Fusion rings from lattice points

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Classification of modular data of integral modular fusion categories up to rank 12

with

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A fusion ring à la Google

Our goals today

- Explain the notion of fusion ring,
- give an overview of fusion ring properties that are crucial for their computation,
- describe the approach of Normaliz to the computation.

Some of the computations have been the hardest mastered challenges for Normaliz so far.

Fusion data and fusion rings

Definition

A fusion datum is a finite set $\{1, 2, ..., r\}$ with an involution $i \to i^*$, and nonnegative integers $N_{i,j}^k$ such that for all i, j, k, t:

- (Ass) $\sum_s N_{i,j}^s N_{s,k}^t = \sum_s N_{j,k}^s N_{i,s}^t$,
- (Unit) $N_{1,i}^j = N_{i,1}^j = \delta_{i,j}$,
- (Auto) $N_{ij}^{k^*} = N_{j^* i^*}^k$,
- (Dual) $N_{i^*,j}^1 = N_{ji^*}^1 = \delta_{i,j}$.

The fusion datum can be denoted as $(N_{i,j}^k)$ (* unique by (Dual)).

A fusion ring R is a free $\mathbb{Z}$-module with a basis $\{b_1, \ldots, b_r\}$ whose multiplication table is given by a fusion datum:

$$b_i b_j = \sum_s N_{ij}^s b_s.$$

Elementary properties of fusion rings

Proposition

Let R be a fusion ring.

- *R is an associative $\mathbb{Z}$-algebra of rank r.*
- *b_1 is its unit element.*
- ** extends linearly to an antiautomorphism of R.*

This follows immediately from the definition of fusion datum.

Proposition (Frobenius reciprocity)

For all i, j, k one has

$$N_{i,j}^{k} = N_{i^*,k}^{j} = N_{j,k^*}^{i^*} = N_{j^*,i^*}^{k^*} = N_{k^*,i}^{j^*} = N_{k,j^*}^{i}.$$

Important for computations: together with the unit axiom this reduces the number of variables N_{ij}^{k} to $\sim (r-1)^3/6$.

Examples of fusion rings: representation theory

We consider a finite group G and $\mathbb{C}$-representations. Then there exist finitely many irreducible representations $V_1, \dots, V_r$ where we choose V_1 as the trivial representation on $\mathbb{C}$ and * is defined by $V_{i^*} \cong V_i^*$. The tensor product of V_i and V_j decomposes:

$$V_i \otimes V_j = \bigoplus_s (V_s)^{\oplus N_{i,j}^s}.$$

Then (N_{ij}^s) is a fusion datum, and the corresponding fusion ring is the Grothendieck ring of the category $\mathrm{Rep}(G)$ of finite-dimensional $\mathbb{C}$-representations of a finite group G.

The symmetric group S_3 has 3 irreducible representations, namely the trivial representation $\mathbb{C}$, the sign representation χ on $\mathbb{C}$, and a $2-$dimensional representation V defined by the decomposition $\mathbb{C}^3 \cong \mathbb{C} \oplus V$ (all self-dual). Then

$$\chi \otimes \chi = \mathbb{C}, \quad \chi \otimes V = V \otimes \chi = V, \quad V \otimes V = \mathbb{C} \oplus \chi \oplus V.$$

The Frobenius–Perron theorem

The multiplication table (N_{ij}^k) of a fusion ring R has nonnegative entries. This has a crucial consequence.

Theorem (Frobenius–Perron)

Let M be a square matrix with nonnegative entries.

- *Then M has a nonnegative real eigenvalue.*
- *Let λ be the maximal real nonnegative eigenvalue. Then $|\mu| \leq \lambda$ for all eigenvaules μ of M.*

Definition

For a base element b_i we let $\mathrm{FPdim}(b_i)$ be the maximal real eigenvalue of the matrix of left multiplication by b_i on R. The Frobenius–Perron dimension $\mathrm{FPdim} : R \to \mathbb{C}$ is defined by $\mathbb{Z}$-linear extension.

Thus $\mathrm{FPdim}(x)$ is an algebraic integer for all $x \in R$. Moreover $\mathrm{FPdim}(b_i) = \mathrm{FPdim}(b_i*) \geq 1$ for all i.

FPdim is a ring homomorphism

Theorem

FPdim : $R \to \mathbb{C}$ *is the unique ring homomorphism that takes nonnegative values on the* b_i.

Let $d_i = \mathrm{FPdim}(b_i)$ for $i = 1, \dots, r$. $(d_1, \dots, d_r)$ is the type of R.
On the one hand $\mathrm{FPdim}(b_i b_j) = d_i d_j$. On the other hand:

$$\mathrm{FPdim}(b_i b_j) = \mathrm{FPdim}(\sum\nolimits_k N_{ij}^k b_k) = \sum\nolimits_k d_k N_{ij}^k.$$

Corollary

- *Let* $(d_1, \dots, d_r)$ *be the type of* R. *Then*

$$d_i d_j = \sum_k d_k N_{ij}^k \qquad i, j = 1, \dots, r.$$

- *In particular, there exist only finitely many fusion rings of a given type.*

Integral and more special type

A fusion ring R is integral if $(d_1, \ldots, d_r)$ is an integral vector. For a given rank r there exist infinitely many integral fusion rings.

Set $\mathrm{FPdim}(R) = \sum_i {d_i}^2$. One calls R 1/2-Frobenius if $\mathrm{FPdim}(R)/d_i^2$ is an algebraic integer for all i.

In the integral case: R is 1/2-Frobenius $\iff$ $\mathrm{FPdim}(R)/d_i^2 \in \mathbb{Z}$ for all i. $\implies$ There exist only finitely many 1/2-Frobenius integral fusion rings of given rank (non-trivial). In principle they can all be computed, and we have succeeded in rank ≤ 12.

Without 1/2-Frobenius one can of course bound $\mathrm{FPdim}(R)$ and then compute all fusion rings within the bound (in progress for a subclass).

Roughly speaking, the integral modular data belong to a subclass of the 1/2-Frobenius integral fusion data that we cannot define here (the modular group $\mathrm{SL}(2, \mathbb{Z})$ plays a role). They are important in another source of fusion rings, conformal field theory (Verlinde 1988).

Categorification

Fusion rings are the combinatorial skeletons of certain tensor categories, distinguished by the existence of a bifunctor called tensor product. Simplest example: the category of vector spaces.

A fusion ring is categorifiable if it is the Grothendieck ring of a fusion category, in the same manner as we have seen it for the Grothendieck ring of the fusion category Rep(G) of the finite-dimensional $\mathbb{C}$-representations of a finite group G.

In the opposite direction one must replace the coefficients N_{ij}^k by vector spaces of dimension N_{ij}^k and find fixed isomorphisms $(X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ that satisfy the so-called pentagon condition. It amounts to a huge degree 3 system of polynomial equations in the F-symbols. Hard to decide solvability.

In the paper we were lucky: all modular fusion data of rank ≤ 12 can be cateogorified by well-known tensor categories.

The constraints of fusion data

The goal is to compute all fusion rings of given type $(d_1, \dots, d_r)$ and duality (1*,. . . ,r*). As seen, the following constraints are necessary and sufficient for a fusion datum of this type and duality:

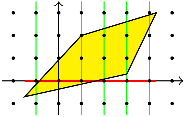
- (NonNeg) N_{ij}^k is an integer ≥ 0 for all i, j, k.
- (Unit) $N_{1,i}^j = N_{i,1}^j = \delta_{i,j}$ for all i, j,
- (Frob) $N_{i,j}^k = N_{i^*,k}^j = N_{j,k^*}^{i^*} = N_{j^*,i^*}^{k^*} = N_{k^*,i}^{j^*} = N_{k,j^*}^i$ for all i, j, k
- (FPdim) $\sum_k d_k N_{ij}^k = d_i d_j$ for all i, j,
- (Ass) $\sum_s N_{i,j}^s N_{s,k}^t = \sum_s N_{j,k}^s N_{i,s}^t$ for all i, j, k.

(Unit) and (Fob) are used to reduce the number of variables. We must then find the lattice points in the polytope defined by (NonNeg) and (FPdim) that satisfy the quadratic equations (Ass).

Basic technique: project-and-lift

We want to compute the lattice points in a polytope P given by linear inequalities. We proceed as follows:

- Project P onto a coordinate hyperplane.
- By recursion we know the lattice points in he projection P'.
- Compute the lattice points in the fibers of the projection over the lattice points x' in P'



The last step is easy: the fiber is a line segment cut out by the inequalities from the total fiber of the projection over x'.

Potentially dangerous: precise computation of P'. But we can relax: it is enough to have a polytope $P'' \supset P'$.

Coarse projection and patching

Coarse projection. In the computation of fusion data there is an obvious relaxation: each equation $a_1x_1 + \cdots + a_nx_n = b$ in (FPdim) has $a_i \geq 0$ and $b \geq 0$. Since the x_i are nonnegative by (NonNeg), at least the implied inequality $a_1x_1 + \cdots + a_nx_n \leq b$ can be restricted to any subset of the variables. x_i.

Patching. Each equation in (FPdim) involves only a small subset of the variables. Therefore we can first compute its lattice solutions restricted to the variables with a nonzero coefficient by coarse projection, and then "patch" the solutions along overlapping coordinates.

Danger: potentially enormous number of solutions of the patches. Better: insert patches successively into the algorithm, compute only extensions of the overlaps of the new patch with the union of the preceding ones and store the extensions.

Damping the combinaorial explosion

Impossible approach: compute the lattice points in the polytope and sieve them by the quadratic equations.

Instead we must stop a chain of extensions as soon as one of the very selective quadratic equations can be applied (because the coordinates in its support have been covered) and is not satisfied.

Look ahead techniques:

- Apply truncations of linear and quadratic equations to inequalities as soon as possible. Helpful effect.
- Derive congruences from the linear equations modulo their coefficients in all possible ways. This gives constraints in fewer variables. Apply them as soon as possible. Has often a significant effect.

Minimization: discard quadratic equations as soon as it is clear that they have no effect. (The system is heavily redundant.)

Automorphisms

An automorphism of fusion rings is a ring homomorphism that permutes the basis $b_1, \dots, b_r$. Equivalently we can start from a permutation π of $\{1, \dots, r\}$ such that

- $\pi(1) = 1, \qquad d_i = d_{\pi(i)}, \qquad \pi(i^*) = \pi(i)^* \qquad$ for all i.

$\implies N_{ij}^k = N_{\pi(i)\pi(j)}^{\pi(k)}$. The normal form of a solution (N_{ij}^k) is the lexicographically largest in its orbit w.r.t. automorphisms. The normal forms represent the isomorphism classes of fusion rings.

The hardest computations are those with many repetitions in the sequence d_i and trivial duality. This weakens the look-head congruences and creates a large automorphism group.

Now Normaliz can exploit the action of the automorphism groups already in the patching process:

- Discard vectors that cannot extend to a normal form.

Often this has an overwhelming effect.

Further aspects

- It is important to find a good insertion order for the patches defined by the linear equations (FPdim). The balance between two principles can be controlled by options:
  - cover enough coordinates as quickly as possible to allow the evaluation of quadratic equations,
  - avoid inserting “bad” patches too early.
- In principle, project-and-lift is a tree search. Normaliz uses a mixture of “depth first” and “breadth first” strategies.
- One can ask Normaliz to compute only simple fusion rings (not containing a proper nontrivial fusion ring).
- For the monsters among the fusion data we have used the high performance cluster at Osnabrück (50 nodes, 128 threads and 1 TB each) with a static splitting strategy that allows successive refinement.

Some computations

	rank FPdim	#var #aut	#lin #qud	#sol #iso	time	hardware
[1,5,5,5,6,7,7]	7 210	56 12	36 240	6 2	0.3 s	laptop
[1,3,3,4,5,5,5,5,5,10,10] $10^* = 11$	11 335	210 240	100 2250	25 2	333 s	laptop
[1,1,2,3,3,6,6,8,8,8,12,12]	12 576	286 48	121 3080	1669 199	108 s 69 s	laptop
M_{12} $[1,11^2,16^2,45,54,55^3,$ $66,99,120,144,176]$, $4^* = 5$	15 95040	546 24	196 8736	72 24	24 h 3.5 h	server
$\mathrm{PSL}(2,17)$ $[1,9^2, 16^4, 17,18^3]$	11 2448	220 288	100 2070	10442 135	n.m. 46 h	HPC
[1,1885,5005,6699,47502, 87087,200970,373230, $870870^2,1306305^3]$	13 $> 6*$ 10^{12}	364 n.u.	144 4422	0 0	290 s	server

exponents: repetitions

lin, # qud: numbers of linear and quadratic equations. n.m.: not measured

iso: number of isomorphism classes, # sol solutions. n.u.: not used

M_{12}, $\mathrm{PSL}(2,17)$: the types in the input files are defined by the Grothendieck rings of $\mathrm{Rep}(G)$ for these classical groups G.

References

M. A. Alekseyev, W. Bruns, S. Palcoux and F. V. Petrov, *Classification of modular data of integral modular fusion categories up to rank 12*, Preprint arXiv:2302.14345.

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