Covering properties of affine monoids

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Berlin, 16.07.07

Affine monoids

Definition

An *affine monoid* M is (isomorphic to) a finitely generated submonoid of $\mathbb{Z}^d$ for some $d \geq 0$, i. e.

- $M + M \subset M$ (M is a semigroup);
- $0 \in M$ (now M is a monoid);
- there exist $x_1, \dots, x_n \in M$ such that $M = \mathbb{Z}_+ x_1 + \cdots + \mathbb{Z}_+ x_n$ (M is a finitely generated)

Often affine monoids are called affine semigroups.

$\mathrm{gp}(M) = \mathbb{Z}M$ is the group generated by M.

$\mathrm{gp}(M) \cong \mathbb{Z}^r$ for some $r = \operatorname{rank} M = \operatorname{rank} \mathrm{gp}(M)$.

Normality

The properties of affine monoids to be discussed imply normality.

Definition

A monoid M is integhrally closed in an overmonoid N $\iff$

$$x \in N,\ kx \in M \text{ for some } k \in \mathbb{Z}, k > 0 \quad \Rightarrow \quad x \in M.$$

M is normal if it is integrally closed in $\operatorname{gp}(M)$.

The simplest example of a normal affine monoid:

Definition

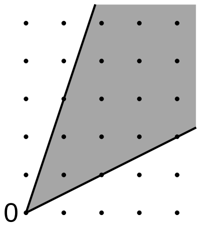
An affine monoid is free if it is generated by linearly independent vectors.

Theorem

$M \subset \mathbb{Z}^d$ is a normal affine monoid $\iff$ $M = C \cap \mathrm{gp}(M)$ where $C = \mathbb{R}_+ M$ is the cone generated by M.

Briefly: **Normal affine monoids are discrete cones.**

In general, $C \cap \mathrm{gp}(M)$ is the normalization of M. It is also an affine monoid by Gordan's lemma.



Positivity and Hilbert bases

Definition

A monoid M is positive if $x, -x \in M \Rightarrow x = 0$.

Proposition

For M affine the following are equivalent:

1. *M is positive;*
2. *$\mathbb{R}_+ M$ is pointed (i. e. contains no full line);*
3. *M is isomorphic to a submonoid of $\mathbb{Z}_+^s$ for some s.*

$x \in M$ is irreducible if $x = y + z \Rightarrow y = x$ or $z = x$.

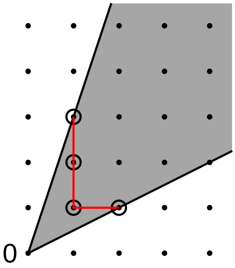
Definition

The set of irreducible elements is called the Hilbert basis of M, denoted by $\operatorname{Hilb}(M)$.

Proposition

$\mathrm{Hilb}(M)$ is the unique minimal system of generators of M. In particular $\mathrm{Hilb}(M)$ is finite.

Typical picture for normal monoids of rank 2: $\mathrm{Hilb}(M)$ is the set of lattice points in the bottom of M.



In higher dimension is situation is complicated.

Question. Can one understand normality in terms of simple combinatorial conditions for $\mathrm{Hilb}(M)$?

Checking isomorphism

The cone $C = \mathbb{R}_+ M \subset \mathbb{R}^d$, $d = \dim C = \operatorname{rank} M$, has finitely many support hyperplanes H_i, $i = 1, \dots, s$, given by uniquely determined linear forms $\sigma_i(x) = \alpha_{i1}\xi_1 + \cdots + \alpha_{id}\xi_d$ such that

- $\sigma_i(x) \geq 0$ for $x \in C$,
- $\alpha_{i1}, \dots, \alpha_{id}$ are coprime integers.

With these data we let $\operatorname{Val}(M)$ denote the $N \times s$ matrix, $N = \#\operatorname{Hilb}(M)$, with entries $\alpha_j(x_i)$, $i = 1, \dots, N$, $j = 1, \dots, s$, $\operatorname{Hilb}(M) = \{x_1, \dots, x_N\}$.

Proposition

Let M and N be positive affine monoids. Then $M \cong N \iff \operatorname{Val}(M)$ and $\operatorname{Val}(N)$ coincide up to permutations of rows and columns.

Covering properties

A submonoid $N \subset M$ is unimodular if it is free, and $\mathrm{gp}(N) = \mathrm{gp}(M)$.

Definition

A positive affine monoid has

(UHC) $\iff$ M is the union of its unimodular submonoids generated by elements of $\mathrm{Hilb}(M)$ (Unimodular Hilbert Cover);

(FHC) $\iff$ M is the union of its free submonoids generated by elements of $\mathrm{Hilb}(M)$ (Free Hilbert Cover);

(ICP) $\iff$ for every $x \in M$ there exist $x_1, \dots, x_d \in \mathrm{Hilb}(M)$, $d = \operatorname{rank} M$, such that $x = a_1x_1 + \cdots + a_dx_d$, $a_i \in \mathbb{Z}_+$ (Integral Carathéodory Property).

Clearly: (UHC) $\Rightarrow$ (FHC) $\Rightarrow$ (ICP).

Note: these properties depend only on the isomorphism type of M.

Proposition

(UHC) $\Rightarrow$ M is normal.

Evident, since M is the union of normal monoids with the *same group of differences*. However, the groups of differences may vary:

Theorem (B.-Gubeladze)

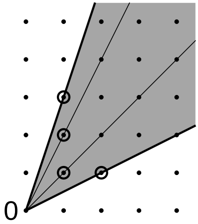
(FHC) $\iff$ (ICP) $\Rightarrow$ M is normal.

The equivalence of (ICP) and (FHC) is of algorithmic importance.

Low dimensions

Proposition

M normal, rank $M \leq 2 \Rightarrow \mathbb{R}_+M$ *has a (unique) triangulation by unimodular Hilbert subcones.*



Theorem (Sebő)

M normal, rank $M = 3 \Rightarrow \mathbb{R}_+M$ *has a triangulation by unimodular Hilbert subcones. In particular, it has (UHC).*

Question. Does every normal M have (uHC), or at least (ICP)?

Counterexamples

The monoid $M_{6,10}$ of rank 6 generated by

$$
\begin{aligned}
z_1 &= (0, 1, 0, 0, 0, 0), & z_6 &= (1, 0, 2, 1, 1, 2),\\
z_2 &= (0, 0, 1, 0, 0, 0), & z_7 &= (1, 2, 0, 2, 1, 1),\\
z_3 &= (0, 0, 0, 1, 0, 0), & z_8 &= (1, 1, 2, 0, 2, 1),\\
z_4 &= (0, 0, 0, 0, 1, 0), & z_9 &= (1, 1, 1, 2, 0, 2),\\
z_5 &= (0, 0, 0, 0, 0, 1), & z_{10} &= (1, 2, 1, 1, 2, 0).
\end{aligned}
$$

is a counterexample to (UHC) and (ICP). Found by B. and Gubeladze in May 1998 as a counterexample to (UHC), shown to fail (ICP) by Henk, Martin, and Weismantel.

Interesting observations

- $M_{6,10}$ is polytopal (i. e. the elements of Hilb lie in a hyperplane. Santos: The underlying polytope is a projection of the Hibi-Ohsugi polytope.
- Aut $M_{6,10}$ has order 20. It acts transitively on Hilb$(M_{6,10}$ and is isomorphic to the Frobenius group of order 20.

The search for further counterexamples (1998 – 2001) yielded only one further monoid $M_{6,12}$ with a Hilbert basis of 12 elements. It, too, violates (ICP). Actually $M_{6,12}$ appeared only once whereas $M_{6,10}$ returned over and over again.

Question. (ICP) $\Rightarrow$ (UHC) ?

The search strategy

Definition

Let M be a normal affine monoid, $H = \operatorname{Hilb}(M)$. $x \in H$ is destructive if (i) $M'\mathbb{Z}_+(H \setminus \{x\})$ is not normal or (ii)$\operatorname{gp}(M') \neq \operatorname{gp}(M)$.

M is tight if every $x \in H$ is destructive.

Lemma

M is a minimal counterexample to (ICP) or (UHC) with respect to (i) dimension, (ii) $\#\operatorname{Hilb}(M)$
$\Rightarrow M$ tight.

The search algorithm

(1) Generate a “random” M, and replace it by its normalization.
(2) Shrink M by successively removing nondestructive elements from the Hilbert basis.
(3) If the tight M' produced by (2) is not free, check whether (UHC) holds. If so, stop. Otherwise go to (1).

Both $M_{6,12}$ to $M_{6,10}$ are tight.

July 2006: Compare $M_{6,12}$ to $M_{6,10}$ . . .

Heureka: $\mathrm{Hilb}(M_{6,12}) \supset \mathrm{Hilb}(M_{6,10})$, there exist interesting objects in the vicinity of $M_{6,10}$.

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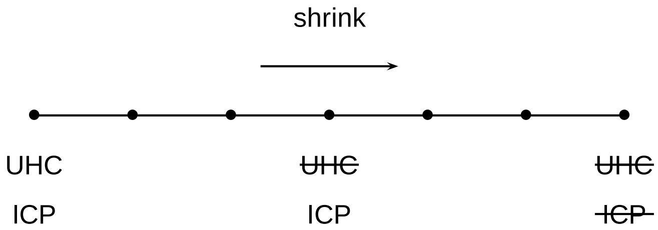
Idea: along a shrink path ending in $M_{6,10}$ (or another counterexample), the stronger property (UHC) should be lost before the weaker property (ICP), at least sometimes:



The idea works, hopefully . . .

An example with (ICP), but violating (UHC)

Extend $\operatorname{Hilb}(M_{6,10})$ by the vectors

$$z'_{11} = (0, -1, 2, -1, -1, 2) \qquad z'_{12} = (1, 0, 3, 0, 0, 3)$$

to the Hilbert basis of $M'_{6,12}$. Then $M'_{6,12}$ is

- tight,
- has (ICP),
- fails (UHC).

Surprising fact: Aut $M'_{6,12}$ is again the Frobenius group of order 20, but the action does not extend that on $M_{6,10}$.

Algorithms for (UHC) and (ICP)

(UHC) is a purely geometric property of the cone $C = \mathbb{R}_+ M$ (of course, the lattice structure has to be fixed).

$D \subset C$ is a *u*-subcone if it is of type $D = \mathbb{R}_+ N$ for a unimodular submonoid N of M. Then

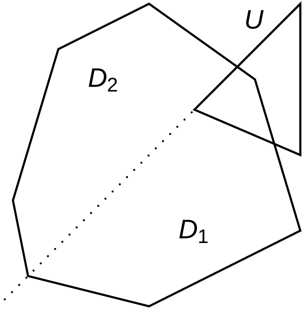
$$\text{(UHC)} \iff C \text{ is the union of its } u\text{-subcones.}$$

The algoritm `UNICOVER` decides whether (UHC) holds.

```
UNICOVER(D, n)
  for i ← n to N
  do
        if int(D) ∩ int(U_i) = ∅
           then continue
        if D ⊂ U_i
           then return
        (D_1, D_2) ← SPLIT(D, U_i)
        UNICOVER(D_1, i)
        UNICOVER(D_2, i)
  OUTPUT( D not u-covered )
  return
```

```
MAIN()
  Create the list U_1, ..., U_N of u-subcones of C
  UNICOVER(C, 1)
```

The function `SPLIT` divides the cone D into the two subcones determined by a support hyperplane of the u-subcone U_i intersecting the interior of D:



For (ICP) we say that $S \subset C$ is an f-subcone if it is generated by a linearly independent subset $x_1, \dots, x_d$ of $\mathrm{Hilb}(M)$. We set

$$Gamma(S) = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_d.$$

In contrast to (UHC), (ICP) is *not a geometric* property: C is always the union of its f-subcones.

In the following algorithm the basic object is a triple $(D, G, \mathscr{R})$:

- D is a subcone of C,
- G is a subgroup of $\mathrm{gp}(M)$,
- $\mathscr{R}$ is a set of residue classes of $\mathrm{gp}(M)/G$.

$\text{CARADEC}(D, G, \mathscr{R}, n)$

for $i \leftarrow n$ **to** N
do if $D \subset S_i$
    then $\mathscr{R}' \leftarrow \emptyset$
        for $y \in G/(G \cap \Gamma(S_i)),\quad x \in \mathscr{R}$
        do if $x + y \notin \Gamma(S_i)$
            then $\mathscr{R}' = \mathscr{R}' \cup \{x + y\}$
        $G \leftarrow G \cap \Gamma(S_i) \quad \mathscr{R} \leftarrow \mathscr{R}'$
        if $\mathscr{R} = \emptyset$
          then return
  if $D \not\subset S_i$ **and** $\operatorname{int}(D) \cap \operatorname{int}(S_i) \neq \emptyset$
    then $(D_1, D2) \leftarrow \text{SPLIT}(D, S_i)$
        $\text{CARADEC}(D_j, G, \mathscr{R}, i) \quad j = 1, 2$
$\text{OUTPUT}($ D not f-covered $)$

$\text{MAIN}()$

Create the list $S_1, \ldots, S_N$ of f-subcones of C
$\text{CARADEC}(C, \mathbb{Z}^d, \{0\}, 1)$

The correctness of the algorithm depends on the following lemma:

Lemma

For every data set $(D, G, \mathscr{R}, n)$ *to which* CARADEC *is applied, one has*

- $G = \bigcap_{i \in U(D,n)} \Gamma(S_i,$
- $\mathscr{R} = \left\{\overline{x} \in \mathrm{gp}(M)/G : x \notin \bigcup_{i \in U(D,n)} \Gamma(S_i)\right\},$

where $U(D,n) = \{i : i < n : S_i \supset D\}$.

For efficiency, one should use the f-subcones in increasing order of $[\mathrm{gp}(M) : \Gamma(S_i)]$.

Note: $[\mathrm{gp}(M) : \Gamma(S_i)] = 1 \iff S_i$ is a u-subcone.

An observation and questions

Observation: all tight counterexamples to (UHC) (with or without (ICP)) , found in a long series of computations (16 in dimension 6, 59 in dimension 7, 10 in dimension 8) contain (in a sense) $M_{6,10}$.

Question 1. Is $M_{6,10}$ the unique minimal counterexample to (ICP) and (UHC) ?

Question 2. Does every normal monoid in dimensions 4 and 5 have (UHC) ?