Polyhedral computations in social choice

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Mathematics in elections

Mathematical models can be applied to two types of elections.

1. Elections of persons: president, mayor, dean etc. Goal: election of the candidate with the highest degree of general approval.
2. Elections of parliaments: parliament of state, city council, university senate etc. Goal: fair representation of "parties".

The theory of fair representation has mainly been driven by the a priori distribution of seats in the US house of representatives to the states.

We will be concerned with the election of persons.

William V. Gehrlein and Dominique Lepelley represent the aspects of social choice that we discuss today.

Preference rankings and the election result

The basic assumption in the mathematics of social choice is the existence of individual preference rankings: every voter ranks the candidates in linear order. (It would be possible to allow indifferences.)

We use capital letters for the candidates. Examples for three candidates:

$$A \succ B \succ C$$

$$C \succ A \succ B$$

For n candidates there exist $N = n!$ preference rankings, usually numbered in lexicographic order.

The result of the election is the N-tuple

$$(v_1, \dots, v_N), \qquad v_i = \#\{\text{voters of preference ranking } i\}.$$

$n = 3, N = 6,\ n = 4, N = 24,\ n = 5, N = 120 \dots$

Impartial Anonymous Culture

In the following we want to compute probabilities of certain events related to election schemes. This requires a probability distribution on the set of election results. We fix the number k of voters.

The Impartial Anonymous Culture (IAC) is the equdistribution on the set of election results: every election result is assumed to have equal probability.

This model does *not* treat the voters as independent individuals. If every voter rolls a dice to choose his/her preference ranking, then the resulting probability distribution of election results is the multinomial distribution!

The IAC lacks certain properties that one would intuitively expect. For example, ignoring one candidate does not map IAC(n) to IAC($n-1$).

The Condorcet paradox

The *Marquis de Condorcet* (1743–1794) was a leading intellectual in France before and during the revolution. He already observed that there is no ideal election scheme and suggested solutions.

We say that candidate A beats candidate B in majority, $A >_M B$, if

$$\#\{\text{voters with } A \succ B\} > \#\{\text{voters with } B \succ A\}.$$

The Condorcet winner (CW) beats all other candidates in majority.

If a single person is to be elected, then there is general agreement that the CW is the person with the largest common approval.

Condorcet realized that a CW need not exist: the relation $>_M$ is not transitive. This phenomenon is called the Condorcet paradox.

It is our guiding example.

Quantization

From a quantitative viewpoint, the most ambitious goal is:

- Given the number of voters k, find the exact number of election results exhibiting the Condorcet paradox (or the opposite).

It is clear that for large k, this number is gigantic. For large k, it is more significant to understand the behavior for $k \to \infty$:

- What is the probability that an election result exhibits the Condorcet paradox?

Since we assume the IAC, this probability is

$$\lim_{k \to \infty} \frac{\#\{\text{electionresults without CW for } k \text{ voters}\}}{\#\{\text{all election results for } k \text{ voters}\}}$$

We can provide exact answers for ≤ 4 candidates, but 5 candidates are out of reach.

Inequalities for the Condorcet winner

An election result for 3 candidates in tabular form:

number of voters	x_{ABC}	x_{ACB}	x_{BAC}	x_{BCA}	x_{CAB}	x_{CBA}
	A	A	B	B	C	C
ranking	B	C	A	C	A	B
	C	B	C	A	B	A

A is the CW if

$$\begin{aligned} A >_M B: &\quad x_{ABC} + x_{ACB} + x_{CAB} > x_{BAC} + x_{BCA} + x_{CBA}, \\ A >_M C: &\quad x_{ABC} + x_{ACB} + x_{BAC} > x_{BCA} + x_{CAB} + x_{CBA}. \end{aligned}$$

If we are only interested in probabilities for $k \to \infty$, we can allow ties and repkace $>$ by $\geq$. Recall: k is always the number of voters.

Lattice points in cones

The election results $(x_1, \dots, x_6)$ are the lattice points (points with integral coordinates) in the positive orthant $\mathbb{R}^6_+$.

The inequalities for "A is the CW" (with ties allowed) cut out a polyhedral cone C from the positive orthant $\mathbb{R}^6_+$.

Our goals:

1. Given the number k of voters, find the number $H(C,k)$ of election results that have A as the Condorcet winner.
2. Less ambitious: Find the asymptotic behavior of $H(C,k)$ as $k \to \infty$.

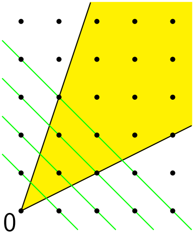
The function $(x_1, \dots, x_6) \mapsto x_1 + \dots + x_6 = k$ is a grading on C, and our goals require the degree wise counting of lattice points in a polyhedral cone.

The Hilbert series

A grading on the (rational polyhedral) cone $C \subset \mathbb{R}^d$ is a surjective $\mathbb{Z}$-linear form $\deg : \mathbb{Z}^d \to \mathbb{Z}$ such that $\deg(x) > 0$ for $x \in C \cap \mathbb{Z}^d$, $x \neq 0$. For simplicity we assume $\dim C = d$.

The Hilbert (or Ehrhart) function is given by

$$H(C,k) = \#\{x \in C \cap \mathbb{Z}^d : \deg x = k\}$$



and the Hilbert (Ehrhart) series is

$$H_C(t) = \sum_{k=0}^{\infty} H(C,k)t^k.$$

Theorem (Hilbert-Serre, Ehrhart)

- *$H_C(t)$ is a rational function.*
- *$H(C,k)$ is a quasi-polynomial of degree $d-1$ for $k \geq 0$.*

The asymptotic behavior

For large numbers of voters we want to find to find the probability of a certain event E, given by the lattice points in a cone $C \subset \mathbb{R}^d$. Because of IAC we can define it by

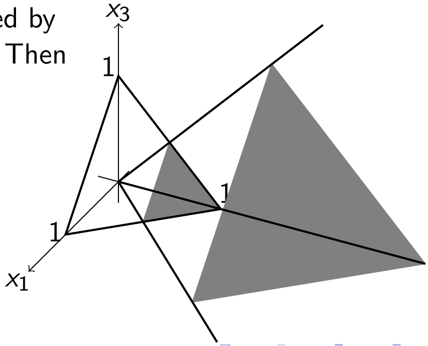
$$\operatorname{prob}(E) = \lim_{k\to\infty} \frac{\#\{\text{degree } k \text{ latt points in } C\}}{\#\{\text{degree } k \text{ latt points in } \mathbb{R}^d_+\}} = \lim_{k\to\infty} \frac{H(C,k)}{\binom{N+k-1}{N-1}}.$$

Let $\mathcal{U}$ be the unit simplex spanned by the unit vectors and $P = \mathcal{U} \cap C$. Then

$$\operatorname{prob}(E) = \frac{\operatorname{vol} P}{\operatorname{vol} \mathcal{U}}.$$

Algebraic formulation:

$$\operatorname{prob}(E) = \operatorname{mult}(C).$$



Symmetrization

Schürmann noticed that the computation of the election models can often be accelerated enormously if one uses symmetrization.

If some preference rankings appear with the same coefficient in all inequalities (depending on the inequality), then one can replace them by their sum.

This amounts to a projection of our cone C onto a cone D of lower dimension. Gain: The combinatorial structure of D is therefore simpler. Loss: lattice points y in D must be counted with their numbers of preimages, given by a polynomial f in y in this case.

Hilbert series $\to$ weighted Ehrhart series with weight f
Volume $\to$ Lebesgue integral of highest homogeneous comp. of f

Our experience: symmetrization pays off.

Commercial break

Normaliz is a tool for computations in discrete convex geometry:

- convex hulls and dual cones
- conversion from generators to constraints and vice versa
- triangulations, disjoint decompositions and Stanley decompositions
- Hilbert basis of rational, not necessarily pointed cones
- normalization of affine monoids
- lattice points of rational polytopes and (unbounded) polyhedra
- Hilbert (or Ehrhart) series and (quasi) polynomials under $\mathbb{Z}$-gradings
- weighted Ehrhart series and Lebesgue integrals of polynomials over rational polytopes

For polynomial arithmetic Normaliz employs CoCoALib.

The Condorcet paradox again

From now on: the election involves $n = 4$ candidates.

The Condorcet paradox is a very easy computation for Normaliz, with or without symmetrization:

$$p(\text{Condorcet paradox}) = \frac{331}{2048} \approx 0.162$$

This was first computed by Gehrlein (2000). Also the Hilbert series is easy:

$$H_C(t) = \frac{1 + 5t^1 + 133t^2 + 363t^3 + \cdots + 481t^{38} + 15t^{39} + 6t^{40}}{(1-t)(1-t^2)^{14}(1-t^4)^9}$$

Without ties, a variant of Ehrhart reciprocity or direct computation yields:

$$\frac{6t^1 + 15t^2 + 481t^3 + 890t^4 + \cdots + 133t^{39} + 5t^{40} + t^{41}}{(1-t)(1-t^2)^{14}(1-t^4)^9}$$

Condorcet efficiency of plurality voting

The election of the Condorcet winner lacks universality, and the probability of failure is significantly > 0. (Whether an election that counts all preference rankings is feasible, is another question.) The easiest way out: plurality voting. That is: the person with the largest number of first places is elected, or the candidates are even ranked by the number of first places.

A measure for the quality of an election scheme is its Condorcet efficiency:

$$\mathrm{CE(scheme)} = \frac{\text{prob(scheme elects CW)}}{\text{prob(CW exists.)}}$$

For plurality one gets:

$$\mathrm{CE(plurality)} = \frac{10658098255011916449318509}{14352135440302080000000000} \approx 0.7426$$

Runoff

Many election schemes use two rounds: a second ballot if the plurality winner has $\leq 50\%$ first places. In the runoff the two candidates with the highest numbers of first places run against each other. This clearly improves the Condorcet efficiency since the CW wins the second round, provided he/she is at least 2nd in the first round:

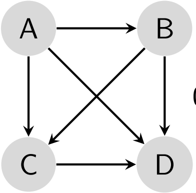
$$\mathrm{CE}(\text{runoff}) = \frac{19627224002877404784030049}{21528203160453120000000000} \approx 0.9117$$

This is a significant increase, justifying the effort of the runoff. On the other hand, one can ask for the probability that the winner of the first round also wins the runoff:

$$\text{prob(1st round winner wins runoff)} = \frac{9185069468583833}{12173449145352192} \approx 0.7545$$

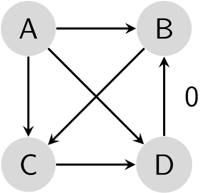
Condorcet patterns

There are 4 isomorphism classes of the majority relation $>_M$. We have computed their probabilities:



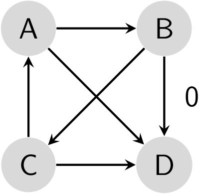
0.7598

linear order



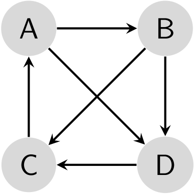
0.07855

CW + 3-cycle



0.07855

3-cycle + CL



0.0831

4-cycle

The Borda paradoxes

The *Chevalier de Borda* (1733 – 1799) was a contemporary of Condorcet. He was particularly concerned about the possibility that plurality voting might elect the worst candidate, the Condorcet loser. We call this result the strong Borda paradox. It occurs with conditional probability

$$\frac{\text{prob(CL is PW)}}{\text{prob(CL exists)}} \approx 0.02268.$$

The reverse strong Borda paradox makes the CW the loser of plurality. Its conditional probability is

$$\frac{\text{prob(CW is PL)}}{\text{prob(CW exists)}} \approx 0.02379.$$

Finally the strict Borda paradox: $>_M$ gives a linear order and plurality reverses it:

$$\frac{\text{prob(Plurality reverses } >_M)}{\text{prob(} >_M \text{ linear)}} \approx 0.00156.$$

Computation times I

For probabilities = (weighted) volumes of polytopes:

	Symmetrize	Laptop 4x	Server 30x
Condorcet paradox	Yes	0.100 s	0.591 s
Plur vs cutoff	Yes	0.33 s	0.76 s
Cond Eff of Plur	Yes	1:11:39 h	8:41 m
Cond Eff of Runoff	Yes	1:48:57 h	15:06 m
$>_M$ linear	No	7.200 s	10.455 s
4-cycle	Yes	0.660 s	1.940 s
strict Borda	No	–	3:57:26 h
strong Borda	Yes	14:51 m	1:39 m
rev strong Borda	Yes	44:54 m	4:17 m

4x: parallelization with 4 threads, 30x: with 30 threads.

Computation times II

For (weighted)Hilbert series:

	Symm	Laptop 4x		Server 30x	
		closed	semi-open	closed	semi-open
Cond paradox	Yes	1.730 s	1.940 s	1.925 s	2.077 s
Plur vs cutoff	Yes	4.400 s	7.64 s	7.010 s	8.440 s
Cond Eff of Plur	Yes	4:50:55 h	4:45:24 h	28:36 m	41:01 m
Cond Eff Runoff	Yes	12:02:42 h	12:47:03 h	1:45:15 h	1:39:19 h
$>_M$ linear	No	16.230 s	28.260 s	24.050 s	34.136 s
4-cycle	Yes	16.770 s	25.810 s	3.156 s	7.967 s
strict Borda	No	–	–	10:08:50 h	37:03:26 h
strong Borda	Yes	1:34:23 h	1:36:16 h	9:01 m	14:13 m
rev strong Borda	Yes	5:56:18 h	5:53:38 h	45:13 m	47:22 m

closed: with ties, semi-open: without ties

Normaliz can do all computations without symmetrization.

References

We are listing only our relevant own publications. They contain numerous references to the social choice literature.

- W. Bruns, B. Ichim and C. Söger, *The power of pyramid decomposition in Normaliz.* J. Symb. Comp. 74 (2016), 513 – 536.
- W. Bruns, B. Ichim and C. Söger, *Computations of volumes and Ehrhart series in four candidates elections*. Preprint arXiv:1704.00153.
- W. Bruns and C. Söger, *Generalized Ehrhart series and Integration in Normaliz*. J. Symb. Comp. 68 (2015), 75–86.