# Stanley decompositions and Hilbert depth in the Koszul complex


Winfried Bruns

FB Mathematik/Informatik
Universität Osnabrück

wbruns@uos.de


Kingston, September 27, 2010

Joint work with

Christian Krattenthaler and Jan Uliczka

to appear in *J. Commut. Algebra* (Fröberg volume)

# The framework

We consider

- $R = K[X_1, \dots, X_n]$, $K$ a field, with the
  - standard grading, $\deg X_i = 1$, indicated by subscript 1,
  - the multigrading, $\deg X_i = e_i \in \mathbb{Z}^n$, indicated by subscript $n$,
- a finitely generated graded $R$-module.

The Hilbert function of $M$ is denoted by

$$H(M,k) = \dim_K M_k, \qquad k \in \mathbb{Z}^m, \quad m = 1 \text{ or } m = n.$$

# The Hilbert series

The Hilbert series are defined by

$$H_M(T) = \sum_{k\in\mathbb{Z}} H(M,k)T^k,$$

$$H_M(T_1,\dots,T_n) = \sum_{k\in\mathbb{Z}^n} H(M,k)T_1^{k_1}\cdots T_n^{k_n},$$

They are rational functions:

$$H_M(T) = \frac{Q_M(T)}{(1-T)^n}$$

$$H_M(T_1,\dots,T_n) = \frac{Q_M(T_1,\dots,T_n)}{(1-T_1)\cdots(1-T_n)},$$

where $Q_M(T) \in \mathbb{Z}[T^{\pm 1}]$ and $Q_M(T_1,\dots,T_n) \in \mathbb{Z}[T_1^{\pm 1},\dots,T_n^{\pm 1}]$

# Stanley decompositions and Stanley depth

## Definition

A Stanley decomposition of $M$ is a finite family

$$\mathscr{D} = (S_i, x_i)_{i \in I}$$

in which $x_i$ is a homogeneous element of $M$ and $S_i$ is a graded $K$-algebra retract of $R$ for each $i \in I$ such that $S_i \cap \operatorname{Ann} x_i = 0$, and

$$M = \bigoplus_{i \in I} S_i x_i \qquad (= \mathscr{D})$$

as a graded $K$-vector space.

## Definition

The *Stanley depth* $\operatorname{Stdepth} M$ of $M$ is the maximal depth of a Stanley decomposition of $M$.

# Hilbert decompositions and Hilbert depth

## Definition

A Hilbert decomposition is a finite family

$$\mathscr{H} = (S_i, s_i)_{i \in I}$$

such that $s_i \in \mathbb{Z}^m$ (where $m = 1$ or $m = n$), $S_i$ is a graded $K$-algebra retract of $R$ for each $i \in I$, and

$$M \cong \bigoplus_{i \in I} S_i(-s_i) \qquad (= \mathscr{H})$$

as a graded $K$-vector space.

## Definition

The *Hilbert depth* Hdepth $M$ of $M$ is the maximal depth of a Hilbert decomposition of $M$.

# Hilbert depth and Hilbert series

Suppose $\mathscr{H} = (S_i, s_i)_{i\in I}$ is a Hilbert decomposition. Then

$$H_M(T_1,\dots,T_n) = \sum_i H_{S_i(-s_i)}(\dots) = \sum_i \frac{T_1^{s_{i1}}\cdots T_n^{s_{in}}}{\prod_{j\in U_i}(1-T_j)}.$$

Thus $\qquad \operatorname{depth}\mathscr{H} \ge d \iff |U_i| \ge d$ for all $i$.

This amounts to the following for the numerator polynomial: It can be written as a sum of terms

$$\text{monomial}\cdot\prod_{k\in V_i}(1-T_k)$$

with $U_i \cup V_i = \{1,\dots,n\}$, $U_i \cap V_i = \emptyset$.

Thus $\qquad \operatorname{depth}\mathscr{H} \ge d \iff |U_i| \le n-d$ for all $i$.

# Stanley depth and Hilbert depth

**Conjecture (Stanley)**

$$\operatorname{Stdepth} M \geq \operatorname{depth} M$$

Since a Stanley decomposition is a Hilbert decomposition:

$$\operatorname{Stdepth}_n M \leq \left\{ \begin{matrix} \operatorname{Hdepth}_n M \\ \operatorname{Stdepth}_1 M \end{matrix} \right\} \leq \operatorname{Hdepth}_1 M$$

Strategy:

- compute a Hilbert decomposition,
- convert it into a Stanley decomposition.

Alas: does not help in the case of fine gradings ($\dim M_k \leq 1$ for all $k$).

**Question**

$$\operatorname{Hdepth} M \geq \operatorname{depth} M \quad ??$$

# Hilbert depth in the standard graded case

Stanley's conjecture was was a theorem in the standard graded case for $|K| = \infty$ (Baclawski and Garsia). Moreover, Hilbert depth can be computed "easily" in the standard graded case:

**Theorem (Uliczka)**

*Then the following numbers coincide:*

1. $\max\{\operatorname{depth} N : H_M(T) = H_N(T)\}$,
2. $\operatorname{Hdepth}_1 M$,
3. *the maximum d such that*

$$H_M(T) = \sum_{e=d}^{n} \frac{Q_e(T)}{(1-T)^e}, \qquad Q_e(T) \in \mathbb{Z}_+[T, T^{-1}],$$

4. $\max\{p : (1-T)^p H_M(T)$ *positive*$\}$,
5. $n - \min\{q : Q_M(T)/(1-T)^q$ *positive*$\}$.

The Koszul complex

$$\mathscr{K}(X_1,\dots,X_n;R) : 0 \to \bigwedge^n R^n \xrightarrow{\partial} \bigwedge^{n-1} R^n \xrightarrow{\partial} \dots \xrightarrow{\partial} R^n \xrightarrow{\partial} R \to 0$$

resolves $K \cong R/\mathfrak{m}$, $\mathfrak{m} = (X_1,\dots,X_n)$.

Let $M(n,k)$ be the $k$-th syzygy module of $K$ $\qquad$ ($M(n,1) = \mathfrak{m}$).

The basis vector $e_{i_1} \wedge \cdots \wedge e_{i_k}$ of $\bigwedge^k R^n$ , $i_1 < \cdots < i_k$, has multidegree $X_{i_1} \cdots X_{i_k}$ and standard degree $k$.

Lower half: $1 \le k < \lfloor n/2 \rfloor$. $\qquad$ Upper half: $\lfloor n/2 \rfloor \le k < n$.

# Multigraded Stanley and Hilbert depth

**Theorem (Biró et al.)**

$$\operatorname{Hdepth}_1 \mathfrak{m} = \operatorname{Hdepth}_n \mathfrak{m} = \operatorname{Stdepth}_n \mathfrak{m} = \lfloor (n+1)/2 \rfloor.$$

Induction arguments yield

**Corollary**

$$\operatorname{Stdepth}_n M(n,k) \geq \lfloor (n+k)/2 \rfloor.$$

However,

**Theorem**

*In the upper half*

$$\operatorname{Hdepth}_1 M(n,k) = \operatorname{Hdepth}_n M(n,k) = n-1.$$

## Proof.

Multigraded numerator polynomial of Hilbert series of $M(n,k)$ is

$$Q(n,k) = \sigma_{n,k} - \sigma_{n,k+1} + \cdots + (-1)^{n-k}\sigma_{n,n}$$

where $\sigma_{n,k}$ is the $k$-th elementary symmetric polynomial in $T_1, \dots, T_n$.

For $j \geq \lfloor n/2 \rfloor$ one has an injective map from the monomials in $\sigma_{n,j+1}$ to those in $\sigma_{n,j}$ that maps each monomial to a divisor! Apply this with $j = k, k+2, \dots$ Consequence:

$Q(n,k)$ can be written as a sum of terms of type

- monomial
- $(1 - T_i)$·monomial

□

# The case $k = n - 3$

For $n = 5$, $k = 2$ we have realized the Hilbert decomposition as a Stanley decomposition. By induction

**Proposition**

$$\operatorname{Stdepth}_n M(n, n-3) = n - 1.$$

**Question**

*Is* $\operatorname{Stdepth} M(n,k) = n - 1$ *in the upper half?*

It is enough to do the case $n$ odd, $k = (n-1)/2$.

# Standard Hilbert depth in the lower half

The numerator polynomial of the standard Hilbert series of $M(n,k)$

$$\binom{n}{k}T^k - \binom{n}{k+1}T^{k+1} + \cdots + (-1)^{n-k}T^n$$

How often do we have so sum (i.e. divide by $1-T$) to get a positive power series? At least

$$v = \left\lceil \binom{n}{k+1} \Big/ \binom{n}{k} \right\rceil \qquad \text{times}$$

## Proposition

$$\operatorname{Hdepth}_1 M(n,k) \le n - \left\lceil \frac{n-k}{k+1} \right\rceil .$$

Is this naive bound sharp? Yes... for $n \le 22$, but not for $n = 23$, $k = 3, 4, 5$. (It is sharp for the powers of $\mathfrak{m}$.)

# Standard Hilbert depth in the lower half

The numerator polynomial of the standard Hilbert series of $M(n,k)$

$$\binom{n}{k}T^k - \binom{n}{k+1}T^{k+1} + \cdots + (-1)^{n-k}T^n$$

How often do we have so sum (i.e. divide by $1-T$) to get a positive power series? At least

$$v = \left\lceil \tbinom{n}{k+1} / \tbinom{n}{k} \right\rceil \qquad \text{times}$$

**Proposition**

$$\operatorname{Hdepth}_1 M(n,k) \le n - \left\lceil \frac{n-k}{k+1} \right\rceil .$$

Is this naive bound sharp? Yes... for $n \le 22$, but not for $n = 23$, $k = 3, 4, 5$. (It is sharp for the powers of $\mathfrak{m}$.)

# Standard Hilbert depth in the lower half

The numerator polynomial of the standard Hilbert series of $M(n,k)$

$$\binom{n}{k}T^k - \binom{n}{k+1}T^{k+1} + \cdots + (-1)^{n-k}T^n$$

How often do we have so sum (i.e. divide by $1-T$) to get a positive power series? At least

$$v = \left\lceil \tbinom{n}{k+1} / \tbinom{n}{k} \right\rceil \qquad \text{times}$$

**Proposition**

$$\operatorname{Hdepth}_1 M(n,k) \le n - \left\lceil \frac{n-k}{k+1} \right\rceil .$$

Is this naive bound sharp? Yes. . . for $n \le 22$, but not for $n = 23$, $k = 3, 4, 5$. (It is sharp for the powers of $\mathfrak{m}$.)

## Proposition

*Let $Q_{n,k}$ be the numerator polynomial of the $\mathbb{Z}$-graded Hilbert series of $M(n,k)$. Then*

$$\frac{Q_{n,k}}{(1-T)^s} = \sum_{j=0}^{\infty} \left( (-1)^j \binom{n-s}{k+j} + \sum_{t=1}^{s} \binom{n-t}{k-1} \binom{s-t+j}{s-t} \right) T^{j+k}$$

Hypergeometric sum . . . E-mail to Krattentahler: help!!!!

Answer: not summable since of type ${}_3F_2$.No explicit bound possible.

## Proposition

*Let $Q_{n,k}$ be the numerator polynomial of the $\mathbb{Z}$-graded Hilbert series of $M(n,k)$. Then*

$$\frac{Q_{n,k}}{(1-T)^s} = \sum_{j=0}^{\infty}\left((-1)^j\binom{n-s}{k+j} + \sum_{t=1}^{s}\binom{n-t}{k-1}\binom{s-t+j}{s-t}\right)T^{j+k}$$

Hypergeometric sum ... E-mail to Krattentahler: help!!!!

Answer: not summable since of type ${}_3F_2$.No explicit bound possible.

## Proposition

*Let $Q_{n,k}$ be the numerator polynomial of the $\mathbb{Z}$-graded Hilbert series of $M(n,k)$. Then*

$$\frac{Q_{n,k}}{(1-T)^s} = \sum_{j=0}^{\infty}\left((-1)^j\binom{n-s}{k+j} + \sum_{t=1}^{s}\binom{n-t}{k-1}\binom{s-t+j}{s-t}\right)T^{j+k}$$

Hypergeometric sum ... E-mail to Krattentahler: help!!!!

Answer: not summable since of type ${}_3F_2$.No explicit bound possible.

# Asymptotic estimate I

**Theorem**

*For a fixed positive integer $k$, we have*

$$\operatorname{Hdepth}_1 M(n,k) = \frac{1}{2}n + \frac{1}{2}\sqrt{(k-1)n\log n} + \frac{1}{4}\sqrt{\frac{(k-1)n}{\log n}}\log\log n + o\left(\sqrt{\frac{n}{\log n}}\log\log n\right),$$

*as $n \to \infty$.*

Without the $o$-term an upper bound for $k \geq 4$, $n \gg 0$.

We have another asymptotic result for $n, k \to \infty$, $n/k$ fixed.

The quality of the asymptotic estimate is indicated by the following numerical results:

| $n$ | $k$ | $\lfloor (n+k)/2 \rfloor$ | Hdepth | asympt bd | $n - \left\lceil \frac{(n-k)}{(k+1} \right\rceil$ |
|---|---|---|---|---|---|
| 50 | 6 | 28 | 41 | 43 | 43 |
| 100 | 6 | 53 | 75 | 77 | 86 |
| 200 | 6 | 103 | 139 | 142 | 172 |
| 500 | 6 | 253 | 318 | 321 | 429 |
| 1000 | 6 | 503 | 602 | 605 | 858 |

# Asymptotic estimate II

Now we let $n$ and $k$ go to infinity simultaneously with $n/k$ fixed:

**Theorem**

*Let $\beta$ be a positive real number with $\beta \le 1/2$. For $k = \beta n + o(n)$, we have*

$$\operatorname{Hdepth}_1 M(n,k) = (1-\gamma)n + o(n), \qquad \text{as } n \to \infty,$$

*where $\gamma$ is the smallest nonnegative solution of the equation*

$$\frac{(\alpha+\gamma)^{\alpha+\gamma}(\alpha+\beta)^{\alpha+\beta}(1-\alpha-\beta-\gamma)^{1-\alpha-\beta-\gamma}}{\alpha^\alpha\beta^\beta\gamma^\gamma(1-\beta)^{1-\beta}(1-\gamma)^{1-\gamma}} = 1,$$

*with*

$$\alpha = \frac{1}{4}\Big(1-2\beta-2\gamma+\sqrt{(1-2\beta-2\gamma)^2-8\beta\gamma}\Big).$$

Again, the quality of the asymptotic estimate is surprising:

| $n$ | $k$ | $\lfloor (n+k)/2 \rfloor$ | Hdepth | asympt bd | $n - \left\lceil \frac{(n-k)}{(k+1} \right\rceil$ |
|---|---|---|---|---|---|
| 48 | 12 | 30 | 44 | 46 | 45 |
| 100 | 25 | 62 | 94 | 96 | 97 |
| 200 | 50 | 125 | 190 | 192 | 197 |
| 500 | 125 | 312 | 479 | 481 | 497 |
| 1000 | 250 | 625 | 960 | 962 | 997 |