Products of Borel fixed ideals of maximal minors

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Preprints

Products of Borel fixed ideals of maximal minors (arXiv:1601.03987)

Linear resolutions of powers and products (arXiv:1602.07996)

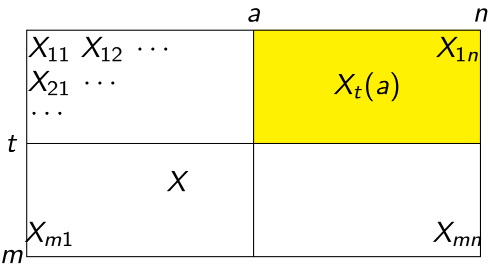
Northeast ideals of maximal minors

K a field, $R = K[x_{ij} : 1 \le i \le m,\ 1 \le j \le n]$, $X = (x_{ij})$.

For $a + t \le n + 1$, the ideal $I_t(a)$ is generated by the t-minors of the northeast submatrix

$$X_t(a) = (x_{ij} : 1 \le i \le t,\ a \le j \le n).$$

$I_t(a)$ is a (row oriented) northeast ideal of maximal minors.



Minors and tableaux

All t-minors involved have the first t rows of X as their rows. Therefore we only need the column indices to denote them:

$$[a_1 \dots a_t] = \det(X_{ia_i} : i = 1, \dots, t).$$

We assume $a_1 < \dots a_t$.

For example, $[1 \dots t]$ is just the determinant of the $t \times t$-submatrix in the left upper corner of X. $I_t(a)$ is generated by all minors $[a_1 \dots a_t]$ with $a_1 \geq a$.

Products of minors will be called tableaux.

The essential point in working with ideals of miors is to single out a class of K-linearly independent tableaux. Usually one chooses the standard (bi)tableaux, but they are not sufficient for our results.

Borel fixedness, diagonal monomial orders

$B_m \subset \mathrm{GL}(m.K)$ Borel subgroup of lower triangular matrices,
$B'_n \subset \mathrm{GL}(n, K)$ Borel subgroup of upper triangular matrices.

$I_t(a)$ fixed by the action of $B_m \times B'_n$ on R via linear substitution: it is a Borel fixed ideal of maximal minors.

Why northeast? Because we want to work with a diagonal monomial order for which the leading monomial of a minor is the product of the diagonal elements, for example the lex order induced by

$$X_{11} > \cdots > X_{1n} > X_{21} > \cdots > X_{2n} > \cdots > X_{mn}.$$

(Used in proofs.) We fix a diagonal monomial order.

In a diagonal monomial order we have

$$\mathrm{in}([a_1 \ldots a_t]) = X_{1a_1} \cdots X_{ta_t}.$$

The main theorem

Theorem

Let $I_{t_1}(a_1), \dots, I_{t_w}(a_w)$ be northeast ideals of maximal minors, and let I be their product. Then

1. *I has a linear resolution.*
2. $\mathrm{in}(I) = \mathrm{in}(I_{t_1}(a_1)) \cdots \mathrm{in}(I_{t_w}(a_w))$, *and the natural generators of I form a Gröbner basis.*
3. *I is integrally closed, and it has a primary decomposition by powers of ideals $I_t(a)$ for various values of t and a.*

Moreover, the multi-Rees algebra associated to the family of ideals $I_t(a)$ is Koszul, Cohen-Macaulay and normal.

In view of statement (1) we say, that the family $(I_t(a))$ has linear products.

The archetype

Theorem

1. *The powers of $I_m(X)$ have a linear resolution (Akin-Buchsbaum-Weyman): $I_m(X)$ has linear powers.*
2. *They are primary and integrally closed (Trung).*
3. $\mathrm{in}(I_m(X)^k) = \mathrm{in}(I_m(X))^k$ *for all k, and the natural generators of $I_m(X)^k$ form a Gröbner basis (Conca).*
4. *The Rees algebra of $I_m(X)$ is Cohen-Macaulay normal (Eisenbud-Huneke) and Koszul.*

This is everything we want to prove for the powers of a single $I_t(a)$.

Intermediate step: products $I_{t_1}(1) \cdots I_{t_w}(1)$ (Berget-B-Conca).

Roughly speaking, standard bitableaux and the KRS correspondence are enough for this case. The general case of northeast ideals is combinatorially harder.

Remarkable classes of ideals

All the following classes generalize the class of principal stable monmial ideals:

1. polymatrodidal (monomial) ideals (Herzog, Hibi, Vladoiu)
2. ideals generated by linear forms (Conca-Herzog)
3. northeast ideals of maximal minors

They share the following properties:

1. linear products
2. "good" primary dedecompositions of products by ordinary powers of primes (with multiplicities given by valuations)
3. normal and Cohen-Macaulay multi-Rees algebras of "fiber type" defined by low degree relations.

The crucial intersection formulas

Let $S = \big((t_1, a_1), \dots, (t_w, a_w)\big)$. Set $J_t(a) = \mathrm{in}(I_t(a))$,

$$I_S = I_{t_1}(a_1) \cdots I_{t_w}(a_w) \quad \text{and} \quad J_S = \mathrm{in}(I_S).$$

Note: $b \le a_i$ and $u \le t_i \iff I_{t_i}(a_i) \subset I_u(b)$. Set

$$e_{ub}(S) = \big|\{i : b \le a_i \text{ and } u \le t_i\}\big|.$$

Theorem

$$J_{t_1}(a_1) \cdots J_{t_w}(a_w) = J_S = \bigcap_{u,b} J_u(b)^{e_{ub}(S)} \qquad (1)$$

$$I_S = \bigcap_{u,b} I_u(b)^{e_{ub}(S)}. \qquad (2)$$

Equation (2) gives a primary decomposition of I_S. The ideals I_S and J_S are integrally closed.

NE-canonical decomposition

The crucial inclusion is

$$\bigcap_{u,b} J_u(b)^{e_{ub}(S)} \subset J_{t_1}(a_1)\cdots J_{t_w}(a_w).$$

Everything else follows from easy arguments.

The inclusion means: a monomial that contains $e_u(b)$ diagonals

$$X_{1j_1}\cdots X_{uj_u}, \qquad b \le j_1 < \cdots < j_u \le n,$$

of "type (u,b)" for all u and b can be factored in a "NE-canonical" way (depending on S!) that fits the decomposition $J_{t_1}(a_1)\cdots J_{t_w}(a_w)$.

It leads to a "NE canonical" representation "of pattern S" of elements in $I_{t_1}(a_1)\cdots I_{t_w}(a_w)$, generalizing the straightening law. Standard tableaux are generalized to "NE-canonical tableaux of pattern S".

NE-canonical tableaux by an example

The NE canonical factorization lets us find a NE-canonical tableaux $\Delta \in I_S$ for a given monomial M such that $M = \operatorname{in}(\Delta)$.

Consider $M = x_{11}x_{12}x_{13}x_{23}x_{24}x_{25}x_{35}$, symbolized by the table

•	•	•		
		•	•	•
				•

It depends on the pattern S which S-canonical tableau has M as its initial monomial.

1. For $S = \big((2,1),(3,2),(2,2)\big)$ the canonical tableau with initial monomial M is
$$[13][245][35].$$
2. For $S = \big((2,1),(2,2),(3,3)\big)$ it is
$$[13][25][345].$$

The NE straightening law

Theorem

Let $S = \big((t_1, a_1), \dots, (t_w, a_w)\big)$ *be a NE-pattern and* $x \in I_S$*. then there exist uniquely determined* S*-canonical NE-tableaux* $M_i\Gamma_i$*,* $i = 0, \dots, p$*, and coefficients* $\lambda_i \in K$ *such that*

$$x = \lambda_0 M_0\Gamma_0 + \lambda_1 M_1\Gamma_1 + \cdots + \lambda_p M_p\Gamma_p$$

and

$$\operatorname{in}(x) = \operatorname{in}(M_0\Gamma_0) > \operatorname{in}(M_1\Gamma_1) > \cdots > \operatorname{in}(M_p\Gamma_p).$$

The multi-Rees algebra

The natural object for the simultaneous investigation of the products $I_{t_1}(a_1)\cdots I_{t_w}(a_w)$ is the multi-Rees algebra defined by the ideals $I_t(a)$:

$$\mathcal{R} = R\bigl(I_t(a) : \text{all } (t,a)\bigr)$$
$$= R[I_t(a)T_{ta} : \text{all } (t,a)] \subset R[T_{ta} : \text{all } (t,a)]$$

It is naturally $\mathbb{Z}^{1+N}$-graded, $N = \#\{\text{all } (t,a)\}$. Since every $I_t(a)$ is generated in a single degree, it is also naturally $\mathbb{Z}$-graded.

It is useful to define partial Castelnuovo-Mumford regularities with respect to the $1+N$ partial degrees. We are mainly intersted in the the 0-th partial degree coming from R and the corresponding regularity reg_0

The theorems of Blum and Römer

Theorem

Let R be a standard graded polynomial ring over the field K. The family $I_1, \dots, I_w$ of ideals in R has linear products if and only if $\operatorname{reg}_0(R(I_1, \dots, I_w)) = 0$.

Implication $\Longleftarrow$ due to Römer, $\Longrightarrow$ by B-Conca-Varbaro.

Theorem

Let $R = K[X_1, \dots, X_n]$ and $I_1, \dots, I_w$ ideals of R such that $R(I_1, \dots, I_w)$ is Koszul. Then the family $I_1, \dots, I_w$ has linear products.

Due to Blum. His theorem actually says more. Roughly speaking, diagonal submodules over diagonal subalgebras of multigraded Koszul algebras have linear resolutions.

Normality and Cohen-Macaulayness

Extend the monomial order from R to

$$R[T_{ta} : \text{all } (t,a)] = K[X, T_{ta} : \text{all } (t,a)].$$

As a subalgebra of of this polynomial ring, $\mathcal{R}$ has a well-defined initial subalgebra $\text{in}(\mathcal{R})$ (generated by a Sagbi basis).

Recall that

$$\text{in}(I_{t_1}(a)) \cdots \text{in}(I_{t_w}(a_w)) = \text{in}\big(I_{t_1}(a)) \cdots I_{t_w}(a_w)\big)$$

This implies

$$\text{in}(\mathcal{R}) = R(\text{in}(I_t(a))) : \text{all } (t,a))$$

Theorem

1. $\text{in}(\mathcal{R})$ *and* $\mathcal{R}$ *are normal.*
2. *Both are Cohen-Macaulay.*

Linear resolutions

Write $\mathcal{R}$ (and/or in($\mathcal{R}$)) as a residue class ring of a polynomial ring $\mathcal{S}$ over K:

$$\Phi : \mathcal{S} \to \mathcal{R}.$$

The "NE straightening law" fits a monomial order on $\mathcal{S}$ that is lifted from $\mathcal{R}$ via Φ with a reverse-lexicographic "tie breaker".

Thus we get

Theorem

1. in($\mathcal{R}$) *and* $\mathcal{R}$ *are defined by Gröbner bases of quadrics.*
2. *Both are Koszul algebras.*
3. *All products* $I_{t_1}(a_1)\cdots I_{t_w}(a_w)$ *and their initial ideals have linear resolutions.*

The essential point: the rewriting of the initial of a tableau in NE canonical decomposition can be done in steps representing degree 2 relations.

Some Gorenstein and some factorial rings

Theorem

Let $I_{t_1}(a_1) \subset I_{t_2}(a_2) \subset \cdots \subset I_{t_p}(a_p)$ such that $\operatorname{ht} I_{t_1}(a_1) = 1$ or 2 and $\operatorname{ht} I_{t_i}(a_i) = 1 + \operatorname{ht} I_{t_{i-1}}(a_{i-1})$ for $i = 2, \dots, p$.

Then the multi-Rees algebra $R\big(I_{t_1}(a_1), \dots, I_{t_p}(a_p)\big)$ is Gorenstein and normal with divisor class group $\mathbb{Z}^{p-1}$ or $\mathbb{Z}^p$, depending on whether $a_1 = n - t_1$ or $a_1 = n - t + 1$.

The proof uses a theorem of Herzog and Vasconcelos. Alternative: toric arguments.

Theorem

Let $t_1 < \cdots < t_p$ and $a_1 \geq \cdots \geq a_p$ and $I_i = I_{t_i}(a_i)$ for $i = 1, \dots, p$. Then the multi-fiber ring $F(I_1, \dots, I_p)$ is factorial and therefore Gorenstein.