Binomial regular sequences and free sums

Winfried Bruns

FB Mathematik/Informatik
Universität Osnabrück

wbruns@uos.de

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Ehrhart series

$P \subset \mathbb{R}^d$, rational polytope.

The Ehrhart series of P is:

$$E_P = E_P(T) = \sum_{k=0} E(P,k)T^k, \qquad E(P,k) = \#(kP \cap \mathbb{Z}^d).$$

It is the power series expansion of a rational function.

If P is integral, then

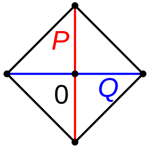
$$E_P = \frac{1 + h_1 T + \cdots + h_s T^s}{(1-T)^{d+1}}, \qquad h_i \in \mathbb{Z}_+,\ d = \dim P \geq s.$$

P Gorenstein $\iff (1, \dots, h_s)$ is palindromic

$P \subset \mathbb{R}^d$, $Q \subset \mathbb{R}^e$ rational polytopes, $0 \in P$, $0 \in Q$

Definition

$R = \text{conv}\big((P \times 0) \cup (0 \times Q)\big) \subset \mathbb{R}^{d+e}$ is the *free sum* of P and Q.



Question (answered by Beck-Jayawant-McAllister)

When do we have

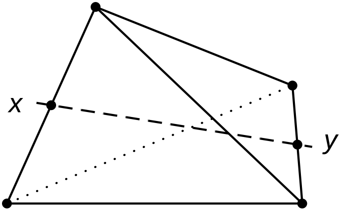
$$E_{P \oplus Q} = (1 - T)E_P E_Q \ ?$$

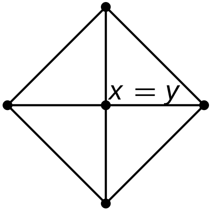
Our goal: to understand the combinatorial theorem via the underlying algebraic structure.

Free join and free sum

$P \vee Q = \operatorname{conv}\big((P \times 0 \times 0) \cup (0 \times Q \times 1)\big)\mathbb{R}^{d+e+1}$ is the free join of P and Q Well-known and easy: $E_{P\vee Q} = E_P E_Q$

$P \oplus Q$ is the image of $P \vee Q$ under the projection along the line joining $x = 0 \times 0 \times 0$ and $y = 0 \times 0 \times 1$:





Algebraically: we pass from a certain monoid (algebra) to a quotient.

Affine monoids and normality

An affine monoid M is a finitely generated submonoid of a lattice $\mathbb{Z}^d$. K a field. Then

$$K[M] = \bigoplus_{x \in M} KX^x, \qquad X^x X^y = X^{x+y}.$$

M affine $\Rightarrow K[M]$ noetherian integral domain

M is normal if $M = \{x \in \mathrm{gp}(M) : kx \in M \text{ for some } k > 0\}$
$\iff M = \mathrm{cone}(M) \cap \mathrm{gp}(M)$
$\iff K[M]$ is a normal domain

Ehrhart series as Hilbert series

$$\mathscr{E}_P = \{(x, k) : x \in kP \cap \mathbb{Z}^p\} \subset \mathbb{Z}^{d+1}.$$

is the Ehrhart monoid of P. It is affine and normal. The Ehrhart series of P is the Hilbert series of $K[\mathscr{E}_P]$:

$$E_P = H_{K[\mathscr{E}_P]} = \sum_{k=0}^{\infty} (\dim K[\mathscr{E}_P]_k) T^k.$$

The initial question: $E_{P \oplus Q} = (1-T) E_P E_Q$?

With $S = K[\mathscr{E}_P \oplus \mathscr{E}_Q]$ We have $H_S = E_P E_Q = E_{P \vee Q}$ and

$$(1-T) E_P E_Q = (1-T) H_S = H_{S/(X^x - X^y)}$$

where the monomials X^x, X^y are the monomials representing the lattice points in $P \vee Q$ that are identified in $P \oplus Q$.

Structural reformulation

We have a natural homomorphism $\mathscr{E}_P \oplus \mathscr{E}_Q \to \mathscr{E}_{P\oplus Q}$. It induces a natural homomorphism $\varphi : K[\mathscr{E}_P \oplus \mathscr{E}_P] \to K[\mathscr{E}_{P\oplus Q}]$. Clearly $\varphi(X^x) = \varphi(X^y)$.

Therefore the question on Ehrhart series amounts to

- When is $K[\mathscr{E}_P \oplus \mathscr{E}_Q]/(X^x - X^y) \cong K[\mathscr{E}_{P\oplus Q}]$?

And it reduces to:

- When is $K[\mathscr{E}_P \oplus \mathscr{E}_Q]/(X^x - X^y)$ a normal domain ?

These questions can be formulated entirely in terms of affine monoids and congruences, but it is useful (and more convenient) to phrase them in terms of monoid algebras.

Generalization

More generally:

Question

M (normal) affine monoid

- *When is $K[M]/(X^x - X^y)$ a (normal) domain ?*
- *When is $K[M]/(X^x - X^y)/(X^{x_1} - X^{x_2}, \dots, X^{x_{n-1}} - X^{x_n})$ a (normal) domain ?*

Note $K[M]/(X^x - X^y)$ is itself an affine monoid domain if $(X^x - X^y)$ is prime:

$$K[M]/(X^x - X^y) = K[M']$$

where M' is the image of M in $K[M]/(X^x - X^y)$.

Integrality

Theorem

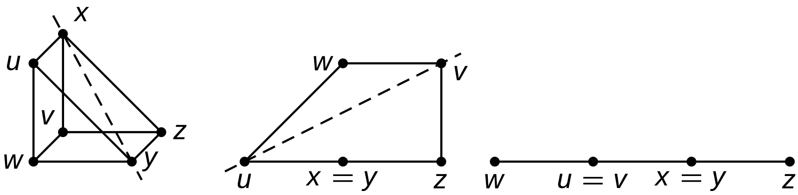
Let K be a field, M an affine monoid, and $x, y \in M$ noninvertible, $x \neq y$.
Then the following statements (1) and (2) are equivalent:

1. $X^x - X^y$ *generates a prime ideal in* $K[M]$.
2. (a) X^x, X^y *is a* $K[M]$*-sequence (*X^y *is not a zero-divisor modulo* X^x*);*
   (b) $\operatorname{gp}(M)/\mathbb{Z}(x-y)$ *is torsionfree.*

(2)(a) $\iff$ no monomial is a zero divisor modulo $X^x - X^y$
$\iff (X^x - X^y) = (X^x - X^y)K[\operatorname{gp}(M)] \cap K[M]$

(2)(b) $\iff (X^x - X^y)K[\operatorname{gp}(M)]$ is a prime ideal.

An example where the theorem can be applied successively:



We take the monoids defined by the point configurations A:
$M = M(A) = \mathbb{Z}_+(A \times 1)$.

In general difficult to check: (2)(a) X^x, X^y is a $K[M]$-sequence

Necessary condition: each facet of $\operatorname{cone}(M)$ contains x or y.

Sufficient if $K[M]$ is Cohen-Macaulay, for example, if M is normal.

Binomial regular sequences

Corollary

With K and M as in the theorem, let $x_1, \dots, x_n$, $n \geq 2$, be noninvertible elements of M.

Then the following statements (1) and (2) are equivalent:

1. $X^{x_1} - X^{x_2}, \dots, X^{x_{n-1}} - X^{x_n}$ *is a $K[M]$-sequence and generates a prime ideal P.*
2. (a) $X^{x_1}, \dots, X^{x_n}$ *is a $K[M]$-sequence;*
   (b) $x_1 - x_2, \dots, x_{n-1} - x_n$ *generate a rank $n-1$ direct summand of* $\operatorname{gp}(M)$.

For the algebraists: if one of the x_i is a unit, (2)(b) $\iff$ (1), provided the units come first in $x_1, \dots, x_n$, although $(X^{x_1} - X^{x_2}, \dots, X^{x_{n-1}} - X^{x_n})$ (and its primeness) do not depend on it.

The case of a direct sum

Consider $M \oplus N$, and $M, N \subset M \oplus M \oplus N$ naturally embedded.

X^x, X^y is a regular sequence for $x \in M$, $y \in N$, x, y noninvertible.

Corollary

Under these assumptions, $K[M \oplus N]/(X^x - X^y)$ is a integral domain if $\operatorname{gp}(M)(\mathbb{Z}x$ or $\operatorname{gp}(N)/\mathbb{Z}y$ is torsionfree.

For example, if $\deg x = 1$ or $\deg y = 1$ in the graded case.

In the situation of free sums of polytopes, all hypotheses of the corollary are satisfied.

A counterexample

We return to Ehrhart series of free sums. One could hope that everything works if one of the polytopes is a $(\pm 1, 0)$-polytope. In general, this condition is not strong enough:

$P \subset \mathbb{R}^3$ the lattice polytope spanned by $-(e_1 + e_2 + e_3)$, e_i, $e_i + e_j$, $i, j = 1, 2, 3$, $i \neq j$.

$Q = [-1, 2] \subset \mathbb{R}$.

Then

$$E_{P \oplus Q} = \frac{1 + 3T + 5T^2 + 4T^3 + 2T^4}{(1 - T)^5} \neq \frac{1 + 3T + 4T^2 + 5T^3 + 2T^4}{(1 - T)^5} = (1 - T) E_P E_Q.$$

Serre's conditions for normality

Serre's conditions for normality allows us to take real advantage of working with monoid algebras.

R a noetherian domain

(S_2): $a, b \in R$, height$(a, b) = 2 \Rightarrow a, b$ is an R-sequence

(R_1): P prime ideal of height 1 $\Rightarrow R_P$ is regular

Theorem (Serre)

Let R be a noetherian domain. Then the following are equivalent:

1. *R is normal (i.e. integrally closed in its field of fractions).*
2. *R satisfies (R_1) and (S_2).*

Serre's conditions for monoid algebras

For affine monoid algebras $K[M]$ both conditions are "combinatorial", i.e. depend only on M.

For affine monoid algebras and regular sequences $a_1, \dots, a_n$, (S_2) is not a problem:

$K[M]$ normal $\Rightarrow$ $K[M]$ Cohen-Macaulay (Hochster) $\Rightarrow$ $K[M]/(a_1, \dots, a_n)$ CM $\Rightarrow$ $K[M]/(a_1, \dots, a_n)$ satisfies all (S_k).

For control of regularity one uses

Lemma

P a prime ideal in $K[M]$, F the face of $\operatorname{cone}(M)$ spanned by all $y \in M$, $X^y \notin P$.
Then the following are equivalent:

1. *$K[M]_P$ is a regular local ring;*
2. *$M[-F]$ is isomorphic to $\mathbb{Z}^{d-n} \oplus \mathbb{Z}_+^n$ for some n, $0 \le n \le d$.*

Normality

Theorem

Let M be a normal affine monoid of rank d, and suppose that $x, y \in M$ satisfy conditions (2)(a) and (b) of the theorem on integrity. Then the following are equivalent:

1. $K[M]/(X^x - X^y)$ *is normal.*
2. *If G is a subfacet of* $\operatorname{cone}(M)$ *such that $x, y \notin G$, then*
   (a) $M[-G] \cong \mathbb{Z}^{d-2} \oplus \mathbb{Z}_+^2$,
   (b) *x or y has (lattice) height* 1 *over one of the exactly two facets F', F'' containing G.*

If both have height 1 *over F' or F'' resp., hen (2)(a) is satisfied.*

Must check (R_1) for $K[M]/(X^x - X^y)$. The only critical question:

P monomial prime ideal in $K[M]$ of height 2, $X^x, X^y \in P$. When is $K[M]_P/(X^x - X^y)$ regular?

Set G be the face of $\operatorname{cone}(M)$ corresponding to $P \Rightarrow G$ subfacet

$K[M]_P/(X^x - X^y)$ regular $\Rightarrow K[M]_P$ regular

$$\Rightarrow M[-G] \cong \mathbb{Z}^{d-2} \oplus \mathbb{Z}^2_+$$

Conversely, $M[-G] \cong \mathbb{Z}^{d-2} \oplus \mathbb{Z}^2_+$ means

$$K[M[-G]] = K[U_1^{\pm 1}, \dots, U_{d-2}^{\pm 1}][V, W] = S.$$

and

$$X^x = \mu V^v,\ Y^y = \nu W^w, \qquad \mu, \nu \text{ monomials in } K[U_1^{\pm 1}, \dots, U_{d-2}^{\pm 1}]$$

$S/(\mu V^v - \nu W^w)$, regular $\iff v = 1$ or $w = 1$.

Normality modulo regular sequence

Corollary

$x_1, \dots, x_n$ as in corollary on integrity. Then the following are equivalent:

1. $K[M]/(X^{x_1} - X^{x_2}, \dots, X^{x_{n-1}} - X^{x_n})$ *is normal;*
2. *for each face F with $\operatorname{rank} M - \dim F = n$ and $x_1, \dots, x_n \notin F$:*
   (a) $M[-F] \cong \mathbb{Z}^{d-n} \oplus \mathbb{Z}^n_+$;
   (b) *at least $n-1$ of the n nonzero numbers $\operatorname{height}_{F_i}(x_j)$ are equal to 1 for the facets $F_1, \dots, F_n$ containing F and $j = 1, \dots, n$.*

In particular, it is sufficient for (1) that all n nonzero $\operatorname{height}_{F_i}(x_j)$ are equal to 1 in the situation of (2).

Back to Ehrhart series

Corollary (Beck-Jayawant-McAllister)

Let $R \subset \mathbb{R}^m$ be a rational polytope that is the free sum of the rational polytopes P and Q, both containing 0. Then the following are equivalent:

1. *At least in one of P or Q the origin has height ≤ 1 over all facets;*
2. $E_{P \oplus Q} = (1-T)E_P E_Q$.

In fact, the critical subfacets of $\operatorname{cone}(\mathscr{E}_P \oplus \mathscr{E}_Q)$ are given by $F' \oplus F''$ where F' is a facet of $\operatorname{cone}(\mathscr{E}_P)$ and $F''$is a facet of $\operatorname{cone}(\mathscr{E}_Q)$.

In particular

Corollary (Braun)

P, Q as above. If P is reflexive with $0 \in \operatorname{int}(P)$, then

$$E_{P \oplus Q} = (1-T)E_P E_Q.$$

From Gorenstein to reflexive polytopes

The following is the main reduction step for B-Römer theorem on h-vectors of normal Gorenstein lattice polytopes.

Corollary

Suppose $K[M]$ is Gorenstein and let X^w, $w \in M$, generate the canonical module$K[M]$.
$x_1, \dots, x_n \in M$ noninvertible elements such that $w = x_1 + \cdots + x_n$.
Then:
$K[M]/(X^{x_1} - X^{x_2}, \dots, X^{x_{n-1}} - X^{x_n})$ is again a Gorenstein normal affine monoid domain.

w ihas height 1 over each facet $\Rightarrow$

“height vectors” of $x_1, \dots, x_n$ are 0-1-vectors with disjoint supports $\Rightarrow$
$K[M]/(X^{x_1} - X^{x_2}, \dots, X^{x_{n-1}} - X^{x_n})$ normal y the theorem

Gorenstein property is preserved modulo regular sequences.

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