Normal lattice polytopes

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Joint work with

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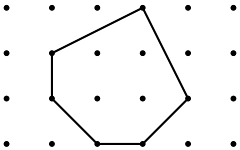
Lattice polytopes

A lattice polytope in $\mathbb{R}^d$ is the convex hull of finitely many lattice points:



We set $\mathrm{L}(P) = P \cap \mathbb{Z}^d$.

$$cP = P + \cdots + P \quad (c \text{ summands})$$

P is normal if

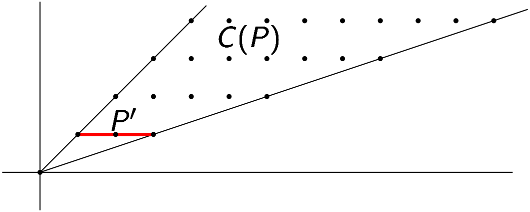
$$\mathrm{L}(cP) = \mathrm{L}(P) + \cdots + \mathrm{L}(P) \qquad (c \text{ summands})$$

This equation means: normal lattice polytopes are the discrete analog of convex polytopes.

Other terminology: P has the integer decomposition property, P is integrally closed

Linearization

The cone $C(P)$ is generated by $P' = P \times \{1\} \subset \mathbb{R}^{d+1}$:



The monoid $C(P) \cap \mathbb{Z}^{d+1}$ is finitely generated by Gordan's lemma.

Proposition

P is normal $\iff$ $\mathrm{L}(P')$ generates $C(P) \cap \mathbb{Z}^{d+1}$.

Easy observation: $P = Q_1 \cup \cdots \cup Q_n$, Q_i normal $\implies$ P is normal

Normality $\implies$ $\mathbb{Z}\mathrm{L}(P') = \mathbb{Z}^{d+1}$.

Simplices

A lattice d-simplex is a lattice polytope S of dimension d with $d+1$ vertices $v_0, \dots . v_d$: a triangle, a tetrahedron,...

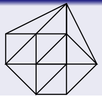
$$S \text{ is unimodular } \iff v_1 - v_0, \dots, v_d - v_0 \text{ are a } \mathbb{Z}\text{-basis of } \mathbb{Z}^d$$
$$\iff \operatorname{vol}(S) = 1/d!.$$

A unimoduler simplex is evidently normal.

S is empty if the vertices are the only lattice points of S.

Theorem

1. *An empty 2-simplex is unimodular.*
2. *Every lattice 2-polytope is normal.*



Proof.

(1) By Pick's formula, an empty 2-simplex has area $1/2$.
(2) Every lattice polytope has a triangulation into empty simplices. □

In general $(d-1)P$ is always normal, $d = \dim P$.

Quantum jumps and partial order

Let $\mathrm{NPol}(d)$ be the set of normal lattice d-polytopes.

A pair (P, Q), $P, Q \in \mathrm{NPol}(d)$, is a quantum jump if $P \subset Q$ and $\#\mathrm{L}(Q) = 1 + \#\mathrm{L}(P)$.

$P < Q \iff$ there are $Q_0, \dots, Q_n$ in $\mathrm{NPol}(d)$ such that $P = Q_0, \dots, Q_n = Q$ and (Q_i, Q_{i+1}) a quantum jump for every i.

Question

Is $<$ the same as $\subset$? Do there exist nontrivial minimal and/or maximal elements in $\mathrm{NPol}(d)$ *?*

Theorem

In $\mathrm{NPol}(2)$ *one has* $P < Q \iff P \subset Q$.

In dim > 3 there exist nontrivial minimal elements: $< \neq \subset$.
Dim 3: nontrivial minimal elements?? Nevertheless $< \neq \subset$.

Characterizations of normality?

Definition

A lattice d-polytope P has

(UC) $\iff$ P is the union of unimodular simplices (Unimodular Cover);

(ICP) $\iff$ $\mathbb{Z}\mathrm{L}(P') = \mathbb{Z}^{d+1}$ and for every $x \in \mathbb{Z}_+\mathrm{L}(P')$ there exist $x_1, \dots, x_{d+1} \in \mathrm{L}(P')$, such that $x = a_1 x_1 + \cdots + a_{d+1} x_{d+1}$, $a_i \in \mathbb{Z}_+$ (Integral Carathéodory Property).

Clearly: (UC) $\implies$ (ICP) & normality.

More difficult: (ICP) $\implies$ normality (B.-Gubeladze)

Question (Sebő)

Does normality imply (UC) or at least (ICP) ?

No characterization of normality

Theorem

Suppose P is a counterexample of minimal dimension d to (UC) or (ICP). Then there exists a descending chain in $\mathrm{NPol}(d)$ down from P to a minimal counterexample with respect to $<$.

So, to find a counterexample, start from some "randomly" chosen polytope, descend from P in $\mathrm{NPol}(d)$ and hope to end at a counterexample. This search strategy was indeed successful:

There exist normal 5-polytopes P_5 and $Q_5 \supset P_5$ with 10 and 12 lattice points, resp., both minimal in $\mathrm{NPol}(5)$, such that

- (B.-Gubeladze, 1998) P_5 fails (UC),
- (Henk-Martin-Weismantel) P_5 fails (ICP),
- (B.,2006) Q_5 fails (UC), but has (ICP).

Open problems on covering and normality

The coordinates of P_5:

$$\begin{aligned} z_1 &= (0, 1, 0, 0, 0, 0), & z_6 &= (1, 0, 2, 1, 1, 2),\\ z_2 &= (0, 0, 1, 0, 0, 0), & z_7 &= (1, 2, 0, 2, 1, 1),\\ z_3 &= (0, 0, 0, 1, 0, 0), & z_8 &= (1, 1, 2, 0, 2, 1),\\ z_4 &= (0, 0, 0, 0, 1, 0), & z_9 &= (1, 1, 1, 2, 0, 2),\\ z_5 &= (0, 0, 0, 0, 0, 1), & z_{10} &= (1, 2, 1, 1, 2, 0). \end{aligned}$$

Question

1. *Does (UC) or at least (ICP) hold in dimension 3 or 4?*
2. *Does every counterexample inherit the failure of (UC) or (ICP) from P_5?*

The height of quantum jumps

Question

Do maximal polytopes exist?

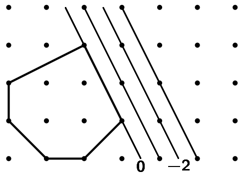
Let (P, Q) be a quantum jump, $z \in \mathrm{L}(Q) \setminus \mathrm{L}(P)$. Then z is also called a quantum jump over P. How far can it be from P? How close is the nearest jump?

Let F be a facet of P. Then $\mathrm{ht}_F : \mathbb{Z}^d \to \mathbb{Z}$ is the unique surjective affine linear function that vanishes on F and is ≥ 0 on $\mathrm{L}(P)$.

F is visible from z if $\mathrm{ht}_F(z) < 0$.

$\mathrm{ht}_P(z) = \max_{F \text{ visible}}\{|\mathrm{ht}_F(z)|\}$

$\mathrm{width}_F P = \max_{x \in P}\{\mathrm{ht}_F(x)\}$



Dimension 2

In dimension 2 the situation is again simple, at least globally:

Proposition

1. *If* $\mathrm{ht}_P(z) = 1$*, then* z *is a jump over* P*.*
2. *The converse holds in dimension 2.*

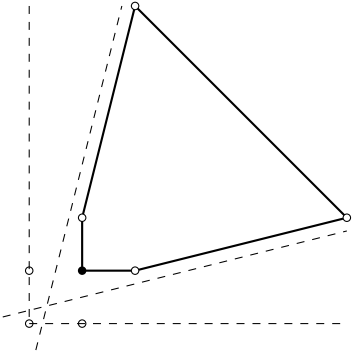
Locally the situation is more complicated, even in dimension 2. Let us say that vertex $x \in P$ is **dark** if it is not visible from a jump.

Proposition

For every n *there exists a 2-polytope* P *with* n *adjacent dark vertices.*

A dark vertex

The dashed lines indicate ht -1 over the facets parallel to them. A point illuminating the origin must have coordinates $(-1, m)$ or $(m, -1)$. Each of them is excluded by one of the other facets.



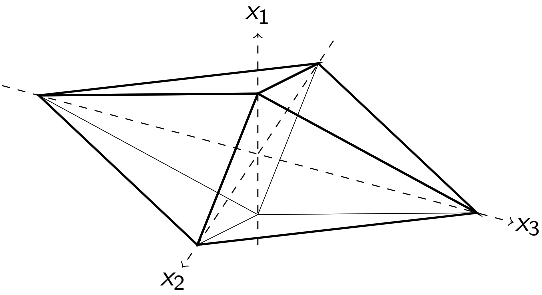
No dim-uniform bound

Theorem

For every n there exists a normal 3-polytope P such that

1. *there is no lattice point of height $< n$ over P;*
2. *there exists a jump z of height n over P.*

One can take the cross-polytope with half axes $n, n+1, n^2+n+1$:



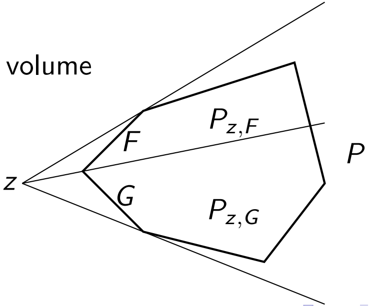
Dimension 3

Theorem

$P \subset Q$ lattice 3-polytopes, $P \in \mathrm{NPol}(3)$, $\#\mathrm{L}(Q) = \#\mathrm{L}(P) + 1$, $z \in \mathrm{L}(Q) \setminus \mathrm{L}(P)$. Then the following are equivalent:

1. *z is a quantum jump over P.*
2. *For each facet F of P visible from z, $P_{z,F}$ contains exactly $\mu(F)$ lattice points y such that $\mathrm{ht}_F(y) = j$, $1 \leq j < |\mathrm{ht}_F(z)|$.*

$\mu(F)$ = multiplicity of F
= lattice normalized volume



The height bound

Theorem

Let z be a quantum jump over P . Then

$$|\operatorname{ht}_F(z)| \le 1 + (d-2)\operatorname{width}_F P$$

for every facet F of P that is visible from z.

Theorem

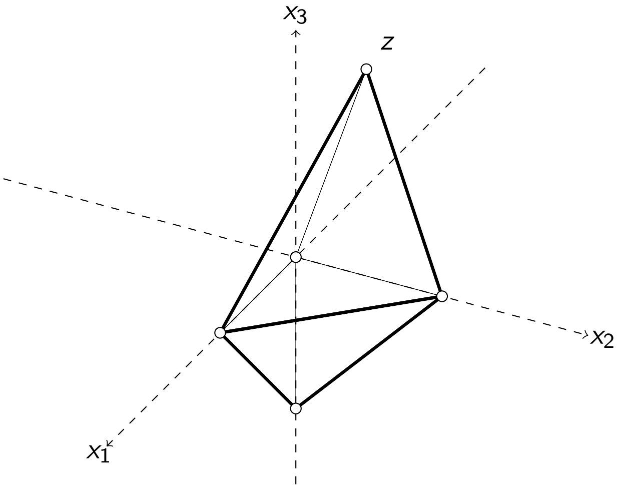
For all $d \ge 2$ and $w \ge 1$ there exists a quantum jump (P, Q) in $\operatorname{NPol}(d)$ *such that:*

1. $z \in \operatorname{L}(Q) \setminus \operatorname{L}(P)$ *is visible from exactly one facet* $F \subset P$,
2. $\operatorname{width}_F P = w$,
3. $|\operatorname{ht}_F(z)| = 1 + (d-2)w$.

A jump of extreme height

$P = \operatorname{conv}(0, \mathbf{e}_1, \dots, \mathbf{e}_{d-1}, -w\mathbf{e}_d)$, $z = (1, \dots, 1, (d-2)w + 1)$

$d = 3$, $w = 1$:



Maximal polytopes

We are not sure in dimension 3, but there is a good chance that maximal polytopes do not exist:

Theorem

No simplex in dimension 3 *is maximal.*

We have no construction that produces a maximal polytope in dimension $d + 1$ from one in dimension d, but there is no doubt that maximal polytopes exist in dimensions ≥ 4:

Theorem

There exist maximal polytopes, even simplices, in dimensions 4 *and* 5*.*

The theorem is based on examples found by a computer search.

A maximal 4-simplex

The simplex with vertices

$$(0,3,2,0)\ (1,1,3,2)\ (2,3,0,4)\ (4,0,0,2)\ (4,4,4,2)$$

is maximal. In order to verify maximality one computes all 125852 lattice points satisfying the height bound, and checks that none of them is a jump. This takes about 2 minutes.

Our program `quantum` does the search and verifies that a potentially maximal polytope is indeed maximal. It uses the library interface of Normaliz.

Search strategies

After long experimenting we found two successful strategies:

1. Start from a "random" normal polytope and extend it successively in such a way that the new polytope has a chance to be maximal. Stop when a maximal polytope is reached or some size bound is reached, and start again.
2. Make a "random" simplex and test it for maximality. If it is not maximal, test the next simplex.

It was a complete surprise that (2) works. We tried it after (1) had found a maximal 5-simplex in dimension 5. One must test MANY polytopes and (2) is fast: mass production beats sophistication.

For (1), all "pure" extension strategies have failed. The following has turned out optimal: if P allows a height 1 jump, take it. If not take the jump for which the new polytope maximizes the average facet multiplicity. Also successful: maximize $\mathrm{vol}(Q \setminus P)$.

Final questions

Question

Do there exist maximal and nontrivial minimal elements in $\mathrm{NPol}(3)$*?*

Question

Do there exist isolated points in $\mathrm{NPol}(d)$*, i. e., polytopes that are both minimal and maximal?*