Sagbi combinatorics of maximal minors and a Sagbi algorithm

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Report on a joint project with

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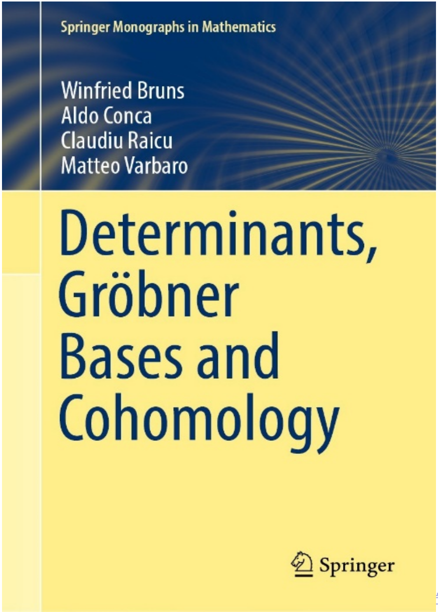
Commercial


Springer Monographs in Mathematics
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Determinants, Gröbner Bases and Cohomology
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Normaliz 3.10.0

So far Normaliz could only compute invariants of normal monomial algebras A: the set of exponent vectors of the monomials in A is the intersection of a cone and a lattice. This allows algorithms based on triangulations and Stanley decompositions.

Version 3.7.10 has added functions for arbitrary monomial algebras (corresponding to arbitrary monoids of exponent vectors):

- Hilbert basis (minimal generating subset of the given generators)
- check for normality
- Markov basis (set of generators) and Gröbner basis for the defining binomial ideal
- Hilbert series
- singular locus
- automorphism groups

Sagbi bases

More fashionable name: *Khovanskii bases*

Let $R = K[X_1, \dots, X_u]$ be a polynomial ring over the field K, endowed with some monomial (or term) order $<$. Let $A \subset R$ be a K-subalgebra. The initial algebra is the vector space

$$\mathrm{in}(A) = \mathrm{in}_<(A) = \sum_{f \in A} K \, \mathrm{in}_<(f).$$

It is automatically a subalgebra. Introduced by Robbiano–Sweedler and Kapur–Madlener $\sim$ 1989. Sagbi $=$ Subalgebra analog of Gröbner bases for ideals.

General problem: $\mathrm{in}(A)$ ned not be finitely generated, even if A is.
General advantage: $\mathrm{in}(A)$ is generated by monomials.

$B \subset A$ is a Sagbi basis if the monomials $\mathrm{in}(f)$, $f \in B$, generate $\mathrm{in}(A)$.

If $\mathrm{in}(A)$ is finitely generated, then it is a toric deformation of A.

The Grassmannian

Let K be a field, $X = (X_{ij})$ be an $m \times n$, $m \le n$, matrix of indeterminates and $R = K[X] = K[X_{ij} : i = 1, \dots, m, j = 1, \dots n]$. Set

$$\mathcal{M} = \mathcal{M}_{m \times n}\{\delta : \delta \text{ is an } m\text{-minor of } X\}$$

The homogeneous coordinate ring of the Grassmannian $G(m, n)$ is the subalgebra

$$K[\mathcal{M}] = K[\mathcal{M}_{mxn}] \subset R.$$

The best monomial orders on R for the exploration of $K[\mathcal{M}]$ are the diagonal ones: $\operatorname{in}(\delta)$ is product of the diagonal elements of δ.

Hodge's standard bitableaux theory $\Longrightarrow$ the maximal minors form a Sagbi basis w.r.t. a diagonal order.

For a diagonal order, $\operatorname{in}(A)$ has all good properties that one can reasonably want: normal, Cohen-Macaulay, Gorenstein, Koszul, rational singularities, ... $\Longrightarrow$ the same for $K[\mathcal{M}]$.

Univeersality ?

By a theorem of Bernstein–Sturmfels–Zelevinsky $\mathcal{M}$ is a universal Gröbner basis of the ideal $I_m(X) \subset R$ generated by $\mathcal{M}$. (Now a simple proof by Conca, De Negri and Gorla.) Universal means: for every monomial order on R.

Question: is $\mathcal{M}$ a universal Sagbi basis of $K[\mathcal{M}]$?
In other words, $K[\text{in}(\mathcal{M})] = \text{in}(K[\mathcal{M}])$? for all monomial orders?
(Note: always $K[\text{in}(\mathcal{M})] \subset \text{in}(K[\mathcal{M}])$.

True for $m = 2$. The answer "no" for $m = 3$ was given by Speyer–Sturmfels (2004): already for 3×6 there exist lexicographic orders for which the m-minors are not a Sagbi basis.

Our starting question: are the m-minors a universally revlex Sagbi basis? What can we say about $K[\text{in}(\mathcal{M})]$ and $\text{in}(K[\mathcal{M}])$?

Experimental approach via CoCoA, Singular and Normaliz.

Findings so far

Test for $K[\mathrm{in}(\mathcal{M})] = \mathrm{in}(K[\mathcal{M}])$: equality of Hilbert series
(necessary: equality of multiplicities, faster to test)

true, false

Universally revlex Sagbi basis: 3×6, 3×7, 3×8.

$K[\mathrm{in}(\mathcal{M})]$ normal
revlex: 3×6, 3×7, 3×8, 3×9
lex: 3×6, 3×7, 3×8, 3×9, 3×10.

$K[\mathrm{in}(\mathcal{M})] = \mathrm{in}(K[\mathcal{M}])$ and not normal: revlex 3×9

Main question: $\mathrm{in}(K[\mathcal{M}])$ finitely generated ???

Refined question: compare $\mathcal{R}(\mathrm{in}(\mathcal{M}))$ and $\mathrm{in}(\mathcal{R}(\mathcal{M}))$.

Experimental approaches

Two experimental approaches: fix format, choose lex or revlex and

1. run a C++ program that creates all candidates and checks them through libnormaliz,
2. create many random orders of the variables and run Singular with the help of Normaliz through normaliz.lib.

To check whether $\mathrm{in}(K[\mathcal{M}]) = K[\mathrm{in}(\mathcal{M})]$ or $\mathrm{in}(K[\mathcal{M}]) = K[E]$ for a set $E \supset \mathrm{in}(\mathcal{M})$ we compare Hilbert series:

- $\mathrm{in}(K[\mathcal{M}])$ has the same Hilbert series as the Grassmannian and can easily be computed by Normaliz since $\mathrm{in}(K[\mathcal{M}])$ is a normal monoid algebra for a diagonal order.
- as long as $K[\mathrm{in}(\mathcal{M})]$ or $K[E]$ is normal, its Hilbert series can also be computed quickly. In the nonnormal case a Gröbner basis of a binomial ideal is needed.

Creating all candidates

It is impossible to run through all monomial orders, lex or revlex: for a 3×6 matrix we have 18! orders of each kind, and even if one takes symmetries into account, there remain too many.

Instead we reverse the process: we select a *matching field*, i.e. a choice of one monomial in each of the minors, and check them for compatibility with lex, revlex or weight orders.

Helpful fact, proved by Sturmfels and Zelevinsky: if the matching field comes from a monomial order, then each row has $m - 1$ entries that do not appear in any initial monomial.

Monomials in the matching field are chosen by tree search: after the choice of n monomials check whether it is lex/revlex compatible. If not go one step back, and pick the next monomial for palce n. If yes, choose the $(n+1)$st monomial.

Tête-a- tête and subduction

For the computation of Sagbi bases one has an algorithm (which need not stop) similar to the Buchberger algorithm for Gröbner bases. Let $\mathcal{F} = \{F_1, \dots, F_n\}$ be a set of *monic* polynomials in the polynomial ring R with a monomial order.

Two operations are needed (terminology of Robbiano–Sweedler):

- (S-polynomial equivalent): find monomials M_1, M_2 in n new variables such that $\mathrm{in}(M_1(F_1, ..., F_n)) = \mathrm{in}(M_2(F_1, ..., F_n))$: the "virtual initial monomial" of $M_1(\mathcal{F}) - M_2(\mathcal{F})$ cancels. ((M_1, M_2) or $M_1 - M_2$ is a tête-a- tête)
- (reduction equivalent): for a monic polynomial $G \in R$ find a monomial M in n variables such that $\mathrm{in}(G) = \mathrm{in}(M(\mathcal{F}))$ and pass to $G - M(\mathcal{F})$ ($M(\mathcal{F})$ subduces G).

Sagbi criteria

Theorem

$\mathcal{F} \subset A$ is a Sagbi basis of A if every tête-a-tête of $\mathcal{F}$ can be lifted to a polynomial relation of $\mathcal{F}$.

Potential application: replace the computation of a defining ideal of A via Gröbner methods by Sagbi and bookkeeping.

Theorem

Let A be the subalgebra generated by homogeneous $\mathcal{F}$ and $A' = K[\mathrm{in}(\mathcal{F})]$. Then

$$\mathrm{in}(A) = A' \qquad \Longleftrightarrow \qquad H_A(t) = H_{A'}(t)$$

where $H_{\dots}(t)$ is the Hilbert series.

$\Longrightarrow$ $H_A(t)$ can control the Sagbi computation.

Application: compute $A' = \mathrm{in}(A)$ and then the Hilbert series of A as the Hilbert series of the monomial algebra A'.

Three variants of the Sagbi algorithm

We have implemented three variants in a Singular library, and Singular falls back on Normaliz for the combinatorial computations.

(Gen) Starting from a system of generators of a subalgebra $A \subset R$, a suitable iteration of tête-a-tête and subduction computes a finite Sagbi basis, provided such exists. It stops if saturation is reached or a preset number of rounds has been completed.

(Deg) It requires a grading, and then proceeds degree by degree. Stops if saturation is reached or a preset degree bound.

(Hilb) It requires the Hiolbert series of A. Details below.

A Hilbert series controlled Sagbi algorithm

Basic assumption: A is a graded subalgebra of a polynomial ring R over a field K, generated by a set G of monic homogeneous polynomials. G is completely subduced, and the Hilbert function $\mathrm{HF}(A, k)$, $k \in \mathbb{N}$, has been computed.

1. Let $B = K[\mathrm{in}(G)]$ and Compute $\mathrm{HF}(B, \ )$.
2. If $\mathrm{HF}(B, k) = \mathrm{HF}(A, k)$ for all k, stop and output G.
3. Otherwise find the smallest degree c for which $\mathrm{HF}(B, c) \neq \mathrm{HF}(A, c)$ and the defect $d = \mathrm{HF}(A, c) - \mathrm{HF}(B, c)$.
4. Find the the degree c tête-a-têtes of G and evaluate them on G to obtain a system T of polynomials. Augment G by T.
5. Check whether $\mathrm{in}(G)$ has d new monomials. If so, go to (1).
6. Otherwise apply subduction to G, augment it and go to (5).

Note: The subduction loop (5)–(6) must stop after finitely many iterations with d new initial monomials.

Discussion

The implementation makes sure that at each step (1) we have a system of generators G of A that is completely subduced: the initial monomials are a minimal system of generators of $B = K[\text{in}(G)]$.

Most critical steps:

- The Gröbner basis computation of the binomial defining ideal of $K[B]$, needed always for tête-a-tête and Hilbert function in the nonnormal case.
  We use a reimplemantation of the project-and-lift algorithm of Hemmecke and Malkin (4ti2).
- Evaluation of monomials M on the system G. For example: degree 11 monomial evaluated on the 84 maxinal minors of a 3×9 matrix. (A degree bound must be set.)

At present we use "broad" evaluation of tête-a-têtes and subduction. Better than go polynomial by polynomial.

Sagbi packages in standard distributions

1. The computations with CoCoA 5 use a variation of the script developed by Anna Bigatti. It realizes (Deg). Will be part of the standard distribution of CoCoA from version 5.4.2 on.
2. The Macaulay2 distribution contains the package SubalgebraBases.m2 by Burr and Duff, version 1.1. It realizes the variant (Deg) and allows a degree bound.
3. The Singular distribution contains the library sagbi.lib by Hackfeld, Pfister and Levandovskyy, version 4.1.1.0. It offers only the variant (Gen) with an optional bound on the number of rounds.

There is now a new M2 package by Burr, Clarke, Duff, Leaman, Nichols and Walker `arXiv:2302.12473`. It appeared after our paper had been submitted.

Test suite

- ($\mathtt{HK}_0$) the subalgebra of $K[X, Y, Z]$ generated by X^6, X^5Y, Y^5Z, XZ^5, $Y^6 + Y^3Z^3$. Han and Kwak found it as a simple counterexample to the Eisenbud–Goto conjecture. char $K = 0$. Monomial order is degrevlex.
- ($\mathtt{HK}_2$) The same, but over a field of characteristic 2.
- ($\mathtt{Pow}_l$) The subalgebra of $K[X, Y, Z]$ generated by the polynomials $X^6 + Y^6 + Z^6$, $X^7 + Y^7 + Z^7$, $X^8 + Y^8 + Z^8$. char $K = 0$, order lex.
- ($\mathtt{Pow}_r$) The same as ($\mathtt{Pow}_l$), but order degrevlex.
- ($\mathtt{2x2}_0$) The subalgebra of $K[X_{ij} : i, j = 1, \dots, 4]$ generated by the 2-minors. The monomial order is diagonal. char $K = 0$.
- ($\mathtt{2x2}_2$) The same, but over a field of characteristic 2.
- ($\mathtt{3x6}$) coordinate ring of Grassmannian $G(3, 6)$, nondiagonal lex order. char $K = 0$.
- ($\mathtt{3x7}$) $G(3, 7)$, nondiagonal lex order.
- ($\mathtt{3x8}$) $G(3, 8)$, nondiagonal lex order.
- ($\mathtt{3x9}_l$) $G(3, 9)$, nondiagonal lex order.
- ($\mathtt{3x9}_r$) $G(3, 9)$, nondiagonal degrevlex order.

Computation times I

	normalized degree				times in minutes			
ecample	bound	Sagbi	(Deg)	#Sagbi	(Deg)	(Hilb)	CoCoA5	M2
$(\mathtt{HK}_0)$	16	—	—	80	1:13.68	1:18.77	3:29.00	357:10.3
$(\mathtt{HK}_2)$	16	—	—	16	0:00.85	0:00.86	0:00.54	0:04.84
$(\mathtt{Pow}_l)$	200	—	—	28	5:57.17	1:14.51	21:27.51	—
$(\mathtt{Pow}_r)$	200	—	—	46	5:31.09	3:26.11	78:58.44	E
$(\mathtt{2x2}_0)$	10	3	7	89	0:39.27	0:11.61	56:17.28	—
$(\mathtt{2x2}_2)$	15	6	13	130	5:15.64	O	931:31.00	—
(3x6)	10	2	4	21	0:00.41	0:00.38	0:0 .40	0:01.46
(3x7)	10	2	4	37	0:01.37	0:00.76	0:08.95	2:44.32
(3x8)	10	3	6	67	0:08.20	0:03.21	0:44.84	H
$(\mathtt{3x9}_l)$	10	3	6	101	0:48.26	0:26.20	14:36.46	—
$(\mathtt{3x9}_r)$	10	7	8	90	M	0:54.52	—	—

M memory overflow, O int overflow, H heap overflow, E M2 error code

Computation times II

	rounds			times in min	
example	bound	Sagbi	#Sagbi	(Gen)	sagbi.lib
$(2x2_0)$	10	3	89	1:18.89	T
$(3x6)$	10	2	21	0:00.53	0:00.22
$(3x7)$	10	2	37	0:01.75	F
$(3x8)$	10	3	65	0:16.31	—
$(3x9_l)$	10	3	101	1:56.62	—
$(3x9_d)$	10	1	84	0:12.20	T

T time $>$ 1 hour, F segmentation fault in Singular

Hardware: Dell xps17 with an Intel i7-11800H at 2.3 GHz, 32 GB.
For CoCoA5 MacBook Pro with an Intel Quad-Core i7 at 2.3 GHz.
xps 17 $=$ 0.73 Macbook.