# The Castelnuovo-Mumford regularity of powers and products

Winfried Bruns

FB Mathematik/Informatik
Universität Osnabrück

wbruns@uos.de

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A remark on regularity

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# Castelnuovo-Mumford regularity

Catelnuovo-Mumford regularity was introduced by Mumford in 1966 for coherent sheaves on projective space. Eisenbud and Goto (1983) translated it into commutative algebra.

**Definition**

Let $R$ be the standard graded $K$-algebra and $M$ a finitely generated graded $R$-module. Then the Castelnuovo-Mumford regularity of $M$ is

$$\operatorname{Reg}_R(M) = \sup\{i + j : H^i_{\mathfrak{m}}(M)_j \neq 0\}$$

where $\mathfrak{m}$ is the irrelevant maximal ideal of $R$.

Always defined and $\neq -\infty$, provided $M \neq 0$.

Essentially independent of $R$: if $S \to R$ is a module-finite extension, then $\operatorname{Reg}_R(M) = \operatorname{Reg}_S(M)$

# Regularity by resolution

A module $M$ as above has a minimal graded free resolution

$$\cdots \to \bigoplus_j R(-j)^{\beta_{ij}} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0j}} \to M \to 0$$

One can base another definition of regularity on it:

**Definition**

$$\operatorname{reg}_R(M) = \sup\{j - i : \beta_{ij}(M) \neq 0\}.$$

$\operatorname{reg}(M)$ is not always finite. But Eisenbud and Goto proved:

**Theorem**

*Let $R = K[x_1, \dots, x_n]$ be the polynomial ring over $K$. Then*

$$\operatorname{reg}_R(M) = \operatorname{Reg}_R(M).$$

# Powers of an ideal

From now on, until further notice, we assume $R = K[x_1, \dots, x_n]$. Let $I$ be a homogeneous ideal. What can be said about $\operatorname{reg}_R(I^a)$?

After some results by Bertram-Ein-Lazarsfeld and Swanson, the following theorem was proved:

**Theorem (Cutcosky-Herzog-Trung 199; Kodiyalam 2000)**

*There exist $\lambda_0, \lambda_1 \in \mathbb{N}$ such that*

$$\operatorname{reg}_R(I^a) = \lambda_0 + \lambda_1 a \qquad \textit{for } a \gg 0.$$

We want to generalize this theorem to products of powers $I_1^{a_1} \cdots I_m^{a_m}$ of homogeneous ideals $I_1, \dots, I_m$.

CHT state en passant: $\operatorname{reg}_R(I_1^{a_1} \cdots I_m^{a_m})$ is linear in $a = (a_1, \dots, a_m)$ for $a \gg 0$ ...

# Notation

It is often more convenient and perhaps necessary to work with the maximal shifts

$$t_i(M) = \sup_j \{\beta_{ij} \neq 0)\}.$$

$t_0(M)$ is the maximal degree of a minimal generator of $M$.

We write

$$I^a = I_1^{a_1} \cdots I_m^{a_m} \qquad \text{for ideals } I_1, \dots, I_m \text{ and } a \in \mathbb{N}^m.$$

Moreover,

$$a \gg 0 \qquad \iff \qquad a_i \gg 0 \text{for all } i$$

# An example

Consider the ideals $I = (x, y^2)$, $J = (x^2, y)$ in the polynomial ring $R = K[x, y]$. For them

$$t_i(I^a J^b) = \begin{cases} \max\{\ 2a+b,\ \ a+2b\ \} + i, & i = 0, 1, (a, b) \neq \{(0, 0),\\ 0, & i = 0, (a, b) = (0, 0). \end{cases}$$

$\Longrightarrow$

$$\operatorname{reg}(I^a J^b) = \max\{\ 2a+b,\ \ a+2b\ \} \text{ for all } (a, b) \in \mathbb{N}^2.$$

First one checks that $I^a J^b$ is minimally generated by

$$\begin{array}{ll} x^a y^b, & x^{a-1}y^{b+2}, x^{a-2}y^{b+4}, \dots, y^{b+2a}, \\ \| & \\ x^a y^b, & x^{a+2}y^{b-1}, x^{a+4}y^{b-2}, \dots, x^{a+2b}. \end{array}$$

This confirms

$$t_0(I^a J^b) = \max\{\ 2a+b,\ \ a+2b\ \} \quad \text{for all } a \in \mathbb{N}^2.$$

# The first syzygy

is given by

$$
\varphi = \begin{pmatrix}
y^2 & -x & & & & & & & & \\
 & y^2 & -x & & & & & & & \\
 & & \ddots & \ddots & & & & & & \\
 & & & y^2 & -x & & & & & \\
x^2 & & & & & -y & & & & \\
 & & & & & x^2 & -y & & & \\
 & & & & & & x^2 & -y & & \\
 & & & & & & & \ddots & \ddots & \\
 & & & & & & & & x^2 & -y
\end{pmatrix}
$$

where the rows are the generators of the syzygy module. Just check the degrees.

# The main result

Let $R = K[X_1, \dots, x_n]$ be standard graded, $I_1, \dots, I_m$ homogeneous ideals, and

$$I_i = (f_{i1}, \dots, f_{ig_i}), \qquad d_{ij} = \deg f_{ij}.$$

**Theorem**

*Then there exist linear functions $L_1, \dots, L_w$,*
$L_k(a) = \lambda_{k0} + (\lambda_{k1}a_1 + \cdots + \lambda_{km}a_m)$, $\lambda_{ki} \in \{d_{i1}, \dots, d_{ig_i}\}$, $\lambda_{k0} \geq 0$,
*such that*

$$\operatorname{reg}_R(I^a) = \max_k L_k(a) \qquad \textit{for all } a \gg 0.$$

The theorem can be extended to all $a \in \mathbb{N}^m$. It is derived from the analogous statement for the individual highest shifts $t_j(I^a)$.

The theorem can be extracted from a paper by Bagheri, Chardin and Tài Hà (2013) as was pointed out to us by Marc Chardin.

# Corollaries

In a special case we indeed have linear behavior:

**Corollary**

*If each ideal $I_i$ is generated in a single degree $d_i$, then there exists $\lambda_0 \geq 0$ such that*

$$\operatorname{reg}_R(I^a) = \lambda_0 + d_1 a_1 + \cdots + d_m a_m, \qquad a \gg 0.$$

**Corollary**

*Let $S$ be an arbitrary standard graded $K$-algebra, and $I_1, \ldots, I_m$ be homogeneous ideals in $S$. Then the theorem and the corollary above hold with $\operatorname{reg}_R$ replaced by $\operatorname{Reg}_S$.*

The last corollary does not follow directly from the theorem, but from a modification of the proof. The same is true for analogous statements on $\operatorname{reg}_R(I^a M)$ or $\operatorname{Reg}_S(I^a M)$ where $M$ is a finitely generated graded module.

# Remarks

(a) The theorem of Cutkosky-Herzog-Trung and Kodiyalam for $m = 1$ follows from the theorem above since for $m = 1$ the maximum of linear functions $L_k(a)$ is one of them for $a \gg 0$.

(b) One can replace the ideals $I_i$ by (minimal) reductions $J_i$, and then the statements hold for $I^a$ with the degrees taken from the generators of the minimal reductions.

The crucial point is that the multi-Rees algebra of $I_1, \dots, I_m$ that we consider below if a finitely generated module multi-Rees algebra of $J_1, \dots, J_m$.

(c) The proof that we sketch now follows CHT, but extends, and simplifies it.

# The multi-Rees algebra

We introduce auxiliary variables $t_1, \dots, t_m$ and set

$$\mathcal{R} = R(I_1, \dots, I_m) = R[I_1 t_1, \dots, I_m t_m] \subset R[t_1, \dots, t_m].$$

This is the multi-Rees algebra of $I_1, \dots, I_m$. It is the right tool for studying all power products $I^a$ simultaneously:

$$\mathcal{R} = \bigoplus_a I^a t^a = \bigoplus_{a_1,\dots,a_m} I_1^{a_1} \cdots I_m^{a_m} t_1^{a_1} \cdots t^{a_m}$$

Note: $\mathcal{R}$ is $\mathbb{Z} \times \mathbb{Z}^m$-graded, with $\mathbb{Z}$ coming from $R$ and $\mathbb{Z}^m$ from $t_1, \dots, t_m$: $\deg f = (\deg_R f, 0)$ for $f \in R$, and $\deg t_i = (0, e_i)$.

Just pretend that $m = 1$.

# The Koszul homology

We want to study

$$\beta_{ij}(I^a) = \dim_K \operatorname{Tor}_i^R(I^a, K)_j.$$

But

$$\operatorname{Tor}_i^R(I^a, K) \cong H_i(I^a) = H_i(I^a, \mathbf{x}), \qquad \mathbf{x} = (x_1, \dots, x_n).$$

The Koszul homologies $H_i(I^a)$ appear as graded pieces of the Koszul homology of $\mathcal{R}$:

$$\bigoplus_a H_i(I^a) = H_i(\mathcal{R}), \qquad H_i(I^a) = H_i(\mathcal{R})_a.$$

This reflects the $\mathbb{Z}^m$-graded structure.

Goal: find the maximal $\mathbb{Z}$-degree $t_i(I^a)$ occurring in $H_i(\mathcal{R})_a$.

# A crucial observation

$H_i(\mathcal{R})$ is a finitely generated $\mathcal{R}$-module. But it is annihilated by $x_1, \dots, x_n \implies$

$H_i(\mathcal{R})$ is a finitely generated module over $\overline{\mathcal{R}} = \mathcal{R}/(x_1, \dots, x_n)$.

Since $\mathcal{R}$ is generated as a $K$-algebra by the $x_i$ and the $f_{ij}t_i$, $\overline{\mathcal{R}}$ is generated by the residue classes $\overline{f_{ij}t_i}$

We now define a polynomial ring mapping onto $\overline{\mathcal{R}}$:

$$A = K[z_{ij} : i = 1, \dots, m, j = \dots, g_i] \to \overline{\mathcal{R}},$$
$$z_{ij} \mapsto \overline{f_{ij}t_i}.$$

It is a homogeneous map of $\mathbb{Z} \times \mathbb{Z}^m$-graded algebras if we set

$$\deg z_{ij} = \deg f_{ij}t_i = (d_{ij}, e_i).$$

# The main lemma

For the remaining argument we can consider a finitely generated $\mathbb{Z} \times \mathbb{Z}^m$-graded $A$-module.

For $a \in \mathbb{Z}^m$ set

$$\rho_M(a) = \sup_i \{(M_{(i,a)} \neq 0\}$$

Then we have a statement like the theorem above:

**Lemma**

*hen there exist linear functions $L_1, \dots, L_w$,*

$L_k(a) = \lambda_{k0} + (\lambda_{k1}a_1 + \cdots + \lambda_{km}a_m)$, $\lambda_{ki} \in \{d_{i1}, \dots, d_{ig_i}\}$, $\lambda_{k0} \geq 0$,

*such that*

$$\rho_M(a) = \max_k L_k(a) \qquad \text{for all } a \gg 0.$$

For the regularity one must go over all Koszul homologies in the place of $M$ and collect all linear functions.

# Sketch of the proof

We want to prove a theorem on $\mathbb{Z} \times \mathbb{Z}^m$-graded Hilbert functions. Therefore we can replace our module by any module with the same Hilbert function.

First step: Filter $M$ by an ascending chain of graded submodules with cyclic quotients.

Second step: Replace every cyclic quotient $A/J$ by $A/\operatorname{in}(J)$ where $\operatorname{in}(J)$ is the initial ideal with respect to some monomial order.

Third step: Filter each $A/\operatorname{in}(J)$ by an ascending chain of monomial ideals whose successive quotients are of type $A/(\text{some } z_{ij})$.

Conclusion: it is essentially enough to consider $A/(\text{some } z_{ij})$. Don't forget the shifts that have been accumulated.

# Final remark

Suppose each ideal is generated in constant degree $d_i$. Then

$$\operatorname{reg}_R(I^a) = \lambda_0 + d_1 a_1 + \cdots + d_m a_m, \qquad a \gg 0.$$

What is $\lambda_0$?

Not hard to see:

$$\operatorname{reg}_R(I^a) \le \operatorname{reg}_x(\mathcal{R}) + d_1 a_1 + \cdots + d_m a_m, \qquad \text{for all } a.$$

First proved by Römer for $m = 1$.

Therefore $\lambda_0 \le \operatorname{reg}_x(\mathcal{R})$, but even for $m = 1$ one can have $\lambda_0 < \operatorname{reg}_x(\mathcal{R})$.

On the other hand: all powers have a linear resolution $\iff$ $\operatorname{reg}_x(\mathcal{R}) = 0$ (B-Conca-Varbaro)