Normaliz: algorithms for rational cones and affine monoids

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Overview

1. introductory examples
2. basic definitions and basic theorems
3. tasks of Normaliz
4. the Normaliz algorithm for Hilbert bases
5. extensions and modifications
6. mastered challenges
7. platforms and systems
8. contributors and references

Affine Monoids

Definition

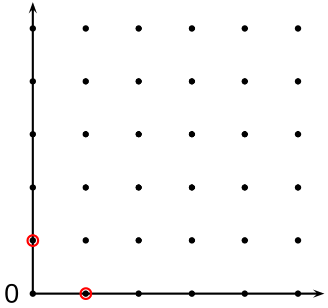
An affine monoid is a finitely generated submonoid of a lattice $\mathbb{Z}^d$, i.e.,

- $M \subset \mathbb{Z}^d$, $M + M \subset M$, $0 \in M$,
- there exist $x_1, \dots, x_n \in M$ such that

$$M = \{a_1 x_1 + \cdots + a_n x_n : a_i \in \mathbb{Z}_+\}.$$

M is positive if $x, -x \in M \Rightarrow x = 0$.

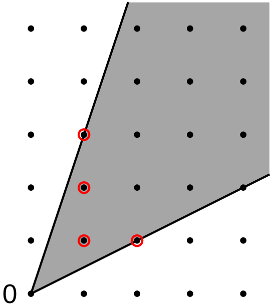
A trivial example



$$M = \mathbb{Z}_+^2$$

(unique) minimal system of generators given by $(1,0),(0,1)$

A not so trivial example



$$M = \{x \in \mathbb{Z}^2 : x \leq 2y,\ 3x \geq y\}$$

(unique) minimal system of generators given by

$$(2,1),\ (1,1),\ (1,2),\ (1,3)$$

Normaliz computes monoids that arise as intersections of cones and lattices (as the examples above):

Definition

A (rational) cone C is a subset

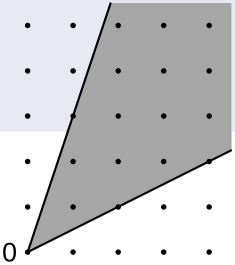
$$C = \{a_1 x_1 + \cdots + a_n x_n : a_1, \ldots, a_n \in \mathbb{R}_+\}$$

with a generating system $x_1, \ldots, x_n \in \mathbb{Z}^d$.

C pointed $\iff (x, -x \in C \Rightarrow x = 0)$.

(For us) a lattice is a subgroup of $\mathbb{Z}^d$.

We will often assume $L = \mathbb{Z}^d$
C pointed and $\dim C = d$.



Basic theorems: : the support hyperplanes

By the theorem of Minkowski-Weyl finitely generated cones can be described by finitely many inequalities:

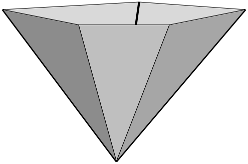
Theorem

For $C \subset \mathbb{R}^d$ the following are equivalent:

- *there are $x_1, \dots, x_n$ such that $C = \{\sum a_i x_i : a_i \mathbb{R}_+\}$;*
- *there are $\lambda_1, \dots, \lambda_s \in (\mathbb{R}^d)^*$ such that $C = \{x \in \mathbb{R}^d : \lambda_i(x) \geq 0\}$.*

In the case $\dim C = d$ (to which we have restricted ourselves) $\lambda_1, \dots, \lambda_s$ for minimal s define the support hyperplanes

$$H_i = \{x : \lambda_i(x) = 0\}$$



and the facets $F_i = C \cap H_i$ of C.

Basic theorems: the Hilbert basis

$x \in M$ is irreducible if $x = y + z \Rightarrow x = 0$ or $y = 0$.

Theorem (van der Corput)

Let M be a positive affine monoid.

- *every element of M is a sum of irreducible elements.*
- *M has only finitely many irreducible elements.*
- *The irreducible elements form the unique minimal system of generators $\mathrm{Hilb}(M)$ of M, the Hilbert basis.*

In particular, monoids of type $C \cap \mathbb{Z}^d$ (C pointed, rational) have a unique minimal finite system of generators, often called $\mathrm{Hilb}(C)$.

Basic theorems: Gordan's lemma

Theorem (Gordan)

Let $C \subset \mathbb{R}^d$ be the cone generated by $x_1, \dots, x_n \in \mathbb{Z}^d$. Then $C \cap \mathbb{Z}^d$ is an affine monoid.

Proof.

Let $y \in C \cap \mathbb{Z}^d$. Then there exist $a_i \in \mathbb{R}_+$ such that

$$y = a_1 x_1 + \dots + a_n x_n.$$

Write $a_i = b_i + q_i$ mit $b_i \in \mathbb{Z}_+$ and $0 \le q_i < 1$. Then

$$y = b_1 x_1 + \dots + b_n x_n + z, \quad z = q_1 x_1 + \dots + q_n x_n \in C \cap \mathbb{Z}^d.$$

Therefore the monoid $C \cap \mathbb{Z}^d$ is generated by $x_1, \dots, x_n$ and the finite set

$$\mathbb{Z}^d \cap \{q_1 x_1 + \dots + q_n x_n : 0 \le q_i < 1\}$$

□

The tasks of Normaliz: Hilbert basis

Normaliz computes (together with other data)

$$\mathrm{Hilb}(C \cap L)$$

Cones C and lattices L can be specified by

- generators $x_1, \dots, x_n \in \mathbb{Z}^d$,
- constraints: homogeneous systems of diophantine linear inequalities, equations and congruences,
- relations: binomial equations.

Normaliz has two algorithms for Hilbert bases: (1) the original Normaliz algorithm, and (2) a variant of an algorithm due to Pottier.

We concentrate on generators as input and the algorithm (1).

The tasks of Normaliz: Hilbert series

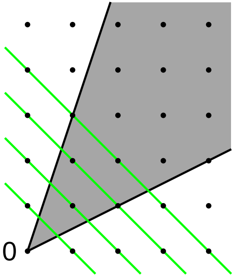
A grading on M is a surjective $\mathbb{Z}$-linear form $\deg : \mathrm{gp}(M) \to \mathbb{Z}$ such that $\deg(x) > 0$ for $x \in M$, $x \neq 0$

The Hilbert (or Ehrhart) function is given by

$$H(M,k) = \#\{x \in M : \deg x = k\}$$

and the Hilbert (Ehrhart) series is

$$H_M(t) = \sum_{k=0}^{\infty} H(M,k)t^k.$$



Theorem (Hilbert-Serre, Ehrhart)

- *$H_M(t)$ is a rational function*
- *$H(M,k)$ is a quasi-polynomial for $k \geq 0$*

System of generators and reduction

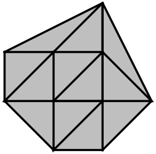
Most likely every algorithm for the computation of Hilbert bases needs two phases:

- the determination of system of generators E,
- the reduction of E to the Hilbert basis.

We need two auxiliary, interleaved steps:

- the computation of the support hyperplanes of the cone,
- a triangulation of the cone.

A triangulation is a decomposition into simplicial cones: $C = \bigcup_{\sigma \in \Sigma} C_\sigma$.



A cone is simplicial if it is generated by linearly independent vectors.

Support hyperplanes: Fourier-Motzkin elimination

This is an incremental algorithm that builds a cone by successive extending the system of generators and determining the support hyperplanes in this process.

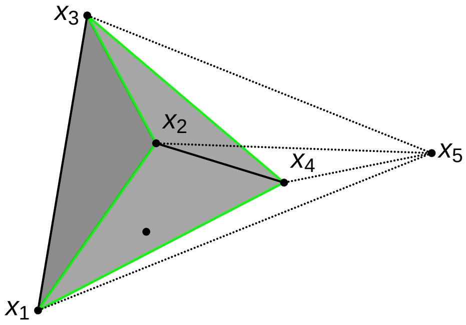
Start: We may assume that $x_1, \dots, x_d$ are linearly independent. The computation of the support hyperplanes is then simply the inversion of the matrix with rows $x_1, \dots, x_d$. (In principle superfluous.)

Extension: we add $x_{d+1}, \dots, x_n$ successively: from the support hyperplanes of $C' = \mathbb{R}_+ x_1 + \cdots + \mathbb{R}_+ x_{n-1}$ we must compute the support hyperplanes of $C = C' + \mathbb{R}_+ x_n$.

We describe this process geometrically.

We determine the boundary V of the part of C' that is visible from x_n and its decomposition into subfacets. Together with x_n these span the new facets of The facets of C' that are visible from x_n ($\lambda_i(x_n) < 0$), are discarded.

In the cross-section of a 4-dimensional cone:

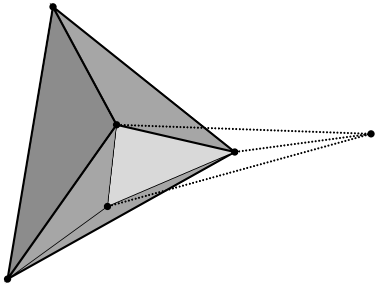


The main problem: find V.

Incremental triangulation

It follows the same inductive scheme, interleaved with Fourier-Moatzkin elimination: we obtain a triangulation of C if we extend the triangulation of C' by the simplicial cones that are spanned by x_n and the facets visible from x_n.

In the cross-section of a 4-dimensional cone:



Main problem: triangulation may be very large. (Ways out: pyramid decomposition and partial triangulation.)

Reduction

The reduction is in principle very simple if one knows the support hyperplanes (or rather the linear forms $\lambda_1, \dots, \lambda_s$).

Theorem

Let E be a system of generators of the positive normal affine monoid M. An element is $x \in M$ reducible if and only if there exists $y \in E$, $y \neq x$, such that $\lambda_i(x - y) \geq 0$ for $i = 1, \dots, s$.

Evidently true, since for $x, y \in \mathbb{Z}^d$ one has $x - y \in C \cap \mathbb{Z}^d$ if and only if $\lambda_i(x - y) \geq 0$ for $i = 1, \dots, s$.

Main problems:

- E is very large and many comparisons are necessary. In this case a sophisticated implementation can help to find y quickly.
- s is very large.

The steps of the algorithm

The (primal) Normaliz algorithm runs as follows:

1. Input of $x_1, \dots, x_n$, preparatory coordinate transformation.
2. Computation of the support hyperplanes, interleaved with the
3. computation of the triangulation Σ.
4. For every cone $C_\sigma \in \Sigma$
   - computation of a system of generators $C_\sigma \cap \mathbb{Z}^d$ $(*)$ and
   - its reduction to the Hilbert basis HB_σ
5. reduction of $\bigcup_{\sigma\in\Sigma} \mathrm{HB}_\sigma$ to $\mathrm{Hilb}(C \cap \mathbb{Z}^d)$.
6. inverse coordinate transformation.

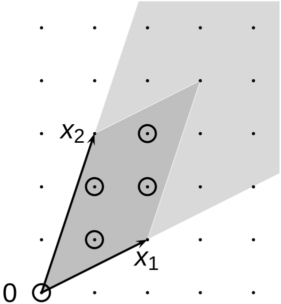
Only step $(*)$ has not yet been explained. Or has it?

Simplicial cones

Let $x_1, \dots, x_d$ be linearly independent and $C = \mathbb{R}_+ x_1 + \dots \mathbb{R}_+ x_d$. In the proof of Gordan's lemma we have learnt:

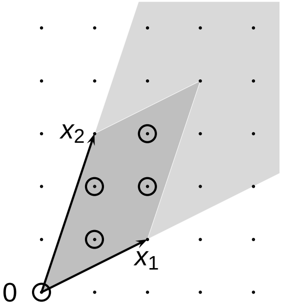
$$E = \{q_1 x_1 + \dots + q_d x_d : 0 \le q_i < 1\} \cap \mathbb{Z}^d$$

together with $x_1, \dots, x_d$ generate the monoid $C \cap \mathbb{Z}^d$.



Easy to see:

Every residue class in $\mathbb{Z}^d/U$, $U = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_d$, has exactly one representative in U.



Representatives of residue classes can be quickly computed ($x_1, \ldots, x_d \to$ triangular matrix) and from an arbitrary representative we obtain the one in E by division with remainder.

Extensions and modifications

- Computation of Hilbert series (since 2000),
- based on Stanley decompositions: Versions 2.0–2.5 based on line shellings, now on a theorem of Köppe–Verdoolaege,
- pyramid decomposition "localizing" triangulation (since version 2.7),
- partial triangulation via "height 1 strategy" for Hilbert basis computations(since version 2.5)
- Pottiers (dual) algorithm (since version 2.1, builds C successively as an intersection of halfspaces)
- parallelization with OpenMP (since version 2.5)

Brand-new: NmzIntegrate computes Lesbesgue integrals of polynomials over rational polytopes and generalized Ehrhart seriees.

Challenges mastered

	Condorcet effic. of plurality voting	linear order polytope for S_6	$5 \times 5 \times 3$ contigency tables
Reference	Schürmann	Sturmfels-Welker	Ohsugi-Hibi
computation	Ehrhart series	volume	Hilbert basis
embed dimension	24	16	55
dimension	24	16	43
# extreme rays	3928	720	75
# Hilbert basis	(25,192)	(720)	75
# supp hyperplanes	30	910	306,955
# full triangulation	347,225,775,338	5,745,903,354	(9,248,527,905)
# eval simpl cones	347,225,775,338	102,526,351	448,645
computation time	218:13:55 h	36:45 min	26:52 min

SUN xFire 4450 (parallelization 20 threads), RAM $<$ 2 GB

Platforms and systems

Normaliz (present public version 2.8) runs on

- Apple
- Linux
- MS Windows

Access to Normaliz from

- Singular (library by WB and Christof Söger)
- Macaulay 2 (package by Gesa Kämpf)
- CoCoA (John Abbott, Anna Bigatti, Christof Söger)
- Sage (optional package by Andrey Novoseltsev)
- polymake (Andreas Paffenholz)
- Regina (for 3-manifolds by Benjamin Burton)

GUI interface jNormaliz (by V. Almendra and B. Ichim)

Contributors

- Robert Koch (1997-2002, implementation in C)
- Witold Jarnicki (2003)
- Bogdan Ichim (2007–2008, cxompletely new implementation in C++, versions 2.0, 2.1, jNormaliz)
- Christof Söger (since 2009)
- Gesa Kämpf (Macaulay 2 package)
- Andreas Paffenholz (polymake interface)
- Andrey Novoseltsev (Sage package)

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