RESEARCH STATEMENT

NEIL EPSTEIN

My research area of study is commutative algebra (i.e. the study of ideals and modules over commutative rings). I have an energetic and wide-ranging research program, with many collaborators representing several distinct research projects. My methods include prime characteristic, homological, and non-commutative algebra. My work has close connections with algebraic geometry, combinatorics, and even complex-valued continuous functions. I hope to continue many of my extant projects in the future, as well as develop new directions at my next place of employment.

Contents

1. Continuous closure, axes closure, and natural closure: compare, contrast

Consider the following definition by H. Brenner.

Definition 1.1. [Bre06] Let $P := \mathbb{C}[x_1, \dots, x_n]$, J an ideal of P, and $R := P/J$. Let $X := \{p \in \mathbb{C}^n \mid g(p) = 0 \text{ for all } g \in J\}$ be the corresponding algebraic set, and let I be an ideal of R. The *continuous closure* I^{cont} of I is the set of elements $g \in R$ that can be written as linear combinations of elements of I with coefficients from the ring of $\mathbb{C}$-valued continuous (in the Euclidean topology) functions on X. That is, if $I = (f_1, \dots, f_k)$, then $g \in I^{\text{cont}}$ if there exist continuous functions $\phi_1, \dots, \phi_k : X \to \mathbb{C}$ such that

$$g = \sum_{i=1}^{k} \phi_i f_i.$$

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This yields an ideal that is sometimes larger than the original ideal. For example, when $R = \mathbb{C}[x, y]$ and $I = (x^3, y^3)$, one has $x^2y^2 \in I^{\text{cont}}$. To see this, note that the functions $\alpha_j : \mathbb{C}^2 \setminus \{\mathbf{0}\} \to \mathbb{C}$ given by

$$\alpha_1(x, y) := \frac{x^2y^2\bar{x}^3}{|x|^6 + |y|^6} \quad \text{and} \quad \alpha_2(x, y) := \frac{x^2y^2\bar{y}^3}{|x|^6 + |y|^6}$$

(where $^-$ denotes complex conjugation) satisfy the property

$$\lim_{(x,y)\to(0,0)} \alpha_j(x, y) = 0.$$

Thus, the maps $\widetilde{\alpha_j} : \mathbb{C}^2 \to \mathbb{C}$, defined by

$$\widetilde{\alpha_j}(x, y) = \begin{cases} \alpha_j(x, y) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

are continuous, and $x^2y^2 = \widetilde{\alpha_1} \cdot x^3 + \widetilde{\alpha_2} \cdot y^3$ as maps from $\mathbb{C}^2 \to \mathbb{C}$.

Brenner also introduced the notion of axes closure I^{ax} in such a ring R. Namely, let Y_n be the union of the n complex coordinate axes in $\mathbb{C}^n$ (i.e. the set of n-tuples with at most one nonzero coordinate), and let $A_n := \mathbb{C}[x_1, \dots, x_n]/(x_ix_j \mid 1 \leq i < j \leq n)$ be its coordinate ring; then the axes closure of $I := (f_1, \dots, f_k)$ is the set of all elements $g \in R$ such that for all $n \geq 1$ and all ring maps $\psi : R \to A_n$, there exist $a_1, \dots, a_k \in A_n$ such that

$$\psi(g) = \sum_{i=1}^{k} a_i \psi(f_i).$$

He showed that I^{ax} always contains I^{cont}, and he raised the question of whether they coincide, showing that they do so for monomial ideals primary to the homogeneous maximal ideal in polynomial rings over $\mathbb{C}$.

In joint work with Melvin Hochster [10], we extend axes closure to general Noetherian rings and define *natural closure* $I^\natural$ for ideals in any Noetherian ring, in such a way that one always has $I^\natural \subseteq I^{\text{ax}}$, and in rings where continuous closure is defined we show that $I^\natural \subseteq I^{\text{cont}} \subseteq I^{\text{ax}}$. Here is a sample of our results concerning their equality:

Theorem 1.2. *Let R be a quotient of a polynomial ring over $\mathbb{C}$, and I an ideal.*

(1) *If R is a locally factorial domain of dimension at most 2, then $I^{\text{cont}} = I^{\text{ax}}$.* [10, Theorem 8.5]
(2) *If R is a polynomial ring over $\mathbb{C}$ and I is a monomial ideal, then $I^\natural = I^{\text{cont}}$.* [10, Theorem 10.3]
(3) *If I has no embedded primes, then $I = I^\natural$ iff $I = I^{\text{ax}}$ iff $I = I^{\text{cont}}$.* [10, Corollary 7.14] *Hence if* $\dim R/I = 0$*, then $I^\natural = I^{\text{ax}} = I^{\text{cont}}$.* [10, Corollary 7.15]

However, there is a monomial ideal I in the ring $R = \mathbb{C}[x, y, z]$ such that $I^{\text{cont}} \neq I^{\text{ax}}$ (thus providing a negative answer to Brenner's question) [10, Example 9.2]*. There are also examples where $I^\natural \neq I^{\text{cont}}$.*

Here are the examples we give to show that $I^\natural$, I^{cont}, and I^{ax} are distinct:

Example 1.3. Consider the polynomial map $\gamma : \mathbb{C}^2 \to \mathbb{C}^4$ given by $\gamma(a, b) := (a^3, a^2, ab, b)$. Let $X := \operatorname{im} \gamma$; then $R := \mathbb{C}[a^3, a^2, ab, b]$ is its coordinate ring. Consider the ideal $I := bR$ of R. In the Euclidean topology, γ is a homeomorphism onto its image. Hence, a is a continuous function on X, which shows that $ab = a \cdot b \in (bR)^{\text{cont}}$. On the other hand, we show that $ab \notin (bR)^{\natural}$.

Example 1.4. Let $R := \mathbb{C}[x, y, z]$ (the coordinate ring of complex 3-space), and $I := (y^2, z^2, x^2yz)$. Then $xyz \in I^{\text{ax}}$ because of the limited structure of axes rings. However, if $xyz \in I^{\text{cont}}$, then after specialization (i.e. setting $x = \lambda$ for particular values of $\lambda \in \mathbb{C}$) it would follow that $yz \in (y^2, z^2)^{\text{cont}}$ (that is, $yz = y^2F + z^2G$ for continuous functions F and G) on a neighborhood of the origin in $\mathbb{C}^2$. But then taking limits as $(y, z) \to (0, 0)$ along the lines $y = z$ and $y = -z$ respectively, one obtains the contradiction that $1 = F(0, 0) + G(0, 0) = -1$.

Along the way, we show that if R is a polynomial ring over $\mathbb{C}$ (so that $X = \mathbb{C}^n$) and $f \in R$ has the property that it vanishes at all the points where its first partial derivatives all vanish, then it is a continuous linear combination of its first partial derivatives. That is, f is in the continuous closure of its Jacobian ideal.

Many additional research projects may be based on our work. This last result in particular provides a good project for a student, even one whose background in commutative algebra is minimal. Namely, investigate (algebraically, geometrically, and/or analytically) the property of a function being in the continuous closure of its Jacobian ideal.

Inspired by our work and that of Brenner, Kollár [Kol10] has worked out an algebraic-geometric description of continuous closure in many important cases. See also [FK11].

2. A version of tight closure that commutes with localization

Tight closure is an operation primarily on rings of prime characteristic (also defined in equal characteristic 0), introduced in a seminal article by Hochster and Huneke [HH90]. In the beginning, it was used in providing surprisingly simple proofs (and interesting extensions) of theorems in apparently disparate areas of commutative algebra, including the Briançon-Skoda theorem and the homological conjectures. Later on, striking connections were found between *test ideals* in characteristic p (ideals of elements that kill the tight closures of all ideals) and *multiplier ideals* (a concept arising in complex analysis and in resolution of singularities). Indeed, tight closure theory has established itself as a central topic of interest in commutative algebra.

For twenty years, an open question in tight closure theory was whether tight closure commuted with localization in finitely generated algebras over locally excellent equicharacteristic Noetherian rings, a central property for operations with geometric content. The surprising answer of "no" by Brenner and Monsky [BM10] raises the question of whether there is a tight closure-like operation that does commute with localization. In joint work with Melvin Hochster [11], we provide such a theory. We define the *homogeneous tight closure* $I^{*\text{h}}$ of an ideal and show the following:

Theorem 2.1. *Let R be an excellent Noetherian ring of equal characteristic. Then for any ideal I and any multiplicative subset W of R, $(W^{-1}I)^{*\text{h}} = W^{-1}(I^{*\text{h}})$.*

This confirms that homogeneous tight closure does not always coincide with ordinary tight closure. In general it is smaller. However, we give a number of important cases where tight closure does coincide with homogeneous tight closure, which in turn provide insight into past localization (and non-localization) results in tight closure theory:

Theorem 2.2. *Let R be an excellent Noetherian ring which is either of prime characteristic $p > 0$ or of equal characteristic 0.*

(1) *If I is a parameter ideal (or more generally, if R/I has* finite phantom projective dimension *as an R-module), then $I^* = I^{*\mathrm{h}}$.*
(2) *If R is a finitely generated and positively-graded k-algebra, where k is an algebraic extension of $\mathbb{F}_p$ or of $\mathbb{Q}$, and I is an ideal generated by forms of positive degree, then $I^* = I^{*\mathrm{h}}$.*
(3) *If R is a* binomial ring *over any field k (that is, $R = k[X_1, \ldots, X_n]/J$, where the X_j are indeterminates and J is generated by polynomials with at most two terms each), then for any ideal I of R, $I^* = I^{*\mathrm{h}}$.*

Moreover, *homogeneous tight closure is very much like ordinary tight closure.* Indeed, we show that homogeneous tight closure captures colons and contractions from finite extensions, yields a Briançon-Skoda theorem, is trivial in regular rings, and may be tested by maps to complete local domains.

If all ideals in an excellent equicharacteristic local ring R are tightly closed, does the same property hold for all its localizations? This question has been open for over two decades, to the great consternation of many researchers. To give an affirmative answer, it would be enough to show that the tight closure of any $\mathfrak{m}$-primary ideal equals its homogeneous tight closure.

3. Reductions, spreads, and special parts of closures

My innovations in this area have caused a change in the way that some people think about tight closure, and to some extent about closure operations (see below) in general. Indeed at least five papers in which I was not involved ([Vra06, Vra08, Vra10, FV10, FVV11]) have used this work.

Let $(R, \mathfrak{m})$ be a Noetherian local ring. In [NR54], Northcott and Rees introduce the idea of a *(minimal) reduction* of an ideal. Namely, if $J \subseteq I$ are ideals, J is a *reduction* of I if $JI^n = I^{n+1}$ for some n (equivalently, J and I have the same integral closure, that is, $J^- = I^-$). A *minimal* reduction is then a reduction that is minimal with respect to inclusion. Cleverly using the Nakayama lemma, they show that any reduction of I contains a minimal reduction of I, and by use of an auxillary ring they show that if $R/\mathfrak{m}$ is infinite, then all minimal reductions of an ideal I have the same size minimal generating set, the *analytic spread*, denoted $\ell(I)$.

In [4], I generalize these notions to an arbitrary closure operation[1] c. Namely, $J \subseteq I$ is a *(minimal)* c*-reduction* if $J^{\mathrm{c}} = I^{\mathrm{c}}$ (resp. if J is minimal with respect to this property), and if all minimal c-reductions have the same size minimal generating set, I call this the c*-spread* $\ell^{\mathrm{c}}(I)$ of I. Not all closure operations c admit minimal reductions (e.g. c=radical does not). However, *Nakayama closures* do. I define a closure c on a local ring $(R, \mathfrak{m})$ to be *Nakayama* if whenever J, I are ideals such that $J \subseteq I \subseteq (J + \mathfrak{m}I)^{\mathrm{c}}$, it follows that $I^{\mathrm{c}} = J^{\mathrm{c}}$. With this tool, we have the following:

[1] A *closure operation* c is a set map on the ideals of R such that $I \subseteq I^{\mathrm{c}} = (I^{\mathrm{c}})^{\mathrm{c}}$ for all I, and such that whenever $J \subseteq I$ we have $J^{\mathrm{c}} \subseteq I^{\mathrm{c}}$. Examples include tight closure, integral closure, continuous closure, axes closure, natural closure, the radical, and homogeneous tight closure. For an overview, see my recent survey article [9], also available on arXiv.

Theorem 3.1. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring.*

1. *Suppose* c *is a* Nakayama *closure. Then for any ideal* I*, any* c*-reduction of* I *contains a minimal* c*-reduction of* I. [4]
2. *Under mild hypotheses, Frobenius closure* [7]*, tight closure* [4]*, and plus closure* [3] *are Nakayama closures; hence they admit minimal reductions as in (1).*

As for c-spread, I show among other things the following:

Theorem 3.2. *Let* $(R, \mathfrak{m})$ *be a complete Noetherian local domain of prime characteristic* p*. Then* $\ell^F(I)$*,* $\ell^+(I)$ *and* $\ell^*(I)$ *are well-defined for every ideal* I *of* R.

To show this, instead of using an auxillary ring, I use Vraciu's theory [Vra02, HV03] of *special tight closure* (expanding it to the more general notion of *special parts of closures* to deal with the Frobenius- and plus-closure cases in [7] and [3] respectively) to show that the minimal generating sets of the minimal c-reductions of an ideal I (where $c = F$, $+$, or $*$) form a *matroid*, and hence are equicardinal. This led to the paper [1] with Joseph Brennan discussed in §9.

In [16], Adela Vraciu and I give a numerical limit to express the $*$-spread of an ideal in a normal ring with perfect residue field. Surprisingly, when the ideal is $\mathfrak{m}$-primary, this number can be expressed entirely in terms of Hilbert-Kunz multiplicities!

Vraciu uses these theorems in [Vra06] to show that if $J \subseteq I$ are tightly closed ideals in a normal ring, there is an intervening ideal I' such that $\lambda(I/I') = 1$; in particular when J, I are $\mathfrak{m}$-primary, she can construct a finite filtration consisting of tightly closed ideals with constrained colengths. In [Vra08] and [Vra10], a new version of $\mathfrak{a}$-tight closure is defined and many properties are developed, using $*$-spread and the Nakayama property of tight closure.

Let c be a Nakayama closure. In [FV10], Fouli and Vassilev define the c-*core* of an ideal I as the intersection of all minimal c-reductions of I. They show among other things that $\ell^*(I)$ generic elements of any given lift of $I/\mathfrak{m}I$ will generate a minimal $*$-reduction of I. Then in [FVV11] with Vraciu, they give a formula for the $*$-core in a number of important cases.

In recent work with Holger Brenner [2], we investigate special parts of closures from the viewpoint of *forcing algebras*.

4. Flatness and injectivity

The most important classes of modules, homologically speaking, are flat, injective, and projective modules. In joint work with Yongwei Yao [17], we take a number of criteria that were generally known to be *implied* by flatness (resp. injectivity), and show that they in fact *characterize* flatness (resp. injectivity) of a module over a Noetherian (reduced) ring. The criteria for flatness are useful even when trying to show that a given finitely generated module over a local ring is free. As a sample,

Theorem 4.1. [17, from Theorem 2.3] *Let* R *be a reduced Noetherian ring, and* M *an* R*-module. The following are equivalent.*

1. M *is flat.*
2. $\operatorname{Ass}(L \otimes_R M) \subseteq \operatorname{Ass}_R L$ *for every* R*-module* L.
3. $P \otimes_R M$ *is torsion-free for every* $P \in \operatorname{Spec} R$.

We use our methods to give a criterion for regularity of Noetherian rings of prime characteristic p (or any ring with a *locally contracting endomorphism*, the theory of which we develop in the article). Namely,

Theorem 4.2. [17, from Theorem 3.10] *Let R be a Noetherian ring of prime characteristic $p > 0$. Let ${}^eR$ denote the left R-module given by restriction of scalars via the ring map $r \mapsto r^{p^e}$. Let $F^e(-)$ be the usual Peskine-Szpiro functor* [PS73] *of tensoring with the action of the Frobenius homomorphism. The following are equivalent.*[2]

(1) *R is regular.*
(2) *${}^eR$ is flat as a left R-module for all $e > 0$.*
(3) *$\exists e > 0$ such that $F^e(\mathfrak{m})$ is torsion-free for every maximal ideal $\mathfrak{m}$ of R.*

Note that our methods involve such commutative algebra notions as torsion-freeness and associated primes, which would seem not strong enough to characterize flatness. Dually, we use the notions of divisibility and *co*associated primes to characterize injectivity, and we prove and make use of a *local criterion for injectivity* (dual to Grothendieck's local criterion for flatness).

In a future project, we hope to extend our results to non-Noetherian rings.

5. Phantom homology

Ian Aberbach [Abe94] invented the notion of phantom M-regular sequences and phantom depth (analogues of M-regular sequences and depth, respectively) to go along with Hochster and Huneke's notion (c.f. the monograph [HH93]) of stable phantom exactness. Aberbach proved a "phantom Auslander-Buchsbaum" theorem, and the three of them together [AHH93] used this theory to get some of the best results to date on the then-unsolved problem of whether tight closure always commutes with localization. However, foundational questions on phantom depth remained: For example, do all maximal phantom M-regular sequence have the same length? Does phantom depth behave well along a flat local homomorphism with a good closed fiber?

In [6], I show that the answer to the first question is "yes" by coming up with alternate characterizations of phantom M-regular sequences and phantom depth. In [5], I built on these observations to prove the following theorem, answering the second question.

Theorem 5.1. *Let $\phi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism of excellent Noetherian local rings of prime characteristic such that R is complete or contains an uncountable field. Suppose that R and S share a weak test element (e.g. if ϕ is smooth) and that $S/\mathfrak{m}S$ is Cohen-Macaulay and F-injective. Then for any finite R-module M,*

$$\text{ph.depth}_R M + \operatorname{depth} S/\mathfrak{m}S = \text{ph.depth}_S(S \otimes_R M).$$

Hence, $\text{ph.depth}_R M = \text{ph.depth}_{\hat{R}} \hat{M}$.

These ideas can be seen in terms of the notion of *almost ring theory* (a concept originally due to Faltings [Fal88] and used in Paul Roberts' work, for instance [Rob08, Rob10]). A good project would be to make this connection explicit.

[2]Special cases of the equivalence (1) $\iff$ (2) are due to Kunz [Kun69], so the point is really that (3) is also strong enough to imply regularity.

6. Hilbert-Kunz multiplicity and generalizations

One of the most important tools of characteristic p algebra precedes tight closure theory – namely, that of the *Hilbert-Kunz multiplicity* (shown to exist as a real number in [Mon83] for all $\mathfrak{m}$-primary ideals). However, it is extremely useful in tight closure theory. If $(R, \mathfrak{m})$ is an excellent quasi-unmixed Noetherian local ring of prime characteristic and $J \subseteq I$ are $\mathfrak{m}$-primary ideals, then $J^* = I^*$ if and only if $e_{\mathrm{HK}}(J) = e_{\mathrm{HK}}(I)$. In other words, the Hilbert-Kunz multiplicity is a numerical measure of whether J is a $*$-reduction of I (see §3). Ideas from the theory of Hilbert Kunz multiplicities were used in Brenner and Monsky's counterexample [BM10] showing that tight closure does not commute with localization.

But what about non-$\mathfrak{m}$-primary ideals? Is there a numerical criterion in the general case? Using the fact that tight closure is a Nakayama closure, Yongwei Yao and I show [18] that there is such a numerical criterion – namely, given two arbitrary ideals $J \subseteq I$, we give a number that is as computable as the Hilbert-Kunz multiplicity itself, and is zero precisely when $J^* = I^*$ [18, §6]. However, we explore a number of more fine-grained measurements of the ideals as well, showing that some of them also characterize when two ideals have the same tight closure in some important cases (and in any case we get a one-way implication). For instance, given an arbitrary pair $J \subseteq I$ of nested ideals in a local ring $(R, \mathfrak{m})$, we define an invariant $u^-(J, I)$ (the *relative multiplicity*), such that the following holds:

Theorem 6.1. *Let R be a Noetherian ring of characteristic p such that $\widehat{R_\mathfrak{p}}$ is equidimensional for all $\mathfrak{p} \in \operatorname{Spec} R$. Let $J \subseteq I$ be a nested pair of ideals.*

(1) *If $u^-(J_\mathfrak{p}, I_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$, then $I \subseteq J^*$.* [18, from Theorem 2.4]
(2) *Conversely, suppose that $I \subseteq J^*$ and that either*
- *J has finite projective dimension and no embedded primes, or*
- *R has* finite F-representation type*, or*
- *a certain module over a certain non-commutative ring derived from R, I, and J is finitely generated.*

then $u^-(J_\mathfrak{p}, I_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$. [18, Proposition 3.1 and Theorems 3.4 and 3.5]

We also give other non-trivial cases [18, §4] where a converse to (1) holds.

In another direction, for an ideal I, we define the so-called *unmixed Hilbert-Kunz multiplicity* of I [19] in such a way that if $J \subseteq I$, we have $e_{un}(J) = e_{un}(I)$ if and only if J and I have the same tight closure "up to unmixedness" – that is iff for all top dimensional associated primes $\mathfrak{p}$ of I, we have $(I_\mathfrak{p})^* = (J_\mathfrak{p})^*$. We extend this moreover to a notion of the Hilbert-Kunz multiplicity of a *triple* of R-modules.

On the other hand, many questions about ordinary Hilbert-Kunz multiplicity of $\mathfrak{m}$-primary ideals are still unanswered. For instance, given $\mathfrak{m}$-primary ideals J and I, what can one say about $e_{\mathrm{HK}}(IJ)$ in terms of $e_{\mathrm{HK}}(J)$ and $e_{\mathrm{HK}}(I)$? What can one say about tight closure containments between I and J when there is no containment between I and J? In joint work with Javid Validashti [15], we give some surprising answers to these questions. Here is a sample of what we prove:

Theorem 6.2. *Let $(R, \mathfrak{m})$ be an excellent quasi-unmixed Noetherian local ring of prime characteristic, and let J, I be $\mathfrak{m}$-primary ideals. Then*

(1) $e_{\mathrm{HK}}(IJ) \leq \ell^*(J) e_{\mathrm{HK}}(I) + e_{\mathrm{HK}}(J)$.

(2) *Suppose* $\ell^*(J) \geq 2$ *(which holds e.g. if* $\dim R \geq 2$*) and equality holds in part (1). Then* $J \subseteq I^*$*.*
(3) *If* $J^* = \mathfrak{a}^*$ *for some parameter ideal* $\mathfrak{a}$*, then*

$$e_{\mathrm{HK}}(IJ) \geq \ell^*(J) e_{\mathrm{HK}}(I+J) + e_{\mathrm{HK}}(J).$$

(4) *Hence, if* J *has the same tight closure as a parameter ideal and* $\dim R \geq 2$*, equality holds in part (1) if and only if* $J \subseteq I^*$*.*

7. Liftable integral closure

In joint work with Bernd Ulrich [14], we define a new closure operation on submodules, called *liftable integral closure* which agrees with ordinary integral closure of ideals and of submodules of finitely generated free modules, but in general is smaller than ordinary integral closure[3]. Using this tool, we investigate the properties of an "integral closure test ideal" $\tau_{\mathcal{I}} := \bigcap_{\text{ideals } I} (I : I^-)$. We obtain the following partial classification theorem.

Theorem 7.1. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring.*

(1) *If* $\dim R = 0$*, then* $\tau_{\mathcal{I}} = \operatorname{ann} \mathfrak{m}$*.*
(2) *Suppose* $\dim R = 1$ *and* $R/\mathfrak{m}$ *is infinite. Then* R *is Cohen-Macaulay if and only if* $\tau_{\mathcal{I}}$ *is the conductor of* R*.*
(3) *Suppose* $\dim R \geq 2$ *and* R *is excellent and equidimensional with no embedded primes. Then* $\tau_{\mathcal{I}} = 0$*.*

A consequence of our work is that for rings like those in part (3), any finite length module may be represented as a quotient of torsionless modules with the same integral closure. That is:

Theorem 7.2. *Let* R *be a Noetherian local ring of dimension at least 2, whose completion is equidimensional with no embedded primes, and let* A *be an finite length* R*-module. Then there are finitely generated* R*-modules* $L \subseteq M \subseteq F$ *such that* F *is free,* M *is integral over* L*, and* $A \cong M/L$*.*

This raises the very interesting question of what *other* sorts of modules lend themselves to being quotients of integrally dependent modules. We hope to answer this in future work.

We also provide an alternate strategy toward solving the homological conjectures, via the work of Dietz [Die10].

8. Tight interior

One of the most important notions in tight closure theory is the notion of a *test ideal*, and duality is of course an important concept in all of mathematics. In joint work with Karl Schwede [13], we take the approach that test ideals are somehow dual to tight closure, and make it explicit in a flexible framework. That is, we define the *tight interior* M_{*R} (or M_* when the ring is clear from context) of any R-module M in such a way that the tight interior of R itself is the (big) test ideal. Then we show that tight interior is dual to tight closure in three different senses – in terms of Matlis duality, annihilation, and in terms of the duality of sum and intersection. By dualizing statements in tight closure theory we obtain sometimes

[3] For the usual notion of integral closure of a submodule, see [EHU03] or [HS06, chapter 16].

surprising correspondences (and explanations and generalizations of known theorems) about test ideals. On the other hand, statements about test ideals can be seen as statements about R_{*R} that can be generalized to statements about M_{*R} for general M. We avoid the machinery of tight closure itself, emphasizing the point of view that tight interior appears to be a more fundamental operation, at least in the F-finite reduced case. Here is a sample of some important properties.

Theorem 8.1. *Let R be a prime characteristic F-finite reduced Noetherian ring, and M an arbitrary R-module.*

1. $(M_*)_* = M_*$
2. *For any multiplicative set W, we have $(M_{*R})_W = (M_W)_{*R_W}$.*
3. *If R is local, $\hat{R} \otimes_R M_{*R} \cong (\hat{R} \otimes_R M)_{*\hat{R}}$.*
4. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal primes of R. Then $M_{*R} = \sum_{i=1}^n (0 :_M \mathfrak{p}_i)_{*(R/\mathfrak{p}_i)}$, thus allowing us to reduce to the domain case.*
5. *Let $R \to S$ be a ring homomorphism such that S_{red} is F-finite. Then if $\varepsilon : \mathrm{Hom}_R(S, M) \to M$ is the natural map, we have $\varepsilon(\mathrm{Hom}_R(S, M)_{*S}) \subseteq M_{*R}$.*

Since the tight interior of the ring is the big test ideal, our work yields results for test ideals and suggests avenues to explore test ideals further. For instance, part (4) of the above theorem suggests (and implies part of) the following.

Theorem 8.2. *Let R be a prime characteristic F-finite reduced Noetherian ring. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal primes of R.*

1. *$\tau_b(R)$ (resp. $\tau(R)$) $\subseteq \sum_{i=1}^n (0 : \mathfrak{p}_i)$, with equality if each $R/\mathfrak{p}_i$ is strongly (resp. weakly) F-regular.*
2. *If the normalization of R is strongly F-regular, then $C(R) = \tau(R) = \tau_b(R)$, where $C(R)$ is the* conductor *of R.*

We also obtain a dual to the theory of phantom homology. Moreover, our framework also provides an explicit construction in many cases for the test submodules of Blickle [Bli09].

This work is wide open to further development and research projects, as it puts test ideals into a whole new framework as a special case of an interior operation.

9. Generic matroids

A *matroid* is a set E along with a nonempty collection $\mathcal{B}$ of finite subsets of E (called its *bases*) such that for all distinct pairs $B, B' \in \mathcal{B}$ and elements $b \in B$, we have $B \nsubseteq B'$ and there exists $b' \in B'$ such that $(B \setminus \{b\}) \cup \{b'\} \in \mathcal{B}$. Examples include edge sets of finite graphs with spanning forests, finite-dimensional vector spaces along with their bases, and field extensions of finite transcendence degree along with their transcendence bases. As I showed in [4, 3, 7], when R is a complete local domain of prime characteristic and I is an ideal, the set I along with the minimal generating sets of its minimal c-reductions (where $\mathrm{c} = *, +, F$ respectively) also forms a matroid. It is a basic fact of matroid theory that all bases have the same cardinality (called the *rank* of a matroid), so this shows that $\ell^{\mathrm{c}}(I)$ exists for these closure operations. Since $\ell^-(I)$ (the analytic spread of [NR54]) also exists, a natural question is whether we get a matroid when $\mathrm{c} = -$.

In joint work with Joseph Brennan [1], after showing the answer to be "no" with a counterexample, we find a new combinatorial-topological construction (the *generic matroid*), jointly generalizing the notions of matroid and topological space.

Intuitively, a generic matroid is a bunch of bona-fide matroids glued together to form an overarching matroid-like structure with some wiggle-room. We exhibit a generic matroid structure not only in the case of ideals and minimal reductions (over a local ring with infinite residue field), but also for graded Noether normalizations of a standard graded k-algebra (k an infinite field) and for minimal complete reductions of a finite set of ideals. This then gives reasons for equicardinality in all these situations.

One of the strengths of matroid theory is the notion of "cryptomorphic" definitions, allowing the theory great combinatorial flexibility. That is, a matroid may be equivalently defined in terms of bases (as above), independent sets, circuits, flats, etc. In future work, we hope to obtain cryptomorphic definitions of generic matroids in the same way.

10. Zero-divisor graphs

Let S be a commutative multiplicative semigroup with zero (i.e. an element 0 such that $s0 = 0$ for all $s \in S$). Then one defines the *zero-divisor graph* $\Gamma(S)$ to be the graph whose *vertices* are the nonzero zero-divisors of S, where there is an *edge* from s to t precisely when $s \neq t$ and $st = 0$ ([DMS02]; originally defined for commutative rings in [AL99]).

In joint work with Peyman Nasehpour [12], we investigate the interplay between the structure of *nilpotent-free* semigroups (i.e. we require for all $s \in S$ that if $s^n = 0$ then $s = 0$) and their corresponding zero-divisor graphs. In doing so, we introduce the notion of an *Armendariz map* between semigroups, which we show preserves many important invariants of the corresponding zero-divisor graphs, including diameter and chromatic number. We give applications to topological spaces (especially those occurring in algebraic geometry), lattice theory, Boolean algebra, contents of polynomials, comaximal ideals, and tensor products.

This topic provides research project potential at all levels, involving as it does so many areas of mathematics.

My publications and preprints

[1] Joseph P. Brennan and Neil Epstein, *Noether normalizations, reductions of ideals, and matroids*, Proc. Amer. Math. Soc. **139** (2011), no. 8, 2671–2680.

[2] Holger Brenner and Neil Epstein, *Special and specific parts of closures*, in preparation, 2011.

[3] Neil Epstein, *Closure operators in commutative algebra: Tight closure, phantom depth, and ∗-spread*, Ph.D. thesis, University of Kansas, 2005.

[4] ______, *A tight closure analogue of analytic spread*, Math. Proc. Cambridge Philos. Soc. **139** (2005), no. 2, 371–383.

[5] ______, *Phantom depth and flat base change*, Proc. Amer. Math. Soc. **134** (2006), no. 2, 313–321.

[6] ______, *Phantom depth and stable phantom exactness*, Trans. Amer. Math. Soc. **359** (2007), no. 10, 4829–4864.

[7] ______, *Reductions and special parts of closures*, J. Algebra **323** (2010), no. 8, 2209–2225.

[8] ______, *Pseudostar operations*, in preparation, 2011.

[9] ______, *A guide to closure operations in commutative algebra*, Progress in Commutative Algebra: Ring Theory, Homology, and Decompositions (Sean Sather-Wagstaff, Christopher Francisco, Lee Klingler, and Janet C. Vassilev, eds.), De Gruyter Proceedings in Mathematics, De Gruyter, to be published February, 2012.

[10] Neil Epstein and Melvin Hochster, *Continuous closure, axes closure, and natural closure*, arXiv:math.AC/1106.3462v1, 2011.

[11] ———, *Homogeneous equational closures, tight closure, and localization*, in preparation, 2011.
[12] Neil Epstein and Peyman Nasehpour, *Zero-divisor graphs of nilpotent-free semigroups*, in preparation, 2011.
[13] Neil Epstein and Karl Schwede, *A dual to tight closure theory*, in preparation, 2011.
[14] Neil Epstein and Bernd Ulrich, *Liftable integral closure*, preprint, 2010.
[15] Neil Epstein and Javid Validashti, *Hilbert-Kunz multiplicity of products of ideals*, in preparation, 2011.
[16] Neil Epstein and Adela Vraciu, *A length criterion of $*$-spread*, Osaka J. Math. **45** (2008), no. 2, 445–456.
[17] Neil Epstein and Yongwei Yao, *Criteria for flatness and injectivity*, Math. Z. (2011), doi:10.1007/s00209-011-0910-y.
[18] ———, *Some extensions of Hilbert-Kunz multiplicity*, arXiv:math.AC/1103.4730v1, 2011.
[19] ———, *Unmixed Hilbert-Kunz multiplicity*, in preparation, 2011.

Other references

[Abe94] Ian M. Aberbach, *Finite phantom projective dimension*, Amer. J. Math. **116** (1994), 447–477.
[AHH93] Ian M. Aberbach, Melvin Hochster, and Craig Huneke, *Localization of tight closure and modules of finite phantom projective dimension*, J. Reine Angew. Math. **434** (1993), 67–114.
[AL99] David F. Anderson and Philip S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), no. 2, 434–447.
[Bli09] Manuel Blickle, *Test ideals via algebras of p^{-e}-linear maps*, arXiv:math.AC/0912.2255, to appear in J. Algebraic Geom., 2009.
[BM10] Holger Brenner and Paul Monsky, *Tight closure does not commute with localization*, Ann. of Math. (2) **171** (2010), no. 1, 571–588.
[Bre06] Holger Brenner, *Continuous solutions to algebraic forcing equations*, arXiv:math.AC/0608611v2, to appear in Internat. J. Math., 2006.
[Die10] Geoffrey D. Dietz, *A characterization of closure operations that induce big Cohen-Macaulay modules*, Proc. Amer. Math. Soc. **138** (2010), no. 11, 3849–3862.
[DMS02] Frank R. DeMeyer, Thomas McKenzie, and Kim Schneider, *The zero-divisor graph of a commutative semigroup*, Semigroup Forum **65** (2002), no. 2, 206–214.
[EHU03] David Eisenbud, Craig Huneke, and Bernd Ulrich, *What is the Rees algebra of a module?*, Proc. Amer. Math. Soc. **131** (2003), no. 3, 701–708.
[Fal88] Gerd Faltings, *p-adic Hodge theory*, J. Amer. Math. Soc. **1** (1988), no. 1, 255–299.
[FK11] Charles Fefferman and János Kollár, *Continuous linear combinations of polynomials*, arXiv:math.CA/1103.0964v1, 2011.
[FV10] Louiza Fouli and Janet C. Vassilev, *The cl-core of an ideal*, Math. Proc. Cambridge Philos. Soc. **149** (2010), no. 2, 247–262.
[FVV11] Louiza Fouli, Janet C. Vassilev, and Adela Vraciu, *A formula for the $*$-core of an ideal*, Proc. Amer. Math. Soc. **139** (2011), no. 12, 4235–4245.
[HH90] Melvin Hochster and Craig Huneke, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), no. 1, 31–116.
[HH93] ———, *Phantom homology*, Mem. Amer. Math. Soc. **103** (1993), no. 490, vi+91.
[HS06] Craig Huneke and Irena Swanson, *Integral closure of ideals, rings, and modules*, London Math. Soc. Lecture Note Ser., vol. 336, Cambridge Univ. Press, Cambridge, 2006.
[HV03] Craig Huneke and Adela Vraciu, *Special tight closure*, Nagoya Math. J. **170** (2003), 175–183.
[Kol10] János Kollár, *Continuous closure of sheaves*, arXiv:math.AC/1010.5480v1, 2010.
[Kun69] Ernst Kunz, *Characterizations of regular local rings for characteristic p*, Amer. J. Math. **91** (1969), 772–784.
[Mon83] Paul Monsky, *The Hilbert-Kunz function*, Math. Ann. **263** (1983), no. 1, 43–49.
[NR54] Douglas G. Northcott and David Rees, *Reductions of ideals in local rings*, Proc. Cambridge Philos. Soc. **50** (1954), no. 2, 145–158.

[PS73] Christian Peskine and Lucien Szpiro, *Dimension projective finie et cohomologie locale. applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck*, Inst. Hautes Études Sci. Publ. Math. (1973), no. 42, 47–119.

[Rob08] Paul C. Roberts, *Almost regular sequences and the monomial conjecture*, Michigan Math. J. **57** (2008), 615–623, Special volume in honor of Melvin Hochster.

[Rob10] ______, *Fontaine rings and local cohomology*, J. Algebra **323** (2010), no. 8, 2257–2269.

[Vra02] Adela Vraciu, $*$-*independence and special tight closure*, J. Algebra **249** (2002), 544–565.

[Vra06] ______, *Chains and families of tightly closed ideals*, Bull. London Math. Soc. **38** (2006), 201–208.

[Vra08] ______, *A new version of* $\mathfrak{a}$-*tight closure*, Nagoya Math. J. **192** (2008), 1–25.

[Vra10] ______, *Drops in joint Hilbert-Kunz multiplicities and projective equivalence*, J. Algebra **323** (2010), no. 8, 2226–2242.

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