THE QUEST FOR COUNTEREXAMPLES IN TORIC GEOMETRY

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ABSTRACT. We discuss an experimental approach to open problems in toric geometry: are smooth projective toric varieties (i) projectively normal and (ii) defined by degree 2 equations? We discuss the creation of lattice polytopes defining smooth toric varieties as well as algorithms checking properties (i) and (ii) and further potential properties, in Particular a weaker version of (ii) asking for scheme-theoretic definition in degree 2.

1. INTRODUCTION

Two of the most tantalizing questions in toric geometry concern the arithmetic normality and the degree of the defining equations of smooth projective toric varieties:

(N) (Oda) Is every equivariant embedding of such a variety $\mathscr{V}$ into projective space arithmetically normal?

(Q) (Bøgvad) Is the ideal of functions vanishing on $\mathscr{V}$ generated in degree 2?

Both questions have affirmative answers in dimension 2, but are open in dimension ≥ 3. They were the major themes of workshops at the Mathematisches Institut Oberwolfach (2007) and the American Institute of Mathematics (2009). The Oberwolfach report [16] gives a good overview of the subject. See Ogata [21], [22] for some positive results in dimension 3.

Toric geometry has developed a rather complete dictionary that translates properties of projective toric varieties into combinatorics of lattice polytopes, and therefore both questions can be formulated equivalently in the language of lattice polytopes and their monoid algebras. In the following, lattice polytopes representing smooth projective toric varieties are called *smooth*, those representing a normal projective toric variety are called *very ample*, and those representing an arithmetically normal subvariety are *normal*. A very brief overview of the connection between toric varieties and lattice polytopes is given in Section 2.

An algorithmic approach for the search of counterexamples was discussed by Gubeladze and the author about 10 years ago, and taken up by Gubeladze and Hoşten in 2003, however not fully implemented. Such an implementation was realized by the author in 2007, and completed and augmented in several steps. A software library on which the implementation is based had previously been developed for the investigation of unimodular covering and the integral Carathéodory property [5], [2]. Moreover, Normaliz [10] proved very useful (and profited from the experience gained in this project).

Unfortunately the search for counterexamples has been fruitless to this day. Nevertheless we hope that a discussion of the algorithmic approach to (Q) and (N) and several related properties of smooth lattice polytopes is welcome.

The main experimental line consists of three computer programs for the following tasks:

(1) the random creation of smooth projective toric varieties via their defining fans;
(2) the computation of support polytopes;
(3) the verification of various properties, in particular (N) and (Q).

The implementation of the first two tasks is described in Section 3.

Testing normality (Section 4) is much easier and faster than testing quadratic generation, and amounts to a Hilbert basis computation that is usually a light snack for the Normaliz algorithm described in [11] and [9]. Quadratic generation requires more discussion (Section 5). One of the results found in connection with the experimental approach is a combinatorial criterion for scheme-theoretic definition in degree 2. In contrast to ideal-theoretic definition in degree 2, as asked for in (Q), it can be tested efficiently for polytopes with a large number of lattice points.

For an arbitrary lattice polytope P the multiples cP are normal for $c \geq \dim P - 1$ and their toric ideals are generated in degree 2 for $c \geq \dim P$ [7]. Therefore one expects counterexamples to have few lattice points, and so we try to reduce smooth lattice points in size without giving up smoothness, of course. In Section 6 we explain the technique of *chiseling*, already suggested by Gubeladze and Hoşten, that splits a smooth polytope in two parts unless it is *robust*. It is then not hard to see that a minimal counterexample to (Q) or (N) must be robust.

After a discussion of some further potential properties of smooth polytopes, in particular the positivity of the coefficients of their Ehrhart polynomials, we widen the class of lattice polytopes by including the very ample ones. In fact, it would already be very interesting to find simple polytopes that are very ample but not normal.

One can interpret the failure of the search for counterexamples as an indication that (N) and (Q) hold. However, the main difficulty is not the investigation of given smooth polytopes: it is their construction for which we depend on the construction of fans, objects that live in the space dual to that of the polytopes. It is doubtful whether we can generate a sufficient amount of complexity in the dual space without loosing the passage to primal space. An argument supporting this viewpoint is given in Section 8.

The software on which our experiments have been based was made public in 2009 and has recently been updated [3]. Its documentation discusses the practical aspects of its use. They will be skipped in the following.

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2. LATTICE POLYTOPES AND TORIC VARIETIES

We assume that the reader is familiar with the basic notions of discrete convex geometry, combinatorial commutative algebra and toric algebraic geometry. These notions are developed in the books by Fulton [14], Oda [19] and Cox–Little–Schenck [12]. We will follow the terminology and notation of [6].

Nevertheless, let us briefly sketch the connection between projective toric varieties, projective fans and lattice polytopes since the experimental approach to the questions (N) and (Q) is based on it. The main actors in the experimental approach are lattice polytopes, and therefore it is natural, but opposite to the conventions of toric geometry, to have them live in the primal vector space $V = \mathbb{R}^d$ whereas (their normal) fans reside in the dual space $V^* = \operatorname{Hom}_{\mathbb{R}}(V,\mathbb{R})$. By L we denote the lattice $\mathbb{Z}^d$, and L^* is its dual in V^*.

Let $P \subset V$ be a lattice polytope. In order to avoid technicalities of secondary importance we will assume that the lattice points in P generate L as an affine lattice whenever we are free to do so. In particular P has dimension d, and we need not distinguish *normal* and *integrally closed* lattice polytopes (in the terminology of [6]).

The set $E(P) = P \times \{1\} \cap \mathbb{Z}^{d+1}$ generates a submonoid $M(P)$ of $\mathbb{Z}^{d+1}$. The monoid algebra $R = K[P] = K[M(P)]$ over a field K has a natural grading and is generated by monomials of degree 1 (that, by construction, correspond to the lattice points of P). Thus $\mathscr{V} = \operatorname{Proj} R$ is a projective subvariety of the projective space $\mathbb{P}^n_K$, $n = \#E(P) - 1$. The natural affine charts of $\mathscr{V}$ are the spectra of monoid algebras obtained by dehomogenizing R with respect to the monomials that correspond to the vertices of P: for such a vertex v the corresponding chart is given by $\operatorname{Spec} K[M(P)_v]$ where the monoid $M(P)_v \subset \mathbb{Z}^d$ is generated by the difference vectors $x - v$, $x \in P \cap L$. (In multiplicative notation, $x - v$ corresponds to the quotient of two monomials of degree 1.)

In classical terminology, a toric variety is required to be normal, and this is the case if and only if all the algebras $K[M(P)_v]$ are normal. A polytope P with this property is called *very ample* since it represents a very ample line bundle on $\mathscr{V}$. Under our assumptions on P, normality of $K[M(P)_v]$ is equivalent to the equality $M(P)_v = C(P)_c \cap L$ where $C(P)_v$ is the *corner* (or *tangent*) cone of P at v: it is generated by the vectors $x - v$, $x \in P$.

The variety $\mathscr{V}$ is smooth if and only if the algebras $K[M(P)_v]$ are polynomial rings. In terms of P, this property can be characterized as follows: exactly d edges emanate from each vertex v, and the d vectors $w - v$, where w is the lattice point next to v on an edge, are a basis of L. Such polytopes are called *smooth*.

The combinatorial approach to the open questions (N) and (Q) is justified by the fact that the homogeneous coordinate rings considered in these questions are all of type $K[P]$ where P is a smooth polytope.

The rather elementary passage from a lattice polytope P to a projective toric variety has been sketched above. In order to justify the last claim we have to reverse the construction. Every projective toric variety of dimension d intrinsically defines a complete projective fan $\mathscr{F}$ in V^* (we remind the reader on our convention on primal and dual space), and the coordinate rings of the equivariant embeddings of $\mathscr{V}$ into projective space are given in the form $\operatorname{Proj} K[P]$ where P is a very ample lattice polytope (satisfying all our basic assumptions) such that $\mathscr{N}(P) = \mathscr{F}$. Such a polytope is called a *support polytope* of

$\mathscr{F}$. The corner cones of P are exactly the cones dual to the cones in $\mathscr{F}^{[d]}$ (the set of d-dimensional faces of $\mathscr{F}$), and since duality preserves unimodularity, $\mathscr{V}$ is smooth if and only if $\mathscr{F}$ is unimodular.

The correspondence between unimodular projective fans and smooth projective toric varieties is not only of fundamental theoretical importance—it offers a way to construct smooth polytopes from “random data”.

3. POLYTOPES FROM FANS

3.A. **Creating unimodular projective fans.** As just explained, the choice of a smooth projective variety is equivalent to the construction of a projective unimodular fan. It can be carried out as follows:

(UF1) Choose vectors $\rho_1,\dots,\rho_s$ in L^* such that the origin is in the interior of $Q = \operatorname{conv}(\rho_1,\dots,\rho_s)$. The cones spanned by the faces of Q (with apex in 0) then form a projective fan $\mathscr{F}$.

(UF2) Choose a regular triangulation of the boundary of Q with vertices in lattice points, and replace $\mathscr{F}$ by the induced simplicial refinement.

(UF3) For each maximal cone of $\mathscr{F}$ compute its Hilbert basis and refine $\mathscr{F}$ by stellar subdivision, inserting all these vectors in some random order.

(UF4) Repeat (UF3) until a unimodular fan is reached.

This is nothing but the algorithm producing an equivariant desingularization of the projective toric variety defined by the choice of $\mathscr{F}$ in (UF1). It terminates in finitely many steps since stellar subdivision by Hilbert basis elements strictly reduces the multiplicities of the simplicial cones. Computing unimodular fans is fast, contrary to the computation of support polytopes.

3.B. **Computing support polytopes.** Once we have a unimodular fan $\mathscr{F}$, we must find support polytopes of $\mathscr{F}$. Let us first assume that $\mathscr{F}$ is just an arbitrary complete fan. The algorithm that we describe in the following will decide whether $\mathscr{F}$ is projective by providing lattice polytopes P such that $\mathscr{N}(P) = \mathscr{F}$ in the projective case, and ending with a negative outcome otherwise. The set of rays of $\mathscr{F}$ is denoted by $\mathscr{F}^{[1]}$, and its elements are listed as $\rho_1,\dots,\rho_s$.

Each support polytope is the solution set of a system of linear inequalities

$$\rho_i(x) \ge -b_i, \qquad b_i \in \mathbb{Z},\ i = 1,\dots,s.$$

(The choice of the minus sign will turn out natural.) We are searching for the right hand sides $b = (b_1,\dots,b_s) \in \mathscr{W} = \mathbb{Z}^s$, such that the following conditions are satisfied:

(LP) For each cone $\Sigma \in \mathscr{F}^{[d]}$ there exists a vector $v_\Sigma \in L$ such that $\rho_i(v_\Sigma) = -b_i$ for $\rho_i \in \Sigma$.

(CP) The points v_Σ are indeed the vertices of their convex hull $P(b)$.

(VA) $P(b)$ is very ample.

In fact, for each $\Sigma \in \mathscr{F}^{[d]}$, the hyperplanes with equations $\rho_i(x) = -b_i$, $\rho_i \in \Sigma$, must meet in a lattice point $v_\Sigma \in L$. Thus condition (LP) selects a sublattice $\mathscr{C}$ of $\mathscr{W}$, and there is a well-defined linear map $\operatorname{vert}_\Sigma : \mathscr{C} \to L$ that assigns $b \in \mathscr{C}$ the prospective vertex v_Σ.

In the unimodular case, (LP) is satisfied for all $b \in \mathscr{W} = \mathscr{C}$, and this simplifies the situation significantly! Also (VA) is automatically satisfied. Therefore we concentrate on condition (CP) which requires that the points v_Σ are in convex position. This is equivalent to the system

$$\rho_j(\mathrm{vert}_\Sigma(b)) \geq -b_j + 1, \qquad j = 1, \dots, s,\ \rho_j \notin \Sigma, \quad \Sigma \in \mathscr{F}. \tag{1}$$

of inequalities being satisfied. Convexity is a local condition, and therefore one can restrict the system to a smaller set of inequalities: one needs to consider only the inequalities $\rho_j(\mathrm{vert}_\Sigma(b)) > -b_j$ such that $\rho_j \notin \Sigma$ is a ray in a cone $T \in \mathscr{F}$ sharing a facet with Σ. (This condition is easily checked algorithmically.) The set of pairs (Σ, j) just defined is denoted by $\mathscr{S}$.

We summarize and slightly reformulate our discussion as follows. Set

$$N' = \{b \in \mathbb{R}^s : \rho_j(\mathrm{vert}_\Sigma(b)) + b_j \geq 0,\ (\Sigma, j) \in \mathscr{S}\}.$$

Then the lattice polytopes we are trying to find correspond to the points in $\mathscr{C} \cap \mathrm{int}(N')$. The cone N' is not pointed since it contains a copy of V, namely the vectors $(\rho_j(v))$. However, we loose nothing if we choose a cone $\Sigma_0 \in \mathscr{F}^{[d]}$ and set $v_{\Sigma_0} = 0$. In this way we intersect N' with a linear subspace U, and the intersection

$$N = N' \cap U$$

is indeed pointed. Moreover, since 0 is one vertex of the polytopes to be found, we have $b_j \geq 0$ for all j and all points $b \in N'$. In particular this implies

$$P(b) \subset P(b + b')$$

for all $b, b' \in N \cap \mathscr{C}$.

Heuristically, and for the reasons pointed out above, the candidates for counterexamples should appear among the *inclusion minimal* polytopes P with $\mathscr{F} = \mathscr{N}(P)$. In view of the inclusion just established, it is enough to determine the minimal system of generators of $\mathscr{C} \cap \mathrm{int}\, N$ as an ideal of the monoid $\mathscr{C} \cap N$. After homogenization of the system (1) and fixing $\mathrm{vert}_{\Sigma_0} = 0$, this amounts to a Hilbert basis calculation in the cone $\tilde{N} \subset \mathbb{R}^{s-d+1}$ defined by the inequalities

$$\begin{aligned} \rho_j(\mathrm{vert}_\Sigma(b)) + b_j - h &\geq 0, \qquad (\Sigma, j) \in \mathscr{S}, \\ h &\geq 0. \end{aligned}$$

From the Hilbert basis computed we extract the elements with $h = 1$, and obtain a collection of polytopes among which we easily find the inclusion minimal ones. (If no such element exists, the fan has proved to be non-projective.)

Remark 3.1. (a) The letters $\mathscr{W}$ and $\mathscr{C}$ have been chosen since $\mathscr{W}$ represents the group of torus invariant Weyl divisors and $\mathscr{C}$ the group of torus invariant Cartier divisors. By fixing one of the vertices at the origin, we have chosen a splitting of the epimorphism $\mathscr{C} \to \mathrm{Pic}(\mathscr{V})$.

Remark 3.2. (a) Since we favor small examples and since very large ones tend to be intractable, we limit the construction of unimodular fans to at most $d + 20$ rays (in other words, $\mathrm{rank}\,\mathrm{Pic}(\mathscr{V}) \leq 20$) and 150 maximal simplicial cones. These numbers can be varied, of course, but they allow the computation of support polytopes.

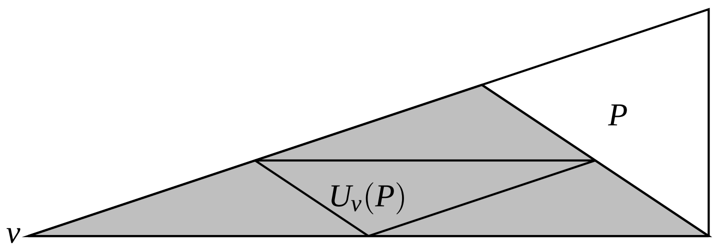
(b) Computing all minimal support polytopes is only possible if the rank of the Picard group is not too large. In practice, 10 is a reasonable bound. One way out is to compute the extreme rays of $\widetilde{N}$ and to select those with $h = 1$. Though there is no guarantee for the existence of such extreme rays, they almost always exist.

(c) Not every element of the Hilbert basis computed defines an inclusion minimal support polytope of the given fan. The reason is that the difference of the corresponding support functions of the fan need not be convex.

4. Normality

4.A. **Checking normality.** For this purpose we use the Hilbert basis algorithm of Normaliz (in the author's experimental library). We refer to [11] and [9] for the details. The algorithm is fast for the polytopes that have been investigated.

4.B. **Extending corner covers.** One reason for which one could expect smooth polytopes P to be normal is that each corner is covered by unimodular simplices—in fact, it is such a simplex—and that it should be possible to extend these covers far enough into P such that P is covered by the extensions. In order to define the extensions one can identify the corner cone $C(P)_v$ with the cone D generated by the sums of unit vectors $e_1, e_1+e_2, \dots, e_1+\dots+e_d$ in $\mathbb{R}^d$ and transfer the Knudsen-Mumford triangulation of D to a triangulation Σ of $C(P)_v$ (compare [6, Ch. 3]. Then we let $U_v(P)$ be the union of those unimodular simplices of Σ that lie in P. If $P = \bigcup_v U_v(P)$, then P is evidently normal. It would be possible to test whether $P = \bigcup_v U_v(P)$. However, a direct test for normality is much faster. We will come back to this point in 7.A below.



5. Generation in degree 2

In the following we must work with the toric ideal of a polytope and its dehomogenizations, and some precise notation is needed. The monoid algebra $K[M(P)]$ has a natural presentation as a residue class ring of a polynomial ring

$$A_P = K[X_x : x \in P \cap \mathbb{Z}^d] \xrightarrow{\phi} K[M(P)], \qquad X_x \mapsto (x,1) \in M(P) \subset K[M(P)].$$

The kernel of ϕ is the toric ideal of P. It is generated by all binomials

$$\prod_x X_x^{a_x} - \prod_x X_x^{b_x} \qquad \text{such that} \qquad \sum_x a_x x = \sum_x b_x x.$$

The lattice points w next to a vertex v of P on an edge of P will be called *neighbors* of v.

5.A. **Testing generation in degree 2.** There seems to be no other way for testing generation in degree 2 than computing the toric ideal via a Gröbner basis method. (For toric ideals, special algorithms have been devised; see [17] or [1].) In order to access also rather large polytopes we only test whether the toric ideal $I(P)$ needs generators in degree 3 as follows, letting J denote the ideal generated by the degree 2 binomials in $I(P)$:

(GB2) We compute the degree 2 component G of a Gröbner basis of J with respect to a reverse lexicographic term order by simply scanning all degree 2 binomials in $I(P)$.
(GB3) Next we compute the degree 3 component of the Gröbner basis of J by the Buchberger algorithm in a specialized data structure.
(HV3) Then we compute the h-vector of J up to degree 3, using the initial ideal from the preceding steps.
(NMZ) Finally the result is compared to the h-vector of $K[P]$ computed by Normaliz. (This h-vector gives the numerator polynomial of the Ehrhart series of P.)

Clearly, $I(P)$ has no generators in degree 3 if and only if the two h-vectors coincide up to hat degree. The time consuming step is (GR3) while the others are very fast.

As we will see next, there is a generalization of generation in degree 2 that is very natural from the viewpoint of projective geometry and much faster to decide.

5.B. **Scheme-theoretic generation in degree 2.** Let $I \subset K[X_1,\dots,X_n]$ be a homogeneous ideal (with respect to the standard grading). We say that I is *scheme-theoretically generated* in degree 2 if there exists an ideal J that is generated by homogeneous elements of degree 2 such that I and J have the same saturation with respect to the maximal ideal $\mathfrak{m} = (X_1,\dots,X_n)$:

$$I^{\mathrm{sat}} = \{x : \mathfrak{m}^k x \in I \text{ for some } k\} = J^{\mathrm{sat}}$$

Equivalently, we can require that I and J define the same ideal sheaf on $\operatorname{Proj} R$. Clearly $J \subset I$ if $I = I^{\mathrm{sat}}$, and this is the case for prime ideals I like toric ideals, whence we may assume that $J = I_{(2)}$ is the ideal generated by the degree 2 elements of I.

Let v be a vertex of P, $R = K[M(P)]$ and $S = R/(X_v - 1)$ be the dehomogenization of R with respect to X_v. The presentation $K[M(P)] = A_P/I(P)$ induces a presentation

$$B_v = A_P/(X_v - 1) \to S \quad \text{with kernel} \quad I(P)B_v.$$

The residue classes of the X_x are denoted by Y_x, and B_v is again a polynomial ring in the Y_x, $x \neq v$. But S is already the image of the subalgebra $B'_v = K[Y_x : x - v \in H_v]$, $H_v = \operatorname{Hilb}(M(P)_v)$. Clearly I_v is generated by the extension of the toric ideal $I(H_v) = I(P)B_v \cap B'_v$ and any choice of polynomials $Y_z - \mu_z$ where $z - v \notin H_v$ and $\mu_z \in B'_v$ is a monomial representing $z - v$ as a $\mathbb{Z}_+$-linear combination of the Hilbert basis H_v. The smooth case is characterized by the condition $I(H_v) = 0$. This simplifies the situation considerably. For the proof of the next theorem one should note that $I(P)B_v = JB_v$ for a homogeneous ideal J in A_P if and only if $I(P)[X_v^{-1}] = J[X_v^{-1}]$.

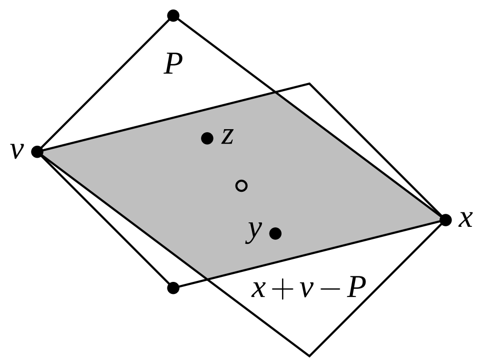
In order to prove that $I(P)$ is the saturation of $I(P)_{(2)}$ have also to check the dehomogenization with respect to indeterminates X_x for non-vertices x of P. However, as we will see, the comparison can be reduced to the consideration of vertices.

Theorem 5.1. *Let P be a smooth lattice polytope. Then the following are equivalent:*

(1) *$I(P)$ is scheme-theoretically generated in degree 2;*

(2) (a) *for all vertices v of P and all lattice points x that are not neighbors of v the polytope $P\cap(x+v-P)$ contains a lattice point $y\neq v,x$, and*

(b) *every non-vertex lattice point x of P is the midpoint of a line segment $[y,z]$, $y,z\in P\cap\mathbb{Z}^d$, $y,z\neq x$.*



Proof. For the implication (2) $\implies$ (1) we note that for every vertex and each non-neighbor x we have a binomial

$$X_vX_x - X_yX_z \in I(P)$$

by (2)(a). Thus $Y_x - Y_yY_z \in I(P)_{(2)}B_v$. As a positive monoid, $M(P)_v$ has a grading, and we can use induction on degree to find monomials μ_x in the Y_w, $w-v\in H_v$, such that $Y_x-\mu_x\in I(P)_{(2)}B_v$. In fact, $\deg(y-v),\deg(z-v)<\deg(x-v)$, so that $Y_y-\mu_y$, $Y_z-\mu_z\in I(P)_{(2)}B_v$. Then $Y_x-\mu_y\mu_z\in I(P)_{(2)}B_v$ as well, and we can set $\mu_x=\mu_y\mu_z$. This argument shows that $I(P)B_v=I(P)_{(2)}B_v$ or, equivalently, $I(P)[X_v^{-1}]=I(P)_{(2)}[X_v^{-1}]$ for all vertices v of P.

It remains to compare $I(P)$ and $I(P)_{(2)}$ after the inversion of X_p for non-vertices p. Let Q be the convex hull of all lattice points w such that X_w is a unit modulo $I(P)_{(2)}[X_p^{-1}]$. One has $Q\neq\emptyset$ since $p\in Q$. Let x be a vertex of Q. If x is a vertex of P, we can invert X_x first and use what has been shown above as a consequence of (2)(a). Suppose that x is a non-vertex of P. Then (2)(b) implies that $X_x^2-X_yX_z\in I(P)$ for suitable lattice points y,z. But only one of y,z can belong to Q, and both are units together with X_x after the inversion of X_p. This is a contradiction, and thus x is a vertex of P.

For (1) $\implies$ (2)(a) we consider the binomial $Y_x-\mu_x\in I(P)B_v$. Homogenizing and clearing denominators with respect to X_v yields a binomial

$$\beta = X_v^kX_x - X_v^p\prod X_w^{a_w}\in I(P).$$

Multiplying by a high power of X_v sends it into $(P)_{(2)}$. So we may assume that β belongs to $(P)_{(2)}$. But then $X_v^kX_x$ must be divisible by a monomial appearing in a degree 2 binomial in $I(P)$. The only potential degree 2 divisors are X_v^2 and X_vX_x. Since v is a vertex, no power X_v^m, $m>1$, can appear in a binomial γ in $I(P)$ as one of the summands (unless γ

is divisible by X_v). This implies that there exists a degree 2 binomial $X_vX_x - X_yX_z$ in $I(P)$, and y and z both belong to $P \cap (x+v-P)$ since they both belong to P.

The argument for (b) is similar. In fact, let F be the smallest face of P containing x. Since x is a non-vertex, F must contain at least one more lattice point w. Modulo $I(P)[X_x^{-1}]$ all X_w for lattice points w in F are units. Since $I(P)[X_x^{-1}] = I(P)_{(2)}[X_x^{-1}]$ by hypothesis, the same holds modulo $I(P)_{(2)}[X_x^{-1}]$. By similar arguments as above this implies the existence of a binomial

$$X_x^k - X_w \mu_w$$

in $I(P)_{(2)}$. But then we must have a nonzero binomial $X_x^2 - X_yX_z \in I(P)$. □

Remark 5.2. The implication (2) $\implies$ (1) can be generalized as follows: smoothness is replaced by the hypothesis that for each vertex v the ideal $I(H_v)$ is generated by homogeneous binomials of degree 2. In fact, *homogeneous* binomials in $I(H_v)$ lift to homogeneous binomials in $I(P)$, and we need only add the binomials $X_vX_x - X_yX_z$ and $X^2 - X_yX_z$ existing by (2) in order to find a degree 2 subideal J of $I(P)$ such that $JB_x = I(P)B_x$ for all lattice points $x \in P$.

For the implication (1) $\implies$ (2) holds for arbitrary lattice polytopes if one restricts (2)(a) to those x for which $x - v \notin H_v$. However, one cannot conclude that the ideals $I(H_v)$ are generated in degree 2. We will come back to this point in Remark 5.4.

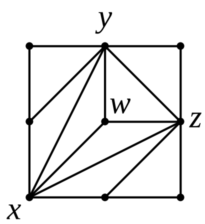
By condition (2) of the theorem, scheme-theoretic generation in degree 2 can be tested very quickly. In most cases y can be chosen as a neighbor of v for (2)(a). Thus the number of lattice points $y \in P$ to be tested is very small on average. However, the theorem very strongly indicates that finding a counterexample to scheme-theoretic generation in degree 2 is extremely difficult.

Remark 5.3. In the author's first formulation of Theorem 5.1 condition (2)(b) was missing. This mistake was pointed out by Mateusz Michalek who also suggested the inclusion of (2)(b). The problem is discussed in [18, Remark 12.2].

5.C. **Abundant degree 2 relations.** All smooth polytopes that have come up in the search for counterexamples satisfy an even stronger condition: let us say that P has *abundant degree 2 relations* if condition (2)(a) continues to hold if we replace the vertex v by an arbitrary lattice point: for all lattice points $v, x \in P$ (including the case $v = x$) the midpoint of the segment $[v,x]$ is also the midpoint of a different line segment $[y,z] \subset P$, apart from the following obvious exceptions: one of v,x is a vertex, say v, and $x = v$ or x is a neighbor of v. In terms of the toric ideal $I(P)$: it contains a binomial $X_vX_x - X_yX_z \neq 0$ for all lattice points v,x unless this is priori impossible. (This includes condition (2)(b).)

The property of having abundant degree 2 relations clearly implies scheme-theoretic generation in degree 2 for smooth polytopes, but it is not clear how it is related to generation in degree 2. For arbitrary lattice polytopes it does not follow from generation in degree 2. As an example one can take the join of two line segments with midpoints whose toric ideal is generated by $X_xX_z - X_y^2$, $X_uX_w - X_v^2$: the midpoint of $[y,v]$, both non-vertices, is not the midpoint of any other line segment since X_yX_v does not appear in the generating binomials.

One is tempted to prove that abundant degree 2 relations imply generation in degree 2 by a Gröbner basis argument. While we cannot exclude that such an argument is possible, it is very clear that its success depends on the choice of the term order. A simple example is the following polytope with the term order that induces the unimodular triangulation: the corresponding Gröbner basis contains $X_xX_yX_z - X_w^3$.



Remark 5.4. An important class of polytopes that have abundant degree 2 relations, but are not known to be quadratically defined, are given by matroids. See [24] for a rather recent result and references for this very hard problem.

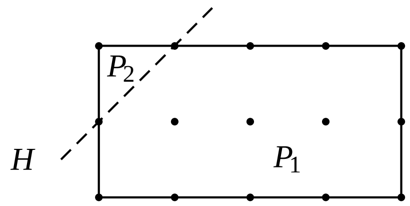
In fact, it is not difficult to see that the symmetric exchange in matroids supplies abundant degree 2 relations. So one could try to apply the generalization 5.2 of Theorem 5.1, (2) $\implies$ (1), in order to show scheme-theoretic generation in degree 2 for matroid polytopes. But this does not work since the ideal $I(H_v)$ need not be generated in degree 2, even if $I(P)$ is.

An example for this phenomenon is given by the matroid whose bases are the triples $[i_1,i_2,i_3]$ indicating the vector space bases contained in the family $u_i = e_i$, and $u_{i+3} = e_j + e_k$, $i = 1,2,3$, $1 \le j < k \le 3$, $i \ne j,k$. For the vertex $v = [1,2,3]$ the ideal $I(H_v)$ is generated by a degree 3 binomial.

5.D. **Squarefree divisor complexes.** Whether the toric ideal $I(P)$ needs generators in a certain multidegree can be tested by checking the connectivity of the squarefree divisor complex of the given degree. For example, see [8] for the terminology and the details. (According to Stanley [25, 7.9], the result goes back to Hochster.) Such a test has been implemented and is applied to a random selection of multidegrees of total degree 3 for polytopes that are too large for the approach described in 5.A. However, it seems rather hopeless to find a counterexample to quadratic generation by this test since it can only deal with a single multidegree at a time.

6. Chiseling

As pointed out above, one should expect counterexamples to be small, but "complicated". In particular, one can try to pass from a smooth polytope P to a smooth subpolytope by splitting P along a suitable hyperplane. In its simplest form this amounts to cutting corners off P as illustrated in the figure below. The operation described in the following is called *chiseling*—it sounds less cruel than “cutting faces”.



Suppose F is a face of the smooth polytope P and $\operatorname{aff}(P)$ the intersection of the support hyperplanes $H_1,\dots,H_s$ where $s = d - \dim F$. Let H_i be given by the equation $\lambda_i(x) = b_i$. Set $\sigma = \lambda_1 + \dots + \lambda_s$. Then σ has constant value $b = b_1 + \dots + b_s$ on F. Let c be the minimum value of σ on the lattice points x of P that do not lie in F, and suppose that $b < c-1$ (clearly $b < c$). In this case the hyperplane H with equation $\sigma(x) = c-1$ splits P into the polytopes

$$\begin{aligned} P_1 &= \{x : x \in P \text{ and } \sigma(x) \geq c-1\}, \\ P_2 &= \{x : x \in P \text{ and } \sigma(x) \leq c-1\}. \end{aligned} \tag{2}$$

Lemma 6.1. *P_1 and P_2 are smooth lattice polytopes of full dimension.*

This is easily seen by considering the normal fans of P_1 and P_2. The normal fan of P_1 is a stellar subdivision of $\mathcal{N}(P)$. The next theorem shows that it suffices to investigate P_1 and P_2 in the search for counterexamples. As in [6] we say that a polytope is *integrally closed* if $P\cap\mathbb{Z}^d$ generates $\mathbb{Z}^d$ affinely and the monoid ring $K[P]$ is normal, or, equivalently, $K[P]$ is integrally closed in $K[\mathbb{Z}^{d+1}]$.

Theorem 6.2. *Let P be a lattice polytope and H a rational hyperplane such that $P_1 = P\cap H^+$ and $P_2 = P\cap H^-$ are lattice polytopes.*

(1) *If P_1 and P_2 are integrally closed, then P is integrally closed.*
(2) *If P_1 and P_2 are integrally closed and the toric ideals of P_1 and P_2 are generated in degrees d_1 and d_2 respectively, then the toric ideal of P is generated in degrees $\leq \max(2, d_1, d_2)$.*

Proof. (1) is obvious.

(2) We define a weight function on the lattice points x of P (or the generators of $M(P)$) by $w(x) = |\lambda(x)|$ where λ is the primitive integral affine linear form defining H by the equation $\lambda(x) = 0$. This weight “breaks” P along H, and it breaks the monoid $M(P)$into the monoidal complex consisting of $M(P_1)$ and $M(P_2)$, glued along H. We refer the reader to [6, Chapter 7] for the terminology just used.

Let I be the toric ideal of P. The normality of P_1 and P_2 implies that $M(P)\cap H^+ = M(P_1)$ and $M(P)\cap H^- = M(P_2)$. By [6, Corollary 7.19] this is equivalent to the fact that $\operatorname{in}_w(I)$ is a radical ideal.

Therefore $\operatorname{in}_w(P)$ is the defining ideal of the monoidal complex by [6, Theorem 7.18]. On the other hand, the defining ideal of the monoidal complex is generated by the binomial toric ideals of P_1 and P_2 and the monomial ideal representing the subdivision of P along H [6, Proposition 7.12]. The latter is of degree 2 since a monomial with support not in P_1 or P_2 must have a factor of degree 2 with this property (as is always the case by subdivisions along hyperplane arrangements). This shows that $\mathrm{in}_w(I)$ is generated in degrees $\le \max(2,d_1,d_2)$ and it follows that I itself is generated in degrees $\le \max(2,d_1,d_2)$. $\square$

For a special case, Theorem 6.2 is contained in an unpublished manuscript of Gubeladze and Hoşten, but with a proof using squarefree divisor complexes.

The following counterexample shows that part (1) of the theorem cannot be reversed, and that part (2) does no longer hold if one omits the assumption that P_1 and P_2 are integrally closed: set

$$x=(0,0,0),\qquad y=(1,0,0),\qquad z=(0,1,0),$$
$$v=(1,1,2),\qquad w=(0,0,-1),$$

and let P be the polytope spanned by them. One can easily check that the given points are the only lattice points of P since P is the union of the simplices $P_1=\mathrm{conv}(x,y,z,v)$ and $P_2=\mathrm{conv}(x,y,z,w)$. The toric ideals of P_1 and P_2 are both 0, but $I(P)$ is generated by the binomial $X_xX_yX_z-X_vX_w^2$. Condition (a) is violated since the lattice points in P_1 do not span $\mathbb{Z}^2$ as an affine lattice.

Let us say that a lattice polytope is *robust* if it cannot be chiseled into two lattice subpolytopes of the same dimension along a hyperplane H, as described by (2). The robust smooth polytopes P can be characterized as follows: from each face F of P there emanates an edge of length 1.

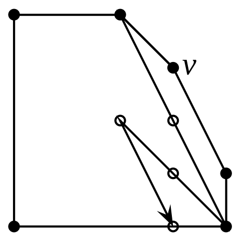
Corollary 6.3. (1) *In every dimension, a minimal counterexample to the normality question is robust.*
(2) *In every dimension, a normal counterexample to generation in degree* 2 *is robust.*

Chiseling has been used in two ways: (1) to reduce the size of polytopes that had been computed from unimodular fans, and (2) to produce polytopes from large smooth polytopes by chiseling them parallel to faces chosen in some random order.

Remark 6.4. In our search, we only investigate P_1 further, although we cannot exclude that "bad" properties of P come from P_2. Neglecting P_2 is however justified if the face F is a vertex. In this case P_2 is a multiple of the unit simplex and $I(P_2)$ is generated in degree 2 since $K[M(P_2)]$ is a Veronese subalgebra of a polynomial ring.

7. Miscellaneous properties of smooth polytopes

7.A. **Superconnectivity and strong connectivity.** Let v be a vertex of P. A Hilb_v-*path* is a sequence of lattice points $v=x_0,x_1,\dots,x_m$ in P such that $x_{i+1}-x_i\in \mathrm{Hilb}(M(P)_v)$ for all $i=0,\dots,m-1$. We say that P is *superconnected* if every lattice point in P is connected to every vertex v by a Hilb_v-path. However, superconnectivity is rarely satisfied, and even in dimension 2 one easily finds counterexamples. Consider the smooth polygon with vertices $(0,0)$, $(4,0)$, $(4,1)$, $(3,3)$, $(2,4)$, and $(0,4)$:



The lattice point $(3,0)$ is reachable by a Hilb_v-path from $v=(3,3)$, but the origin is not.

On the other hand, every smooth polytope encountered in the search satisfies the following weaker condition: for each lattice point x there exists at least one vertex v to which x is connected by a Hilb_v-path. We call such polytopes *strongly connected*. If strong connectivity should fail for a smooth polytope P, then P has a lattice point $x \notin \bigcup_v U_v(P)$, and in particular $P \neq \bigcup_v U_v(P)$ (compare 4.B).

Superconnectivity of P is equivalent to the following condition: the lattice point y in condition (2)(a) of Theorem 5.1 can always be chosen among the neighbors of v. Therefore superconnectivity can be considered to be a strong form scheme-theoretic generation in degree 2 for *smooth* polytopes without non-vertex lattice points.

The matroid polytopes discussed briefly in Remark 5.4 are superconnected since the symmetric exchange of single elements produces neighbors.

7.B. **Positivity of coefficients of the Ehrhart polynomial.** Since we are computing the Ehrhart series of smooth polytopes anyway, checking the Ehrhart polynomial for positivity of its coefficients costs no extra time. In fact, for all smooth polytopes found in our search these coefficients proved to be positive. Therefore it seems reasonable to ask the following question:

Question 7.1. Do the Ehrhart polynomials of smooth polytopes have positive coefficients?

See De Loera, Haws and Köppe [13] for a discussion of the positivity question for another class of polytopes.

8. Very ample polytopes

Smooth polytopes can be considered as special instances of the following class:

(HC) for each vertex v of the (simple) polytope P the points x such that $x-v \in \mathrm{Hilb}(M(P)_v)$ are contained in the polytope spanned by v and its neighbors.

Such polytopes P are automatically very ample, provided the lattice points in P span $\mathbb{Z}^d$ affinely. (As usual, a polytope is *simple* if exactly d hyperplanes meet in each of its vertices.)

Generalizing question (N), one may ask whether polytopes of class (HC) are normal. As the counterexample below shows, the answer is “no” for non-simple polytopes, but seems to be unknown for simple ones. In fact, we do not know of any simple very ample, but non-normal polytope.

The question (HC) fits into a line of research that relates normality of polytopes to the length of their edges; see Gubeladze [15] for an edge length bound guaranteeing normality.

8.A. **Very ample non-normal polytopes.** The first example of a very ample, but non-normal lattice polytope was given in [4]. Its vertices are the 0-1-vectors representing the triangles of the minimal triangulation of the real projective plane. The polytope has dimension 5.

Very ample non-normal polytopes can easily be found by *shrinking*. One starts from a normal polytope P, chooses a vertex v, and checks whether the polytope Q spanned by $(P \cap \mathbb{Z}^d) \setminus \{v\}$ is very ample. If so, P is replaced by Q. If not, we test another vertex of P. If no vertex of P can be removed without violating very ampleness for Q, one stops at P. The very ample polytopes encountered in the process are checked for normality, and surprisingly often non-normal very ample polytopes pop up, and even smooth ones do. (This technique was originally applied to find normal polytopes without unimodular cover; see [5].)

By shrinking we found the following polytope: $P \subset \mathbb{R}^3$ is the convex hull of

$$\big((0,0) \times I_1\big) \cup \big((0,1) \times I_2\big) \cup \big((1,1) \times I_3\big) \cup \big((1,0) \times I_4\big).$$

with $I_1 = \{0,1\}$, $I_2 = \{2,3\}$, $I_3 = \{1,2\}$, $I_4 = \{3,4\}$ (see [6, Exerc. 2.24]). This polytope has 4 unimodular corner cones and 4 non-simple ones. One can check by hand that at each vertex v the vectors $w - v$, w a neighbor of v, form $\operatorname{Hilb}(M(P)_v)$. This example has recently been generalized by Ogata [23] in several ways. Very ampleness for these polytopes can be proved by applying the following criterion to each of the non-unimodular corner cones:

Proposition 8.1. *Let C be a rational cone of dimension d generated by $d+1$ vectors $w_1, \dots, w_{d+1}$. For each facet F of C suppose that the $w_i \in F$ together with one of the (at most two) remaining ones generate $\mathbb{Z}^d$. Then $w_1, \dots, w_{d+1}$ are the Hilbert basis of C.*

Proof. The hypothesis guarantees that $w_1, \dots, w_{d+1}$ generate $\mathbb{Z}^d$. Moreover, the monoid $C \cap \mathbb{Z}^d$ is integral over the monoid M generated by the w_i. Therefore it is enough to show that M is normal. The hypothesis on the generation of $\mathbb{Z}^d$ by the $w_i \in F$ and *one* additional vector implies that the monoid algebra $K[M]$ satisfies Serre's condition (R_1) (compare [6, Exerc. 4.16]). Serre's condition (S_2) is satisfied since an affine domain of dimension d generated by $d+1$ elements is Cohen-Macaulay. Normality is equivalent to (R_1) and (S_2). □

8.B. **The search for very ample simple polytopes.** Finding random simple polytopes has turned out as difficult as finding random smooth polytopes. Constructing such polytopes from simplicial fans follows the algorithm outlined in Section 3, except that one does not refine a simplicial cone to a unimodular one. However, the property (LP) now comes into play, and the "right hand sides"' b that yield lattice polytopes (and not just rational ones) form a proper sublattice $\mathscr{C}$ of $\mathscr{W}$. It is not hard to describe $\mathscr{C}$ by congruences that its members must satisfy. However, often the Hilbert basis computation is arithmetically much more complicated than for unimodular fans, and the way out described in Remark 3.2 does not work well.

Despite of the fact that simple lattice polytopes are usually not normal, those that we have obtained from fans have all been normal. This fact reveals the most problematic aspect of our search: creating polytopes from random simplicial or even smooth fans seems to produce only harmless examples since one cannot reach the complication, arithmetically or combinatorially, that $\mathscr{N}(P)$ needs for P to be non-normal. It should be much more promising to define polytopes in terms of their vertices.

Simplices are the only class of simple polytopes that can easily be produced by choosing vertices at random. In dimension ≥ 3 they are usually non-normal, but we have not yet been able to find a very ample such simplex. The only result known to us that indicates that simplices are special in regard to very ampleness is a theorem of Ogata [20]: if P is a very ample simplex of dimension d, then the multiples cP are normal for $c \geq n/2-1$. In particular, very ample 3-simplices are normal.

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