathfrak{f}_R$ , a result recoverable from our theorem since one-dimensional integral domains are Cohen-Macaulay.

### Higher dimension.

**Theorem 4.5.** *Let  $(R, \mathfrak{m})$  be either excellent or the homomorphic image of a Gorenstein local ring. Suppose that  $\dim R \geq 2$  and that  $R$  is equidimensional with no embedded primes. Then  $\tau_{\mathcal{M}} = \tau_{\mathcal{I}} = 0$ .*

*Proof.* Under the given hypotheses, it is shown in [HH94, Propositions 2.7 and 3.8] that there is a ring  $S$ , into which  $R$  embeds as a subring, such that  $S$  is module-finite over  $R$  and satisfies Serre’s condition  $(S_2)$ , both as an  $R$ -module and as a ring in its own right.

Accordingly, let  $x, y \in \mathfrak{m}$  form a regular sequence on  $S$  as an  $R$ -module. Fix a positive integer  $n$ , and let  $I = (x^{2n}, y^{2n})$ . Note that



$x^n y^n \in I^-$ , as  $(x^n y^n)^2 = (x^{2n})(y^{2n})$  gives an equation of integrality. Also note that the fact that  $x, y$  form a regular sequence on  $S$  makes it easy to show that  $IS :_S x^n y^n = (x^n, y^n)S$ . So we have

$$\begin{aligned} \tau_{\mathcal{I}} &\subseteq (I :_R I^-) \subseteq (I :_R x^n y^n) \\ &\subseteq (IS :_S x^n y^n) = (x^n, y^n)S \subseteq \mathfrak{m}^n S. \end{aligned}$$

However,  $n$  was arbitrary and  $S$  is a finitely generated  $R$ -module. So by the Krull intersection theorem,  $\tau_{\mathcal{M}} \subseteq \tau_{\mathcal{I}} \subseteq \bigcap_n \mathfrak{m}^n S = 0$ .  $\square$

## 5. SOME SURPRISING CONSEQUENCES

**Proposition 5.1.** *Let  $(R, \mathfrak{m}, k, E)$  be a local ring of dimension at least 2. Suppose that  $\hat{R}$  is equidimensional and has no embedded primes. Then  $0_E^- = E$ .*

*Proof.* First assume  $R = \hat{R}$ .

By Theorem 4.5,  $0_E^-$  is a faithful  $R$ -module. However, any faithful  $R$ -submodule  $L$  of  $E$  must equal  $E$ .<sup>3</sup> To see this, consider the short exact sequence

$$0 \rightarrow L \rightarrow E \rightarrow E/L \rightarrow 0.$$

Taking the Matlis dual, we get the short exact sequence

$$0 \leftarrow R/J \leftarrow R \leftarrow J \leftarrow 0,$$

where  $J = (E/L)^\vee$  and  $R/J = L^\vee$ . But then

$$L = (R/J)^\vee = \text{Hom}_R(R/J, E) \cong (0 :_E J),$$

so that  $JL = J \cdot (0 :_E J) = 0$ . So since  $L$  is faithful,  $J = 0$ , whence  $J^\vee = E/L = 0$ , so that  $E = L$ .

Finally, we treat the general case, where  $R$  is not necessarily complete. Note that any  $R$ -submodule  $L$  of  $E$  is in fact an  $\hat{R}$ -submodule, and that  $L = \hat{R} \otimes_R L$ . In particular,  $0_E^- = \hat{R} \otimes_R 0_E^-$  (with closures being taken as  $R$ -modules)  $= 0_{\hat{R} \otimes_R E}^-$  (by Proposition 2.5 (9))  $= 0_E^-$  (with closures being taken as  $\hat{R}$ -modules)  $= E$  (by the first part of the proof).  $\square$

The above result is surprising because this could never happen in a finitely generated module. We find this sufficiently interesting to merit a concrete example:

<sup>3</sup>This particular claim can fail when  $R$  is not complete. Indeed, if  $J$  is any ideal of  $\hat{R}$  that contracts to 0 in  $R$ , let  $M := (0 :_E J)$ . Then  $\text{ann}_{\hat{R}} M = J$  (so that  $M \neq E$ ), but  $\text{ann}_R M = R \cap (\text{ann}_{\hat{R}} M) = R \cap J = 0$ , so that  $M$  is faithful as an  $R$ -module. To see that such a  $J$  can exist, let  $R$  be any local domain which is not analytically irreducible, and let  $J$  be any minimal prime of  $\hat{R}$ . To be even more concrete, set  $R := k[x, y]_{(x, y)} / (x^2 - y^2 - y^3)$ , where  $k$  is a field of characteristic  $\neq 2$  and  $x, y$  are indeterminates over  $k$ , and let  $J := (x - y\sqrt{1+y})$ .

**Example.** Let  $R := k[[x, y]]$ ,  $k$  a field, and  $E$  the injective hull of the residue field. Let  $F := \bigoplus_{i \geq 1} R t_i$ , where the  $t_i$  are free generators. For each  $i \geq 1$ , let  $a_i := x^i t_i$ ,  $b_i := y^i t_i$ ,  $c_i := (xy) t_{i+1} - t_i$ , and  $d_i := (xy)^i t_{2i} - t_i$ . Let  $U$  be the submodule of  $F$  generated by the  $a_i$ s,  $b_i$ s, and  $c_i$ s. Note that  $E \cong F/U$  (indeed this is essentially the “inverse powers” presentation of Macaulay) and one checks readily that every  $d_i \in U$ . For each  $i \geq 1$ , the following equation holds in the symmetric algebra of  $F$  over  $R$ :

$$t_i^2 + 2d_i t_i + (d_i^2 - a_{2i} b_{2i}) = 0.$$

This shows that  $t_i$  is integral over  $U$  for every  $i$ . Hence  $U_F^- = F$ , so that by definition  $0_E^- = E$ .

The general fact yields the following consequence:

**Theorem 5.2.** *Let  $R$  be a Noetherian local ring of dimension at least two, whose completion is equidimensional with no embedded primes. Let  $M$  be an Artinian  $R$ -module. Then there exist torsionless<sup>4</sup>  $R$ -modules  $L \subseteq T$  such that  $M \cong T/L$  and  $T$  is integral over  $L$ . Moreover, if  $M$  is finitely generated (and hence, has finite length),  $T$  and  $L$  may also be chosen to be finitely generated.*

*Proof.* We do the general case first. Since  $M$  is Artinian, there is some positive integer  $n$  such that  $j : M \hookrightarrow E^{\oplus n}$ . We consider this injection to be an inclusion of modules. Let  $\pi : F \twoheadrightarrow E^{\oplus n}$  be a surjection from a free module,  $U := \pi^{-1}(M)$ , and  $K := \ker \pi$ . Then since  $K_F^- = F$  (by Proposition 5.1), we have that  $U + K$  is also integral over  $K$ , and  $(U + K)/K \cong M$ .

If  $M$  has finite length, in particular it is finitely generated, so we can pick elements  $u_1, \dots, u_t \in U$  that generate  $U$  modulo  $K$ . Let  $U' := \sum_{i=1}^t R u_i$ . Fix a basis of  $F$  as a free module over  $R$ , and let  $\mathcal{F}$  denote the set of finitely generated free submodules  $G$  of  $F$  generated by parts of this basis such that  $U' \subseteq G$ . Note that  $\mathcal{F}$  is a direct limit system under inclusion, whose directed union equals  $F$ . Also,  $M \subseteq \pi(G)$  for any  $G \in \mathcal{F}$ , and  $U' \subseteq F = K_F^-$ . Then by Lemma 2.4, there is some  $G \in \mathcal{F}$  such that  $U' \subseteq (K \cap G)_G^-$ . Let  $L := K \cap G$  and  $T := L + U'$ . Then  $T$  is integral over  $L$  in the finite free module  $G$ , and  $T/L \cong M$ .  $\square$

We can take this a bit further. Namely:

**Theorem 5.3.** *Let  $R$  be a Noetherian ring. Let  $M$  be a finite  $R$ -module, and let  $\mathcal{A}$  be the set of minimal primes  $\mathfrak{p}$  of  $M$  such that  $\text{ht } \mathfrak{p} \geq 2$  and  $\widehat{R}_{\mathfrak{p}}$  is equidimensional with no embedded primes. Assuming  $\mathcal{A} \neq \emptyset$ , there exist finite torsionless  $R$ -modules  $L \subseteq T$  such that  $M \cong T/L$  and such that for all  $\mathfrak{p} \in \mathcal{A}$ ,  $T_{\mathfrak{p}}$  is integral over  $L_{\mathfrak{p}}$  (as  $R_{\mathfrak{p}}$ -modules).*

<sup>4</sup>Recall that a module is *torsionless* if it is a submodule of a free module.

*Proof.* We replace the  $E^{\oplus n}$  in the proof of Theorem 5.2 with the injective hull of  $M$ , so that we get a (canonical) injective map  $j : M \hookrightarrow E := E_R(M)$ . But since  $R$  is Noetherian and  $M$  finitely generated, we have  $E = \bigoplus_{\mathfrak{p} \in \text{Supp } M} E_R(R/\mathfrak{p})^{\oplus \mu(\mathfrak{p}, M)}$ , where  $\mu(\mathfrak{p}, M)$  is the 0th Bass number of  $M$  with respect to  $\mathfrak{p}$ , and is always a nonnegative integer. Also, this works well with localization, so that for any minimal prime  $\mathfrak{p}$  of  $M$ , we have  $E(M)_{\mathfrak{p}} = E_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})^{\oplus \mu(\mathfrak{p}, M)}$ , where  $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$  is the residue field of  $\mathfrak{p}$ . Thus, whenever we localize at any  $\mathfrak{p} \in \mathcal{A}$ , we are in the situation of Theorem 5.2. Now pick a free module surjection  $\pi : F \twoheadrightarrow E$ ,  $U := \pi^{-1}(M)$ , and  $K := \ker \pi$ .

Now pick elements  $u_1, \dots, u_t \in U$  that generate  $U$  modulo  $K$ , and let  $U' := \sum_{i=1}^t Ru_i$ . Let  $\mathcal{F}$  be as in the proof of Theorem 5.2. For each  $\mathfrak{p} \in \mathcal{A}$ ,  $(U+K)_{\mathfrak{p}}$  is integral over  $K_{\mathfrak{p}}$  in  $F_{\mathfrak{p}}$ . Thus, for *each* such  $\mathfrak{p}$ , there is some  $G \in \mathcal{F}$  such that  $U'_{\mathfrak{p}} \subseteq ((K \cap G)_{\mathfrak{p}})_{G_{\mathfrak{p}}}^-$  as  $R_{\mathfrak{p}}$ -modules. Let  $H$  be the sum of all such  $G$ 's. By construction,  $H$  is also finitely generated and free, and we have  $U'_{\mathfrak{p}} \subseteq ((K \cap H)_{\mathfrak{p}})_{H_{\mathfrak{p}}}^-$  as  $R_{\mathfrak{p}}$ -modules, for *all*  $\mathfrak{p} \in \mathcal{A}$ . Now let  $L := K \cap H$  and  $T := L + U'$ , and the inclusion  $L \subseteq T$  (as submodules of  $H$ ) satisfies the conclusion of the theorem.  $\square$

[[FURTHER COMMENTS...]]

## 6. AN APPROACH TO PROVING THE EXISTENCE OF BIG COHEN-MACAULAY MODULES

The so-called ‘‘direct summand conjecture’’ of M. Hochster (which is a theorem in equal characteristic) is one of the long-standing open problems of mixed characteristic commutative algebra. A proof of the existence of big Cohen-Macaulay modules for complete local domains would imply a positive answer to the direct summand conjecture. In [Die10], Dietz gave conditions, in terms of the existence of a certain kind of closure operation, for the existence of big Cohen-Macaulay modules:

**Theorem 6.1** ([Die10, Theorem 3.16]). *Let  $(R, \mathfrak{m})$  be a complete local domain. Let  $\mathfrak{h}$  be an operation on submodules of finitely generated  $R$ -modules which satisfies the following axioms for all inclusions  $N \subseteq M$  of finitely generated  $R$ -modules:*

- (1)  $N_M^{\mathfrak{h}}$  is a submodule of  $M$  that contains  $N$ .
- (2)  $(N_M^{\mathfrak{h}})_M^{\mathfrak{h}} = N_M^{\mathfrak{h}}$ .
- (3) If  $W$  is a finite  $R$ -module containing  $M$ , then  $N_W^{\mathfrak{h}} \subseteq M_W^{\mathfrak{h}}$ .
- (4) Let  $f : M \rightarrow W$  be a module homomorphism. Then  $f(N_M^{\mathfrak{h}}) \subseteq f(N)_W^{\mathfrak{h}}$ .
- (5) If  $N_M^{\mathfrak{h}} = N$ , then  $0_{M/N}^{\mathfrak{h}} = 0$ .
- (6)  $\mathfrak{m}_R^{\mathfrak{h}} = \mathfrak{m}$  and  $0_R^{\mathfrak{h}} = 0$ .

- (7) Let  $x_1, \dots, x_{k+1}$  be part of a system of parameters for  $R$ , and  $J := (x_1, \dots, x_k)$ . Let  $M$  be a finitely generated  $R$ -module. Suppose there exists a surjective homomorphism  $f : M \rightarrow R/J$  and  $v \in M$  such that  $f(v) = x_{k+1}$  modulo  $J$ . Then  $(Rv)_M^{\natural} \cap \ker f \subseteq (Jv)_M^{\natural}$ .

Then  $R$  has a big Cohen-Macaulay module.

We have shown that liftable integral closure satisfies axioms 1-5 above, and when  $R$  is a complete local domain, axiom 6 is well-known for ordinary integral closure (and hence for our operation). Axiom 7, called *generalized colon-capturing* in Dietz's paper, is then the main point. We can prove the following special case:

**Proposition 6.2.** *Axiom (7) holds (for liftable integral closure) in case  $v \in M \setminus \mathfrak{m}M$ . That is, given  $R$ ,  $x_1, \dots, x_{k+1}$ ,  $J$ ,  $f$ , and  $v \in M$  as above such that  $v \notin \mathfrak{m}M$ , it follows that  $(Rv)_M^- \cap \ker f \subseteq (Jv)_M^-$ .*

*Proof.* Since  $v$  is a minimal generator of  $M$ , we can choose a surjection  $\pi : F = R^{\oplus n} \twoheadrightarrow M$  such that  $\pi(\mathbf{e}_1) = v$ , where  $\mathbf{e}_1, \dots, \mathbf{e}_n$  is a minimal basis for  $F$  as a free  $R$ -module. Since  $F$  is projective, we may lift the surjection  $f \circ \pi$  through the natural surjection  $R \twoheadrightarrow R/J$  to a map  $g : F \rightarrow R$ , and we may choose it in such a way that  $g(\mathbf{e}_1) = x_{k+1}$ . Now, pick an element  $t \in (Rv)_M^- \cap \ker f$ . Let  $K := \ker \pi$ . Then by definition of liftable integral closure, there is some  $u \in F$  such that  $\pi(u) = t$  and  $u \in (R\mathbf{e}_1 + K)_F^- \cap g^{-1}(J)$ .

Note first that, in terms of the given basis of  $F$ ,  $g$  is uniquely representable as a row-vector (that is, a  $1 \times n$  matrix) of elements of  $R$ . Say  $g = [g_1 \ \cdots \ g_n]$ . We have  $g_1 = g(\mathbf{e}_1) = x_{k+1}$ , so that in fact,  $g = [x_{k+1} \ g_2 \ \cdots \ g_n]$ . Next, if we let  $F' := \bigoplus_{i=2}^n R\mathbf{e}_i$ , so that  $F = R\mathbf{e}_1 \oplus F'$  in a natural way, and  $K' := K \cap F'$ , then in fact  $R\mathbf{e}_1 + K = R\mathbf{e}_1 \oplus K'$ , and  $u \in (R\mathbf{e}_1 + K)_F^- = R\mathbf{e}_1 \oplus (K')_{F'}^-$ . Now let  $g' := [g_2 \ \cdots \ g_n] : F' \rightarrow R$ . Similarly, there is a unique representation of  $u$  in terms of the given basis, say  $u := [u_1 \ \cdots \ u_n]^{\text{tr}}$ , where "tr" means transpose. Then let  $u' := [u_2 \ \cdots \ u_n]^{\text{tr}} \in F'$ . By the direct sum decomposition we derived above, we have  $u' \in (K')_{F'}^-$ . Now,

$$x_{k+1}u_1 + g'(u') = g(u) \in J.$$

But  $g'(K') \subseteq g(K) \subseteq J$ , so that by functoriality of liftable integral closure,  $g'(u') \in g'((K')_{F'}^-) \subseteq (g'(K'))_R^- = \overline{g'(K')} \subseteq \bar{J}$ . This coupled with the fact that  $x_{k+1}u_1 + g'(u') \in J$  implies that  $x_{k+1}u_1 \in \bar{J}$ . That is,  $u_1 \in (\bar{J} : x_{k+1})$ , so that by Ratliff's theorem ([Rat74, Theorem 2.12], as stated in [HS06, Theorem 5.4.1]), we have  $u_1 \in \bar{J}$ . Finally, we have  $u = u_1\mathbf{e}_1 + u' \in \bar{J}\mathbf{e}_1 + K$ , so that  $t = \pi(u) \in \bar{J}v \subseteq (Jv)_M^-$ , as was to be shown.  $\square$

Analyzing the proof of Dietz's theorem, it is clear that the main case for purposes of proving the existence of big Cohen-Macaulay modules is when  $v \in \mathfrak{m}M$  anyway, so our proposition does not contribute substantially to the goal. It is suggestive, however. We include this section simply to indicate that liftable integral closure provides a tempting new way to approach these conjectures which have been at the center of research in commutative algebra for a very long time.

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