Initial algebras of determinantal rings, Cohen-Macaulay and Ulrich ideals

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1 Determinantal ideals and rings

K is a field,

X an $m \times n$ matrix of indeterminates over K.

$K[X]$ is the polynomial ring generated by all the X_{ij}.

Let $r \leq \min\{m,n\}$

$I_{r+1} = I_{r+1}(X)$ is the determinantal ideal generated by all $(r+1)$-minors of X if $r < \min\{m,n\}$ (and $I_{r+1} = (0)$ otherwise).

Let $R_{r+1} = R_{r+1}(X)$ be the determinantal ring $K[X]/I_{r+1}$.

2 Standard bitableaux

Approach: consider all minors of the matrix X as generators for the K-algebra $K[X]$ (not only the 1-minors X_{ij}).

Advantage: products of minors appear as “monomials”; the $I_{r+1}(X)$ are generated by “indeterminates”.

Disadvantage: these “monomials” are not linearly independent.

Way out: find a combinatorially manageable subset of these “monomials” that form a K-basis of $K[X]$.

This subset consists of the standard bitableaux.

For $1 \le t \le \min\{m,n\}$ let

$$[a_1 \dots a_t | b_1 \dots b_t] = \det(X_{a_i b_j} : i = 1, \dots, t,\ j = 1, \dots, t).$$

We require that $1 \le a_1 < \cdots < a_t \le m$ and $1 \le b_1 < \cdots < b_t \le n$.

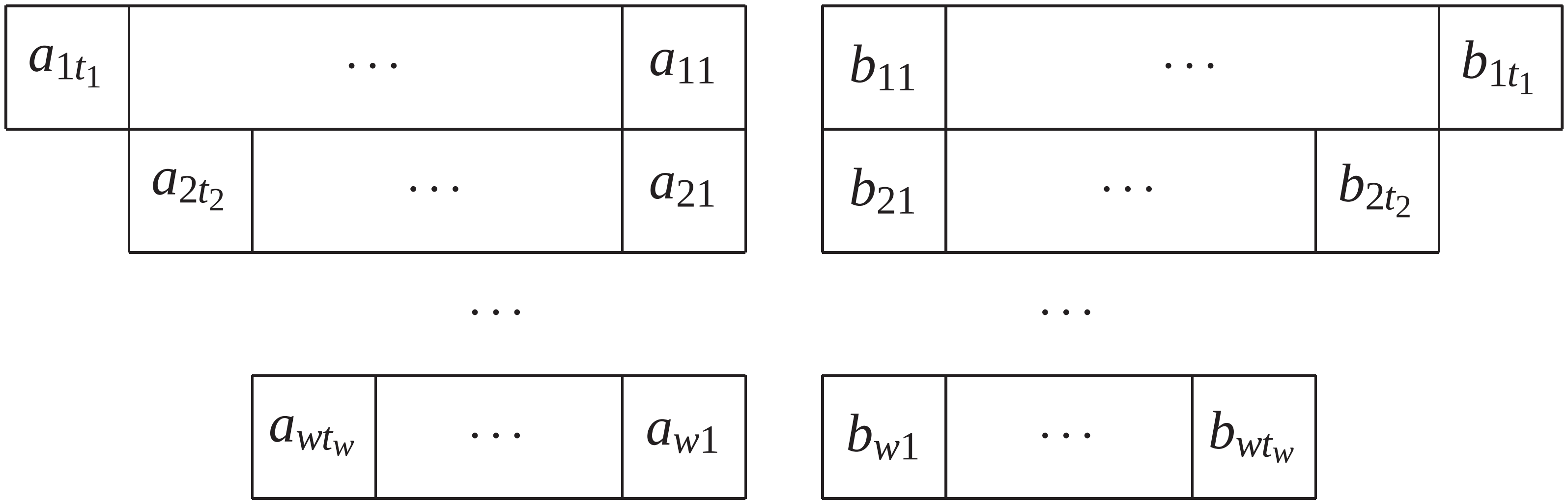
$[a_1 \dots a_t | b_1 \dots b_t]$ minor of X.

A bitableau Δ is a product of minors

$$\prod_{i=1}^{w} [a_{i1} \dots a_{it_i} | b_{i1} \dots b_{it_i}] \qquad \text{such that } t_1 \ge \cdots \ge t_w.$$

We write $\Delta = (a_{ij} \,|\, b_{ij})$.

Graphical description of Δ as a pair of Young tableaux:



Explains the name bitableau.

We consider a partial order on the set of all bitableau:

$$[a_1 \dots a_t | b_1 \dots b_t] \preceq [c_1 \dots c_u | d_1 \dots d_u]$$
$$\iff \quad t \geq u \quad \text{and} \quad a_i \leq c_i, \;\; b_i \leq d_i, \;\; i = 1, \dots, u.$$

A product $\Delta = \delta_1 \cdots \delta_w$ of minors $\delta_i = [a_{i1} \dots a_{it_i} | b_{i1} \dots b_{it_i}]$ is a standard bitableau if

$$\delta_1 \preceq \cdots \preceq \delta_w,$$

i. e. in each "column" of the bitableau the indices are non-decreasing from top to the bottom.

Theorem 2.1 (Hodge). *The (residue classes of the) standard bitableaux $\Sigma \in \mathscr{S}_r$ generate R_{r+1} as a vector space over K.*

3 Generic points and initial algebra

The classical "generic point" for R_{r+1} is the homomorphism

$$\varphi : R_{r+1} \to K[Y,Z]$$

where Y is an $m \times r$, Z is an $r \times n$ matrix of indeterminates, and

$$\varphi : X_{ij} \mapsto (YZ)_{ij}$$

φfactors through R_{r+1} since $\operatorname{rank}(YZ) = r$.

On $K[Y,Z]$ we introduce degree reverse lexicographic order induced by

$Y_{m1} > Y_{m-11} > \cdots > Y_{11} > Y_{m2} > \cdots > Y_{1r} > Z_{1n} > \cdots > Z_{11} > Z_{2n} > \cdots > Z_{r1}$.

Restrictions of the term orders to $K[Y]$ and $K[Z]$ are diagonal: the initial term of a minor of Y or Z is the product of its main diagonal elements.

Lemma 3.1. *Let $1 \le t \le r$. Then*

$$\begin{aligned}\operatorname{ini}\Big([a_1 \dots a_t \,|\, b_1 \dots b_t]_{YZ}\Big)& \\ &= \operatorname{ini}\Big([a_1 \dots a_t \,|\, 1 \dots t]_Y\Big) \cdot \operatorname{ini}\Big([1 \dots t \,|\, b_1 \dots b_t]_Z\Big) \\ &= Y_{a_1 1} \cdots Y_{a_t t} \cdot Z_{1 b_1} \cdots Z_{t b_t}\end{aligned}$$

$\Longrightarrow$ initial monomials of standard bitableaux are pairwise different.

Corollary 3.2. *Let $K[YZ]$ denote the K-algebra generated by the entries of the product matrix YZ.*

(a) *The homomorphism $\varphi : R_{r+1} \to K[YZ]$ is an isomorphism.*

(b) *The standard bitableaux $\Sigma \in \mathscr{S}_r$ form a K-basis of R_{r+1}.*

Theorem 3.3.

(a) *The initial algebra* $D_{r+1} = \mathrm{ini}(R_{r+1}) \subset K[Y,Z]$ *is generated by the monomials* $Y_{a_1 1} \cdots Y_{a_t t} Z_{1 b_1} \cdots Z_{t b_t}$ *with* $1 \le t \le r$*,* $a_1 < \cdots < a_t$ *and* $b_1 < \cdots < b_t$*.*

(b) D_{r+1} *is a normal semigroup ring.*

(c) R_{r+1} *is a normal domain, Cohen-Macaulay, with rational singularities in characteristic* 0*, and* F*-rational in characteristic* $p > 0$*.*

(c) is well-known (Hochster-Eagon,...)

4 Cohen-Macaulay and Ulrich ideals

Suppose that $1 \leq r < \min\{m,n\}$ $(I_{r+1}(X) \neq 0)$

$\mathfrak{p} =$ ideal generated by the r-minors of the first r rows,

$\mathfrak{q} =$ ideal generated by the r-minors of the first r columns.

$\mathrm{Cl}(R) \cong \mathbb{Z}$, generated by $[\mathfrak{p}]$, and $[\mathfrak{p}] = -[\mathfrak{q}]$.

Moreover, $\mathfrak{p}^{(j)} = \mathfrak{p}^j$, $\mathfrak{q}^{(j)} = \mathfrak{q}^j$ for all j.

J reflexive rank 1 module $\implies$ $J \cong \mathfrak{p}^j$ or $J \cong \mathfrak{q}^j$ for some j.

In particular: J Cohen-Macaulay rank 1 module $\implies$ $J \cong \mathfrak{p}^j$ or $J \cong \mathfrak{q}^j$ for some j.

M Cohen-Macaulay $\implies \mu(M) \leq \text{rank}(M) \cdot e(R)$

M Ulrich module $\iff M$ Cohen-Macaulay and $\mu(M) = \text{rank}(M) \cdot e(R)$.

$e(R)$ is known and the $\mu(\mathfrak{p}^j)$ and $\mu(\mathfrak{q}^j)$ can be computed (Hodge postulation formula).

Theorem 4.1.

$\mathfrak{p}^t$ *(resp.* $\mathfrak{q}^t$*) Cohen-Macaulay ideal* $\iff t \leq m-r$ *(resp.* $t \leq n-r$*).*

$\mathfrak{p}^{m-r}$ *and* $\mathfrak{q}^{n-r}$ *are Ulrich ideals.*

Proof. (i) $\mu(\dots) > e(R)$ outside the range given for t.

(ii) $\mathrm{ini}(\mathfrak{p}^t) \subset \mathrm{ini}(R_{r+1})$ is a "conic" ideal for $t = 1, \dots, m-r$ and therefore Cohen-Macaulay

$\implies \mathfrak{p}^t$ Cohen-Macaulay.

Similarly for $\mathfrak{q}^t$, $1 \leq t \leq n-r$.

(iii) $\mu(\mathfrak{p}^{m-r}) = \mu(\mathfrak{q}^{n-r}) = e(R)$.

$\square$