Relations of minors

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Joint work with

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The Grassmannian

$G(m,n)$ Grassmannian of m-dimensional subspaces of K^n with the Plücker embedding. For its homogeneous coordinate ring choose

- X as an $m \times n$ matrix of indeterminates, $m \le n$,
- $K[X]$ as the polynomial ring $K[X_{ij} : 1 \le i \le m,\ 1 \le j \le n]$,
- $G(X)$ as the subalgebra of $K[X]$ generated by the maximal minors of X: it is the coordinate ring of $G(m,n) \subset \mathbb{P}(\bigwedge^m K^n)$.

Theorem

The ideal of polynomials vanishing on $G(m,n)$ is generated in degree 2 by the Plücker relations.

Simplest Plücker relation: $X = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \end{pmatrix}$,

$$[12][34] - [13][24] + [14][23] = 0, \qquad [ij] = \det \begin{pmatrix} X_{1i} & X_{1j} \\ X_{2i} & X_{2j} \end{pmatrix}.$$

And what for $t < m$?

Let $V = K^m$, $W = K^n$, consider

$$\Lambda_t \;:\; \mathrm{Hom}_{\mathbb{k}}(W, V) \to \mathrm{Hom}_{\mathbb{k}}\Big(\bigwedge^t W, \bigwedge^t V\Big), \quad \Lambda_t(\phi) = \bigwedge^t \phi,$$

and let

$$\mathbb{V}_t = \mathbb{V}_t(m,n) = \text{Zariski closure of } \operatorname{Im} \Lambda_t.$$

Then the coordinate ring is

$$A_t = A_t(m,n) = K[t\text{-minors of } X] \subset K[X].$$

with the grading in which t-minors have degree 1. Notation for t-minors:

$$[i_1, \dots, i_t | j_1, \dots, j_t] = \det \begin{pmatrix} X_{i_1,j_1} & \dots & X_{i_1,j_t} \\ \vdots & & \vdots \\ X_{i_t,j_1} & \dots & X_{i_t,j_t} \end{pmatrix}$$

Properties of A_t

- $\dim A_t = mn$ if $t < m$

If $\operatorname{char} K = 0$ or $> \min(t, m-t, n-t)$ (*non-exceptional*), then

- A_t has a basis of standard monomials (De Concini-Eisenbud-Procesi, B-Vetter)
- A_t factorial $\iff$ $t = 1$ or $t = m$ (Grassmannian) or $t = m - 1 = n - 1$ (**excluded from now on**)
- is a normal Cohen-Macaulay domain, $\operatorname{Cl}(A_t) = \mathbb{Z}$ (B-Conca 2001)
- $\operatorname{cl}(\omega_{A_t}) = mn - t(m+n)$
- in particular, A_t Gorenstein $\iff$ $\dfrac{1}{t} = \dfrac{1}{m} + \dfrac{1}{n}$.

Main tool: Gröbner basis theory for ideals I_t^j based on KRS $\to$ Sagbi bases $\to$ toric deformation.

Note: depth $A_2(4,4) = 1$ if $\operatorname{char} K = 2$.

Properties of $\mathbb{V}_t$

$K = \mathbb{C}$ from now on. Set $E = \bigwedge^t V$, $F = \bigwedge^t W$,

$$\operatorname{Hom}_{\Bbbk}(W, V) = W^* \otimes V \qquad \operatorname{Hom}_{\Bbbk}(F, E) = F^* \otimes E$$

$$\Lambda_t \ : \ W^* \otimes V \to F^* \otimes E = (\bigwedge W^* \otimes \bigwedge V)_{(t,t)}$$

$$\Lambda_t(\phi) = \bigwedge^t \phi = \phi^t \qquad \mathbb{V}_t = \operatorname{closure}(\operatorname{Im} \Lambda_t)$$

$GL(V) \times GL(W^*)$ acts naturally on A_t and $\mathbb{V}_t$.

- $\mathbb{V}_t$ decomposes into $t(m-t)+2$ $GL(V) \times GL(W^*)$-orbits classified by rank and "small rank" (B-Conca 2009)
- $\Lambda_t^{-1}(\Lambda_t(x)) = \langle \zeta_t \rangle x$ if $\operatorname{rank} x > t$, and $SL(t, K)$ if $\operatorname{rank} x = t$
- $\operatorname{Sing} \mathbb{V}_t = \{y \in \mathbb{V}_t : \operatorname{rank} y \le 1\} \cong \operatorname{cone}(G(t,m) \times G(t,n))$
- $\bigcup_t \mathbb{V}_t = (\bigcup_t \operatorname{Im} \Lambda_t) \cdot (W^* \times V)$

And the equations?

What are the polynomials vanishing on $\mathbb{V}_t$? What are the relations between the t-minors if $t < m$?

Degree 2 is not enough: for $m = 3$, $n = 4$, $t = 2$

$$\det \begin{pmatrix} [12|12] & [12|13] & [12|14] \\ [13|12] & [13|13] & [13|14] \\ [23|12] & [23|13] & [23|14] \end{pmatrix} = 0$$

In fact: $x_1, x_2, x_3, x_4 \in K^3 \implies x_1 \wedge x_2, x_1 \wedge x_3, x_1 \wedge x_4$ linearly dependent.

More difficult to see: this degree 3 relation is not in the ideal generated by the degree 2 relations.

Hilbert series arguments show that we need degree 3 relations.

Going one step to the left

We want to understand the defining ideal of A_t as a residue class ring of a polynomial ring. Note

$$(A_t)_1 = E \otimes F^*, \qquad E = \bigwedge^t V,\ F = \bigwedge^t W.$$

So we have a natural projection $\operatorname{Sym}(E \otimes F^*) \to A_t$ and want to understand its kernel $J_t = J_t(m, n)$. Go one step back:

$$\bigotimes(E \otimes F^*) \to \operatorname{Sym}(E \otimes F^*) \to A_t$$

The natural homomorphisms are $\operatorname{GL}(V) \times \operatorname{GL}(W^*)$-equivariant. From the viewpoint of representation theory of $\operatorname{GL}(V) \times \operatorname{GL}(W^*)$

computable $\to$ essentially unknown $\to$ well known

$\bigotimes(E \otimes F^*) = \bigoplus_k \bigotimes^k E \otimes \bigotimes^k F^*$: "separates rows and columns"

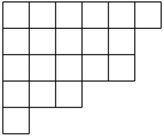
Representations and Young diagrams

Irreducible polynomial representations of $\mathrm{GL}(V)$

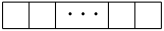
$\leftrightarrow$ partitions $\lambda = (\lambda_1, \dots, \lambda_s)$ with $\lambda_1 \le \dim V$, $\lambda_1 \ge \dots \ge \lambda_s$

$\leftrightarrow$ Young diagrams with first row $\le \dim V$

If $e = \lambda_1 + \dots + \lambda_s$, we write $\lambda \vdash e$. The irreducible representation corresponding to λ is the *Schur module* $L_\lambda V$ with highest weight λ^*.

Example: $(6, 5, 5, 3, 1) =$ 

In our (Italian) convention

$\bigwedge^t V \leftrightarrow (t) \leftrightarrow$ 

Predecessors and successors

Definition

A partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash e$ is *(t-)admissible* if $e = td$ for some d and $k \leq d$.

Definition

An admissible partition $\alpha \vdash td$ is a *(t)-predecessor* of the (admissible) partition $\lambda \vdash t(d+1)$ (or λ is a *(t-)successor* of α) if

$$\alpha_i \leq \lambda_i, \; i \geq 1, \qquad \lambda_i \leq \alpha_{i-1}, \; i \geq 2.$$

.

Example: $t = 2$, $(3,1)$ has successors

$$(5,1), \quad (3,3), \quad (4,2), \quad (4,1,1), \quad (3,2,1)$$

Pieri's rule describes the decomposition of $L_\lambda V \otimes \bigwedge^t V$: if λ is admissible, then

$$L_\lambda V \otimes \bigwedge^t V \cong \bigoplus_\mu L_\mu V \qquad \mu \text{ successor of } \lambda$$

Important consequence: the irreducibles occurring in $\bigotimes^e(E)$, $E = \bigwedge^t V$, are the Schur modules $L_\lambda V$, $\lambda \vdash et$ admissible, $\lambda_1 \le \dim V$

$$G = \mathrm{GL}(V) \times \mathrm{GL}(W^*)$$

Fortunately the representation theory of G results from that of its factors: the irreducible polynomial representations are

$$L_\lambda V \otimes L_\mu W^*,$$

parametrized by *bi-diagrams* $(\lambda|\mu)$, and we an speak of *bi-predecessors*, *bi-successors* etc.

Representation theory of A_t

The *Cauchy rule* says:

Theorem

$$K[X] = \bigoplus L_\lambda V \otimes L_\lambda W^*, \qquad \lambda_1 \le m$$

By De Concini-Eisenbud-Procesi:

$$A_t \cong \bigoplus L_\lambda V \otimes L_\lambda W^*, \qquad \lambda_1 \le m,\ \lambda\ t\text{-admissible}$$

Recall: we want to find the defining ideal of A_t, kernel J_t of $\mathrm{Sym}(E \otimes F^*) \to A_t$.
Consequences:

- symmetric bi-diagrams that have multiplicity 1 in $\mathrm{Sym}(E \otimes F^*)$ do not appear in the ideal J_t of relations
- all asymmetric bi-diagrams in $\mathrm{Sym}(E \otimes F^*)$ belong to J_t

Degree 2

Standard decomposition

$$\bigotimes^2 E = \mathrm{Sym}^2(E) \oplus \bigwedge^2 E,$$

and similarly for $\bigotimes^2 F^*$ and $\bigotimes^2 (E \otimes F^*)$. Consequence:

$$\mathrm{Sym}^2(E \otimes F^*) = \mathrm{Sym}^2(E) \otimes \mathrm{Sym}^2(F^*) \oplus \bigwedge^2 E \otimes \bigwedge^2 F^*$$

Easy to see:

$$\mathrm{Sym}^2(E) = \bigoplus L_\lambda V, \qquad \lambda = (\lambda_1, \lambda_2) \vdash 2t, \ (\lambda_1 - \lambda_2)/2 \text{ even},$$

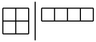
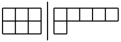
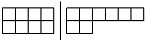
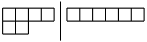
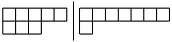
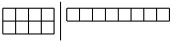
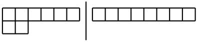
$$\bigwedge^2 E = \bigoplus L_\lambda V, \qquad \lambda = (\lambda_1, \lambda_2) \vdash 2t, \ (\lambda_1 - \lambda_2)/2 \text{ odd},$$

Set $\tau_u = (t+u, t-u)$ Then

$$(J_t)_2 = \bigoplus L_{\tau_u} V \otimes L_{\tau_v} W^*, \qquad u+v \text{ even}, \ u \neq v.$$

For $t = 2$: $(\tau_u|\tau_v)$ is one of $(4|2,2)$, $(2,2|4)$, both Plücker relations.

For $t = 3$: $(5,1|3,3)$, $(3,3|5,1)$ Plücker, $(6|4,2)$, $(4,2|6)$ non-Plücker

	$t=2$	$t=3$	$t=4$	
$(\tau_0\|\tau_2)$				
$(\tau_1\|\tau_3)$				
$(\tau_0\|\tau_4)$				
$(\tau_2\|\tau_4)$				

Relations in degree 2

$L_{\tau_u} V \otimes L_{\tau_v} W^*$ as a G-module generated by highest weight vector

$$\mathbf{f}_{u,v} = \sum_{\substack{I,J\\H,K}} (-1)^{I,J}(-1)^{H,K}[1,\dots,t-u,I|1,\dots,t-v,H]$$
$$[1,\dots,t-u,J|1,\dots,t-v,K]$$

$I \cup J\{t-u+1,\dots,t+u\}$, $H \cup K = \{t-v+1,\dots,t+v\}$

To show: they are invariant under the action of the unipotent subgroup $\mathrm{U}_-(V) \times \mathrm{U}_+(W^*) \subset \mathrm{GL}(V) \times \mathrm{GL}(W^*)$.

Shape relations

Lemma

Suppose that all bi-predecessors $(\alpha|\beta)$ *of* $(\gamma|\lambda)$, $\gamma \neq \lambda$, *in* $\mathrm{Sym}(E \otimes F^*)$ *are symmetric and have multiplicity* 1*. Then* $L_\gamma V \otimes L_\lambda W^* \subset J_t$ *occurs in* $J_t/\mathfrak{m}J_t$, $\mathfrak{m} \subset \mathrm{Sym}(E \otimes F^*)$ *the irrelevant maximal ideal generated by* $E \otimes F^*$.

Definition

$(\gamma|\lambda)$ as in the lemma is called a *shape relation*.

All degree 2 relations are shape relations, and in fact T-shape relations.

Definition

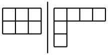
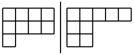
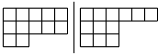
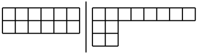
$(\gamma|\lambda)$ is a *T-shape relation* if all its predecessors in $\bigotimes(E \otimes F^*)$ are symmetric and have multiplicity 1.

T-shape relations

Theorem

All T-shape relations have degree 2 or 3, those in degree 3 given by $(\gamma_u|\lambda_u)$, $u = 1, \dots, \lfloor t/2 \rfloor$,

$$\gamma_u = (t+u,\ t+u,\ t-2u),$$
$$\lambda_u = (t+2u,\ t-u,\ t-u).$$

	$t=2$	$t=3$	$t=4$
$(\gamma_1\|\lambda_1)$			
$(\gamma_2\|\lambda_2)$			

Introducing another player

Let H be the "big" group, namely

$$H = \mathrm{GL}(E) \times \mathrm{GL}(F^*)$$

The H-decomposition of $\mathrm{Sym}(E \otimes F^*)$ is well-understood (Cauchy rule), as well as the H-stable ideals in it (De Concini-Eisenbud-Procesi) .
Our ideal J_t is not H-stable, and not even the direct sum of its intersections with the H-irreducibles, but nevertheless H helps. For example

$$\mathrm{Sym}^3(E \otimes F^*) = \bigwedge^3 E \otimes \bigwedge^3 F^* \oplus L_{(2,1)} E \otimes L_{(2,1)} F^* \oplus \mathrm{Sym}^3(E) \otimes \mathrm{Sym}^3(F^*)$$

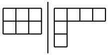
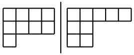
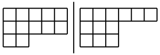
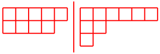
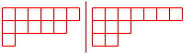
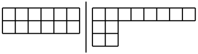
Using it one finds ...

Further shape relations in degree 3

Theorem

The only further shape relations in degree 3 *are of type* $(\rho_u|\sigma_u)$, $u = 2, \dots, \lceil t/2 \rceil$,

$$\rho_u = (t+u,\ t+u-1,\ t-2u+1),$$
$$\sigma_u = (t+2u-1,\ t-u+1,\ t-u).$$

	$t=2$	$t=3$	$t=4$
$(\gamma_1\|\lambda_1)$			
$(\rho_2\|\sigma_2)$			
$(\gamma_2\|\lambda_2)$			

The conjecture

Conjecture

The degree 2 relations and the degree 3 shape relations generate J_t.

Based on computer calculations (Singular, Lie, own programs):

- Conjecture holds for $t = 2$ and if $m \leq 4$ or $m = n = 5$
- No further minimal degree 3 relations for $t = 2$ and $t = 3$
- No minimal generators in degree 4 for $t = 2$
- No further shape relations in the cases: (i) $t = 2, 3$, $d \leq 5$ and (ii) $t = 4, 5$, $d \leq 4$

Regularity doesn't help: $\operatorname{reg} A_t \approx mn - \frac{mn}{t}$.

What to do next

Obstructions to prove the conjecture:

- the plethysm problem—what is the G-decomposition of $\mathrm{Sym}(E \otimes F^*)$?
- lack of understanding the G-ideals in $\mathrm{Sym}(E \otimes F^*)$

Bonus for the patient audience: determinantal type relations

The representation $\bigwedge^k E \otimes \bigwedge^k F^*$ is H-irreducible.By the Cauchy rule $\bigwedge^k E \otimes \bigwedge^k F^*$ appears in $\mathrm{Sym}(E \otimes F^*)$.

Let $\mathcal{U} = \mathrm{U}_-(E) \times \mathrm{U}_+(F^*)$ the unipotent subgroup determined by an order of the standard base elements $e_I = e_{i_1} \wedge \cdots \wedge e_{i_t}$ in $E = \bigwedge^t V$ and similarly for W^* and $F^* = \bigwedge^t W^*$.

The corresponding highest weight vector in $\bigwedge^k E \otimes \bigwedge^k F^*$ is a $k \times k$ determinant $\det(e_I \otimes f_J)$ where e_I runs over the first k base elements in E and f_J over the first k base elements in F^*.

The highest weight vector for $\mathcal{U}$ is also a highest weight vector for $\mathrm{U} = \mathrm{U}_-(V) \times \mathrm{U}_+(W^*)$, provided the homomorphism $G \to H$ satisfies $\mathrm{U} \hookrightarrow \mathcal{U}$.

But the orders for the e_I and f_J are not uniquely determined by the requirement $\mathrm{U} \hookrightarrow \mathcal{U}$. For $t = 2$ we can choose

$$e_1 \wedge e_2 < e_1 \wedge e_3 < e_2 \wedge e_3 < \dots$$
$$f_1 \wedge f_2 < f_1 \wedge f_3 < f_1 \wedge f_4 < \dots$$

This choice gives the determinantal relation we started with, corresponding to the bi-diagram