The computation of generalized Ehrhart series and integrals in Normaliz

Winfried Bruns

FB Mathematik/Informatik
Universität Osnabrück

wbruns@uos.de

Berlin, November 2013

Normaliz and NmzIntegrate

The computer program Normaliz has been developed in Osnabrück since 1997. Current team members:

Bogdan Ichim (Bucharest), Christof Söger (OS), WB

Now supported by the DFG SPP "Experimental methods in algebra, geometry and number theory"

Normaliz computes two types of data:

1. Hilbert bases of rational cones
(normalizations of affine monoid domains)
2. Ehrhart series of rational polytopes
(Hilbert series of normal monoid domains)

Normaliz has an offspring NmzIntegrate (by WB and C. Söger) for the computation of generalized (or weighted) Ehrhart series. NmzIntegrate is based on CoCoALib.

Ehrhart series

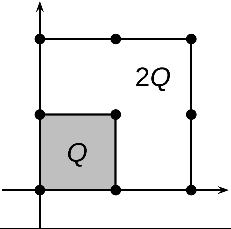
$P \subset \mathbb{R}^d$ rational polytope.

$$E(P,k) = \#(kP \cap \mathbb{Z}^d) = \#\left(P \cap \frac{1}{k}\mathbb{Z}^d\right)$$

is called the Ehrhart function of P. The generating function

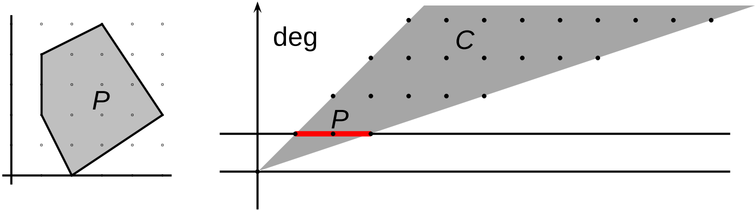
$$E_P(t) = \sum_{k=0}^{\infty} E(P,k)t^k$$

is the Ehrhart series. Example: $Q \subset \mathbb{R}^2$ unit square. Then



$$E(Q,k) = (k+1)^2 \qquad E_Q(t) = \frac{1+t}{(1-t)^3}$$

Data: C pointed polyhedral rational cone in $\mathbb{R}^d$, monoid $M = C \cap \mathbb{Z}^d$, grading $\deg : \mathrm{gp}(M) \to \mathbb{Z}$ surjective, $\deg x > 0$ for $x \in M$, $x \neq 0$



Set $E_M(t) = \sum_{x \in M} t^{\deg x}$. Then

$$E_M(t) = E_P(t) \qquad P = \{y \in C : \deg y = 1\}$$

$$= H_{K[M]}(t) \qquad K \text{ a field, } H \text{ Hilbert series}$$

(deg extended to $\mathbb{R}M$)

Generalized Ehrhart series

$M \subset \mathbb{Z}^d$ a graded monoid as above, f a polynomial in d variables (with rational coefficients). Then

$$E_{M,f}(t) = \sum_{x \in M} f(x) t^{\deg x}$$

is the f-generalized (or f-weighted) Ehrhart series of M. For rational polytopes

$$E(P,f,k) = \sum_{x \in kP \cap \mathbb{Z}^d} f(x), \qquad E_{P,f}(t) = \sum_{k=0}^{\infty} E(P,f,k) t^k.$$

Typical application: $\pi : \mathbb{R}^d \to \mathbb{R}^e$ is a projection, $P \subset \mathbb{R}^d$, $Q = \pi(P)$, $f(x) = \#\pi^{-1}(x)$. Then

$$E_P(t) = E_{Q,f}(t).$$

If $\dim Q \ll \dim P$, this is usually much faster.

Theorem

$$E_{M,f}(t) = \frac{1 + h_1 t + \cdots + h_s t^s}{(1 - t^\ell)^{\operatorname{rank} M + \deg f}}, \qquad h_i \in \mathbb{Q}, s < \ell(\operatorname{rank} M + \deg f).$$

where ℓ *is the lcm of* $\deg x_i$, $x_i \in \mathbb{Z}^d$, $i = 1, \dots, m$, *generating* $\mathbb{R}_+ M$.

Equivalently

Theorem

There exists a quasipolynomial $q = q_{M,f} \in \mathbb{Q}[X]$ *of degree* $g < \operatorname{rank} M + \deg f$ *and period* $\pi \mid \ell$ *such that*

$$E(P, f, k) = q(k), \qquad k \geq 0.$$

A quasipolynomial is a “polynomial ” with periodic coefficients:

$$q(k) = c_g^{(k)} k^g + \cdots + c_1^{(k)} k + c_0^{(k)}, \qquad c_i^{(k)} = c_i^{(j)} \text{ if } j \equiv k \ (\pi).$$

Lead coefficient and Lebesgue integral

The “virtual” leading coefficient of the quasipolynomial is constant:

Proposition

$$c_{\operatorname{rank} M+m-1} = \int_P f_m \, d\mu, \qquad m = \deg f,$$

where f_m is the leading homogeneous component of f, and μ is the Lebesgue masure under which a basic lattice cube in $\operatorname{aff}(P)$ has volume 1.

Often the leading coefficient is the most important information, since it controls the growth order of $E(M, f, k)$.

NmzIntegrate also computes Lebesgue integrals of (not necessarily homogeneous) polynomials over rational polytopes.

Application: The Condorcet paradox

A paradigmatic paradox of the theory of social choice has been named after the *Marquis de Condorcet* (1741–1794).

Consider an election with 3 candidates, called A,B,C. Each of the k fixes a preference order, in other words, a linear order of the candidates. There 6 such orders:

P_1	P_2	P_3	P_4	P_5	P_6
A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

Election result: x_i of the k voters choose the preference order P_i. Then $x_1 + \cdots + x_6 = k$, $x_i \in \mathbb{Z}$, $x_i \geq 0$ for $i = 1, \ldots, 6$.

Who is the winner?

Usually the majority winner, the candidate with most first places.

But one could also take the Condorcet winner (CW): A *beats* B if the number of voters that prefer A to B is larger than the number of voters with the opposite preference.

A is the Condorcet winner if he beats B and C.

Observation of Condorcet (and others): there are election results for which “beats“ is not transitive! In particular, there need not exist a CW.

This is the *Condorcet paradox*.

The standard assumption for the following quantification: a priori, all election results have equal probability.

Quantification

A is the Condorcet winner if

$$\lambda_1(x) = x_1 + x_2 - x_3 - x_4 + x_5 - x_6 > 0$$
$$\lambda_2(x) = x_1 + x_2 + x_3 - x_4 - x_5 - x_6 > 0$$

If there are k voters, then the probability of “A is the CW” is

$$p_k(\text{A CW}) = \frac{\#\{x : \lambda_i(x) > 0, i = 1,2\}}{\#\{\text{all } x\}} = \frac{\#\{x : \lambda_i(x) > 0, i = 1,2\}}{\binom{k+5}{6}}$$

The probability that there is a CW at all then is

$$p_k(\text{CW}) = 3 \cdot p_k(\text{A CW})$$

Obviously, for large k the limit as $k \to \infty$ is (more) interesting:

$$p(\text{CW}) = \lim_{k\to\infty} p_k(\text{CW}).$$

Election results as lattice points

The potential election results for a fixed number of k voters are exactly the lattice points in the simplex

$$S_k = \{x \in \mathbb{R}^6 : x_i \geq 0, \sum x_i = k\}$$

und the results that have a CW are the lattice points in the subset

$$C_k = \{x \in S_k : \lambda_i(x) > 0, i = 1, 2\}$$

Attention: C_k is a semiopen polytope since the lattice points in some faces (in this case: faects) do not belong to C_k:

$$C_k = \overline{C_k} \setminus \text{(union of some faces)}$$

Normaliz and NmzIntegrate can now (current development versions) deal with semiopen polytopes and “semiopen” monoids.

Symmetries

Once more the inequalities for “A is the CW” (3 candidates):

$$\lambda_1(x) = x_1 + x_2 - x_3 - x_4 + x_5 - x_6 > 0$$
$$\lambda_2(x) = x_1 + x_2 + x_3 - x_4 - x_5 - x_6 > 0$$

Achill Schürmann’s observation: the inequalities are symmetric with respect to $x_1 \leftrightarrow x_2$ and $x_4 \leftrightarrow x_6$. Simplification:

$$\mu_1(y) = y_1 - y_2 - y_3 + y_4 > 0 \qquad y_1 = x_1 + x_2, y_3 = x_3$$
$$\mu_2(y) = y_1 - y_2 + y_3 - y_4 > 0 \qquad y_2 = x_4 + x_6, y_4 = x_5$$

Hence we consider the projection $\pi : \mathbb{R}^6 \to \mathbb{R}^4$ and replace the polytope C by $D = \pi(C)$.

Gain: simpler geometry. Loss: more difficult counting problem.

Computations for 3 candidates are very easy, but for 4 candifates they are already extremely hard without symmetrization.

The enormous gain of symmetrization

The following table shows the partly extreme acceleration of computations by use of the symmetrization.

	dim	deg f	# triang/ # dec	time	
				E series	lead coeff
Cond paradox	24	0	$1.3 * 10^6$ / $1.8 * 10^6$	4.2 sec	3.2 sec
Cond. parad. symm	8	16	17 / 21	2.9 sec	0.07 sec
Cond. eff. plurality	24	0	$3.5 * 10^{11}$ / $4.1 * 10^{12}$	218 h	
Cond. eff. pl. symm	13	11	17,953/ 23,453	3:34 h	25:53 min
Plur. vs. cutoff	24	0	$2.5 * 10^{11}$ / $4.1 * 10^{12}$	175 h	
Plur. vs. cut symm	6	18	3 / 4	13.5 sec	0.18 sec

SUN xFire 4450, 20 parallel threads

Back to generalized Ehrhart functions

We restrict ourselves to closed polytopes and monoids: the semiopen case is reduced to the closed case via inclusion/exclusion.

Recall that we want to compute

$$E_{M,f}(t) = \sum_{x \in M} f(x) t^{\deg x} = \frac{1 + h_1 t + \cdots + h_s t^s}{(1 - t^{\ell})^{\operatorname{rank} M + \deg f}}$$

for a monoid $M = C \cap \mathbb{Z}^d$ with grading deg.

We proceed in 3 steps:

1. $M = \mathbb{Z}_+$: direct approach
2. $M = \mathbb{Z}_+^d$: induction on d
3. the general case by decomposing M into suitable images of $\mathbb{Z}^d$ (Stanley decomposition)

$M = \mathbb{Z}_+$

By linearity, it is enough to consider $\sum_{k=0}^{\infty} k^m t^{uk}$. Example:

$$\sum_{k=0}^{\infty} kt^k = \sum_{k=0}^{\infty} (k+1)t^k - \sum_{k=0}^{\infty} t^k = \left(\frac{1}{1-t}\right)' - \frac{1}{1-t} = \frac{t}{(1-t)^2}.$$

So we write k^m as a linear combination of the rising factorials

$$(k+1)_j = (k+1)(k+2)\cdots(k+j), \qquad j = 0,\dots,m$$

(coefficients of k^m: Stirling numbers of the second kind) and use

$$\sum_{k=0}^{\infty} (k+1)_j \, t^k = \left(\frac{1}{1-t}\right)^{(j)} = \frac{j!}{(1-t)^{j+1}}.$$

After substitution $t \mapsto t^u$:

$$\sum_{k=0}^{\infty} k^m t^{uk} = \frac{A_m(t^u)}{(1-t^u)^{m+1}} \qquad \text{(Eulerian numbers)}$$

$M = \mathbb{Z}_+^d$

If $f(x) = g(y)h(z)$, $y = (x_1, \dots, x_r)$, $z = (x_{r+1}, \dots, x_d)$, then

$$E_{M,f}(t) = \sum_{x \in \mathbb{Z}_+^d} f(x) t^{\deg x} = \left(\sum_{y \in \mathbb{Z}_+^r} g(y) t^{\deg y} \right) \left(\sum_{z \in \mathbb{Z}_+^{d-r}} h(z) t^{\deg z} \right)$$

In the general case we split off the last variable:

$$f(x) = \sum_i f_i(x_1, \dots, x_{d-1}) x_d^i,$$

and use

$$E_{M,f}(t) = \sum_i \left(\left(\sum_{x' \in \mathbb{Z}_+^{d-1}} f_i(x') t^{\deg x'} \right) \left(\sum_{k=0}^{\infty} k^i t^{ui} \right) \right)$$

$$= \sum_i \left(\frac{A_i(t^u)}{(1 - t^u)^{i+1}} \sum_{x' \in \mathbb{Z}_+^{d-1}} f_i(x') t^{\deg x'} \right)$$

with $u = \deg e_d$.

Stanley decomposition = discrete triangulation

Theorem (Stanley)

Let $M = C \cap \mathbb{Z}^d$, $\operatorname{rank} M = d$. There exists a finite decomposition

$$M = \bigcup_s \alpha_s(\mathbb{Z}_+^d) \qquad \textit{(disjoint)}$$

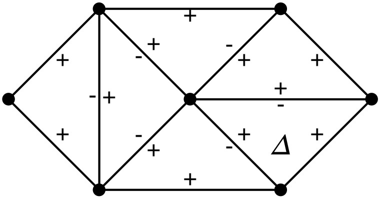
where $\alpha_s : \mathbb{Z}_+^d \to M$ is a $\mathbb{Z}$-linear affine map:

$$\alpha_s(x_1, \dots, x_d) = u^{(s)} + \sum_{i=1}^{d} x_i v_i^{(s)}.$$

Consequence:

$$E_{M,f}(t) = \sum_s t^{\deg u^{(s)}} \left(\sum_{x \in \mathbb{Z}_+^d} f(\alpha_s(x)) t^{\deg_s(x)} \right) \quad deg_s(x) = \sum x_i \deg v_i^{(s)}$$

The crucial argument for Stanley decomposition



Theorem

Let Σ be a triangulation of a polytope P. Then there exists a decomposition

$$P = \bigcup_{\Delta \in \Sigma,\ \dim \Delta = \dim P} \Delta \setminus S_\Delta$$

where S_Δ is a union of (at most $\dim P$) facets of Δ.

Stanley's proof uses a line shelling (Bruggesser-Mani). Since version 2.7, Normaliz uses a theorem of Köppe-Verdoolaege.

Integration

In the paper of Jeffries, Montaño and Varbaro on j-multiplicities and ε-multiplicities we find the formula

$$e(A_t(X)) = c \int\limits_{\substack{[0,1]^m \\ \sum z = t}} (z_1 \cdots z_m)^{n-m} \prod_{1 \le i < j \le m} (z_j - z_i)^2 \mathrm{d}\mu$$

This is an interesting example of an integral of a polynomial over a rational polytope.

NmzIntegrate handles such integrals as follows:

1. triangulation of the polytope,
2. linear transformation from a rational simplex to the unit simplex,
3. integration over the unit simplex.

Integration over the unit simplex

We consider the unit $(d-1)$-simplex Δ naturally embedded into $\mathbb{R}^d$ as the convex hull of the unit vectors. Integration over the unit simplex is done monomial by monomial:

Proposition

$$\int_\Delta y_1^{m_1} \cdots y_d^{m_d} \, d\mu = \frac{m_1! \cdots m_d!}{(m_1 + \cdots + m_d + d - 1)!}.$$