Normaliz: a tutorial

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Part 1: Overview of Normaliz

The objectives of Normaliz

Normaliz computes

- in geometric terms: lattice points of polyhedra,
- in algebraic terms: solutions of linear diophantine systems.

The polyhedron and the lattice can be defined

- by generators: extreme rays of cones, vertices of polyhedra, generators of the lattice,
- by constraints: inequalities, equations, congruences.

The conversion between generators and constraints is an important part of Normaliz.

In Part 1 we restrict ourselves to the homogeneous case: the polyhedron is a cone, $0 \in$ lattice, constraints are homogeneous.

Offspring NmzIntegrate computes weighted Ehrhart series and integrals of polynomials over rational polytopes.

Implementation, platforms, contributors, interfaces

Normaliz decomposes into a front end (input and output) and the kernel libnormaliz, a C++ library. Parallelized with OpenMP, uses GMP for unbounded integer precision. In 3.0 SCIP (integer optimization) can be used optionally. It is compiled for Linux (32 and 64 bit), Mac OS and MS Windows (32 and 64 bit).

First C++ code by Bogdan Ichim, present version mainly by Christof Söger and WB, contributions by Richard Sieg.

Normaliz has interfaces for

- Singular (WB and Christof Söger)
- Macaulay 2 (Gesa Kämpf)
- CoCoA (John Abbott, Anna Bigatti, Christof Söger)
- Sage (optional package by Andrey Novoseltsev)
- GAP (Sebastian Gutsche, Max Horn, Christof Söger)
- polymake (polymake team)
- Regina (for 3-manifolds by Benjamin Burton)

Cones and lattices

Definition

A (rational) cone $C \subset \mathbb{R}^d$ is a subset

$$C = \{a_1x_1 + \cdots + a_nx_n : a_1, \ldots, a_n \in \mathbb{R}_+\}$$

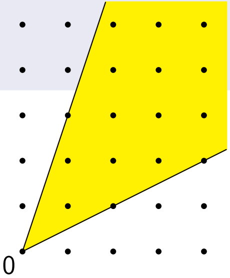
with a generating system $x_1, \ldots, x_n \in \mathbb{Z}^d$.

C pointed $\iff (x, -x \in C \implies x = 0)$.

(For us) a lattice is a subgroup of $\mathbb{Z}^d$.

We will often assume $L = \mathbb{Z}^d$,
C pointed and $\dim C = d$.

(In Normaliz: preliminary coordinate transformation.)



Basic theorems: Minkowski-Weyl

By the theorem of Minkowski-Weyl finitely generated cones can be described by finitely many inequalities:

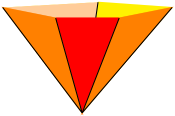
Theorem

For $C \subset \mathbb{R}^d$ the following are equivalent:

- *there are $x_1, \dots, x_n$ such that $C = \{\sum a_i x_i : a_i \mathbb{R}_+\}$;*
- *there are $\lambda_1, \dots, \lambda_s \in (\mathbb{R}^d)^*$ such that $C = \{x \in \mathbb{R}^d : \lambda_i(x) \geq 0\}$.*

In the case $\dim C = d$ (to which we have restricted ourselves) $\lambda_1, \dots, \lambda_s$ for minimal s define the support hyperplanes

$H_i = \{x : \lambda_i(x) = 0\}$



and the facets $F_i = C \cap H_i$ of C.

Basic theorems: Gordan's lemma

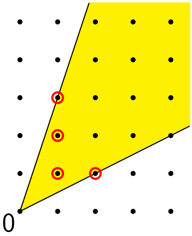
Theorem (Gordan)

Let $C \subset \mathbb{R}^d$ be the cone generated by $x_1, \dots, x_n \in \mathbb{Z}^d$. Then $M = C \cap \mathbb{Z}^d$ is a finitely generated monoid.

$x \in M$ is irreducible if $x = y + z \implies y = 0$ or $z = 0$.

Theorem

The (finitely many) irreducible elements of M (as above) form the unique minimal system of generators $\mathrm{Hilb}(M)$ of M, the Hilbert basis.



Basic theorems: the Hilbert series

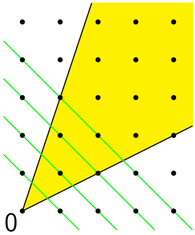
A grading on M is a surjective $\mathbb{Z}$-linear form $\deg : \mathrm{gp}(M) \to \mathbb{Z}$ such that $\deg(x) > 0$ for $x \in M$, $x \neq 0$

The Hilbert (or Ehrhart) function is given by

$$H(M,k) = \#\{x \in M : \deg x = k\}$$

and the Hilbert (Ehrhart) series is

$$H_M(t) = \sum_{k=0}^{\infty} H(M,k)t^k.$$



Theorem (Hilbert-Serre, Ehrhart)

- *$H_M(t)$ is a rational function*
- *$H(M,k)$ is a quasi-polynomial for $k \geq 0$*

The main computation goals of Normaliz

In the homogeneous case when an intersection $C \cap L$ is to be computed:

1. Conversion: lattice generators $\longleftrightarrow$ equations and congruences
2. Conversion: cone generators $\longleftrightarrow$ inequalities
3. Hilbert basis of $C \cap L$
4. Hilbert series of $C \cap L$ (if a grading is given)

Important variants:

1. lattice points in polytopes
2. multiplicities (volumes): normalized leading coefficient of Hilbert (quasi)polynomial

The computation goals can be selected via command line options or in the input file.

The tools of Normaliz

1. linear algebra over $\mathbb{Z}$ (rank, determinant, linear systems of equations, inversion of matrices, normal forms)
2. Fourier-Motzkin elimination (cone generators $\longleftrightarrow$ inequalities)
3. pyramid decomposition and triangulation
4. evaluation of simplicial cones
5. reduction of a system of generators to the Hilbert basis
6. Stanley decomposition (for Hilbert series)
7. a variant of Pottier's "dual" algorithm for Hilbert bases

Algorithmic variants can be selected via command line options

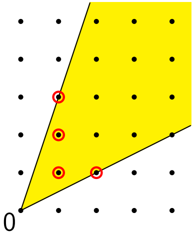
The original Normaliz "primal" algorithm is composed as an interleaved succession of tools (2)–(5) (and (6)).

The dual algorithm is of type "pair completion".

Part 2: Some examples

A cone in dimension 2

Goal: compute the Hilbert basis of the cone in the figure:



Input (syntax of Normaliz 3.0):

```
amb_space 2
cone
2
2 1
1 3
```

Command for running Normaliz (on a Linux platform; 'official" directory structure):

```
./normaliz -c example/2cone
```

Alternative: use the GUI interface jNormaliz (V. Almendra and B. Ichim)

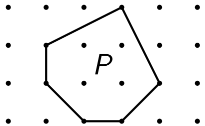
The output

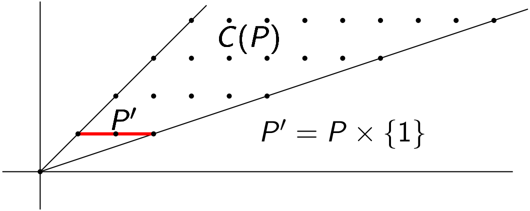
```
4 Hilbert basis elements
2 extreme rays
2 support hyperplanes

embedding dimension = 2
rank = 2 (maximal)
external index = 1
internal index = 5
original monoid is not
        integrally closed

size of triangulation   = 1
resulting sum of |det|s = 5

No implicit grading found
rank of class group = 0
finite cyclic summands:
5: 1
```

```
4 Hilbert basis elements:
1 1
1 2
1 3
2 1

2 extreme rays:
1 3
2 1


2 support hyperplanes:
-1  2
 3 -1
```

A lattice polytope

Lattice polytopes are affine objects. One does computations with them via linearization via a homogenizing coordinate:

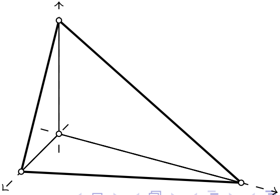




Often one wants to compute not only the lattice points in P, but also those in the multiples kP: compute the cone $C(P)$ together with a grading by the homogenizing coordinate.

Input file (typeset in 2 columns):

```
amb_space 4
polytope
4
0 0 0
2 0 0
0 3 0
0 0 5
```



The Hilbert series

```
grading:
0 0 0 1
multiplicity = 30

Hilbert series:
1 14 15
denominator with 4 factors:
1: 4
degree of Hilbert Series as rational function = -2

Hilbert polynomial:
1 4 8 5
with common denominator = 1
```

$$H(t) = \frac{1 + 14t + 15t^2}{(1-t)^4}$$

$$P(k) = 1 + 4k + 8k^2 + 5k^3$$

One more interesting point:

```
1 further Hilbert basis elements of higher degree:
1 2 4 2
```

Computing in a numerical semigroup

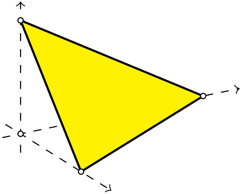
Let $S = \langle 6, 10, 15 \rangle$. How can 97 be written as a sum in the generators?

In other words: we want to find all nonnegative integral solutions to the equation

$$6x_1 + 10x_2 + 15x_3 = 97$$

Input

```
am_space 3
inhom_equations
1
6 10 15 -97
```



The equation cuts out a triangle from the positive orthant.

The set of solutions is a module over the monoid M of solutions of the homogeneous equation $6x_1 + 10x_2 + 15x_3 = 0$. So $M = 0$.

The solutions

```
6 module generators
0 Hilbert basis elements of recession monoid
3 vertices of polyhedron
0 extreme rays of recession cone
3 support hyperplanes of polyhedron
```

```
6 module generators:
2 1 5 1
2 4 3 1
2 7 1 1
7 1 3 1
7 4 1 1
12 1 1 1
```

```
3 vertices of polyhedron:
  0  0 97 15
  0 97  0 10
 97  0  0  6
```

Note: vectors have denominators in inhomogeneous computations.

The Condorcet paradoxon

In social choice elections each of the k voters picks a preference order of the n candidates. There are $n!$ such orders.

We say that candidate A beats candidate B if the majority of the voters prefers A to B. As the Marquis de Condorcet (and others) observed, "beats" is not transitive, and an election may exhibit the Condorcet paradoxon: there is no Condorcet winner.

What is the probability for $k \to \infty$ that there is a Condorcet winner for $n = 4$ candidates?

The event that A is the Condorcet winner can be expressed by linear inequalities on the election outcome (a point in 24-space).

The wanted probability is the volume of the polytope cut out by the inequalities at $k = 1$.

Input for the Condorcet paradoxon

```
amb_space 24
inequalities
3
1 1 1 1 1 1 -1 -1 -1 -1 -1 -1   1  1 -1 -1  1 -1   1  1 -1 -1  1
1 1 1 1 1 1  1  1 -1 -1  1 -1  -1 -1 -1 -1 -1 -1   1  1  1 -1 -1
1 1 1 1 1 1  1  1  1 -1 -1 -1   1  1  1 -1 -1 -1  -1 -1 -1 -1 -1
nonnegative
total_degree
Multiplicity
```

We have listed the 24 permutations in lexicographic order. The first inequality expresses that A beats B etc.

`nonnegative` indicates that we add the inequalities $x_i \geq 0$ to the 3 inequalities preceding it.

`total_degree` says that every coordinate is counted with the same weight 1.

Probability of the Condorcet paradoxon

```
size of triangulation   = 1344671
resulting sum of |det|s = 1816323

grading:
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

degrees of extreme rays:
1: 6  2: 132  4: 96

multiplicity = 1717/8192
```

The probability that A is the Condorcet winner is $1717/8192$. So the probability that there is a Condorcet winner is $1717/2048$.

Normaliz easily computes the Hilbert series in this case (omit `Multiplicity`), or the Hilbert series if one excludes ties (replace `inequalities` by `excluded_faces`).