Normaliz – a tool for discrete convex geometry

Winfried Bruns

FB Mathematik/Informatik
Universität Osnabrück

wbruns@uos.de

Salt Lake City, November 2016

Lattice points in a polyhedron

Normaliz computes the set N of lattice points in a rational polyhedron.
Main computation goals:

- Generation: Describe N by generators.
- Enumeration Given a grading, count the elements of degree k.

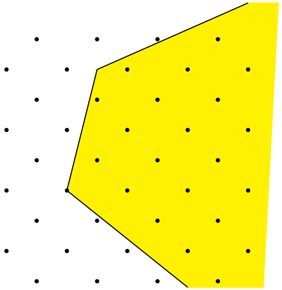
A rational polyhedron is defined by (inhom)

- linear inequalities with coefficients from $\mathbb{Z}$.

An affine lattice is defined by (inhom)

- diophantine linear equations and
- linear congruences.

I.e., Normaliz solves linear diophantine systems.



A classical enumeration problem

A magic square is (for us) a square matrix $A = (a_{ij}) \in \mathbb{Z}^{n\times n}$ with nonnegative entries such that all row and column sums are equal to the magic constant k. In 1968 Ananad, Dumir aand Gupta made some conjectures about the counting function

$$H(M, k) = \#\{A \in \mathbb{Z}^{n\times n} \text{ magic of cinstant } k\}.$$

Clearly we are counting degree k points in the intersection $M = P \cap L$ where

- P is the positive orthant in $\mathbb{R}^{n\times n}$ and
- L is the lattice of solutions to a homogeneous system of diophantine linear equations.

The conjectures caught Richard Stanley's attention. He started combinatorial commutative algebra and provd them.

A classical generation problem

A group in Australia develops chronomorphic music,

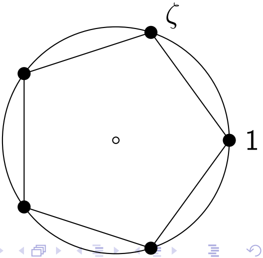
http://www.dynamictonality.com/xronomorph.htm,

based on polygons inscribed to the unit circle with vertices that are nth roots of unity ζ^k, $k = 0, \ldots, n-1$, summing to to $0 \in \mathbb{C}$. Finding such polygons is the problem of vanishing sums of roots of unity.

The lattice L is spanned by the cyclotomomic polynomial Φ_n and its multiples $\zeta^k \Phi_n$, $k = 1, \ldots, n-1$, but the nonnegative elements are not known in general.

Normaliz has been used to find the vanishing sums of roots of unity for various n, for example $n = 66, 70, 78, 102$ (dual algorithm).

On asimov.math.utah.edu we could compute the 24,140,411 extreme (rays of the cone of) vanishing sums of 105th roots of unity.



Code, platfoms, people

Current version: 3.1.3. Technical aspects:

- Code: C++, GMP, OpenMP, SCIP (optional), CoCoALib (NmzIntegrate)
- Platforms: Linux, MacOS, MSWindows
- API: libnormaliz (C++ class library)

Developers

- currently: Winfried Bruns, Christof Söger (2009 – 2016)
- past: Robert Koch (Normaliz 1, $\sim$ 2000), Bogdan Ichim (Normaliz 2, $\sim$ 2008)

Connections

- Normaliz: Tim Römer, Richard Sieg
- libraries and interfaces: John Abbott, Anna Bigatti (CoCoA), Sebastian Guttsche, Max Horn (GAP), Gesa Kämpf (Macaulay 2), polymake team, Ben Burton (Regina), SecDec-3.0 team.

Access from and use in other systems

Normaliz has interfaces for

- Singular
- Macaulay 2
- CoCoA
- Python (under construction)
- Sage (under construction)
- GAP

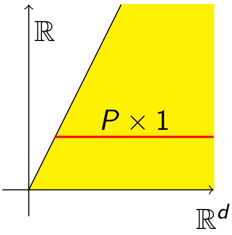
Normaliz is esed by

- polymake (polymake team)
- Regina (3-manifolds, Benjamin Burton)
- SecDec-3.0 (multi-scale integrals, S. Borowka et al.)
- Zeta (topological zeta functions, T. Rossmann)
- HeLP (GAP-package, units in integral group rings, A. Bächle, L. Margolis)

Reduction to the core case

Recall: Normaliz takes any polyhedron and any affine lattice in $\mathbb{R}^d$ as input. We want to discuss Generation. Like Enumeration it is reduced to the core case by 3 coordinate transformations.

1. Homogenization by which we pass from affine geometry to linear (or projective) geometry: the resulting polyhedron is a cone and the lattice is a subgroup of $\mathbb{Z}^d$.
2. Passage to the effective lattice: we can assume $\dim C = d$ and $L = \mathbb{Z}^d$.
3. Passage to the quotient modulo the maximal linear subspace of C.



Core case: $P = C \subset \mathbb{R}^d$ is a pointed rational cone of dimension d. Coimpute the monoid $M = C \cap \mathbb{Z}^d$.

Basic theorems: : Minkowski-Weyl

Theorem

For $C \subset \mathbb{R}^d$ the following are equivalent:

- *there are $x_1, \dots, x_n \in \mathbb{Z}^d$ such that $C = \{\sum a_i x_i : a_i \mathbb{R}_+\}$;*
- *there are $\lambda_1, \dots, \lambda_s \in (\mathbb{R}^d)^*$ extended from $(\mathbb{Z}^d)^*$ such that $C = \{x \in \mathbb{R}^d : \lambda_i(x) \geq 0\}$.*

C is a rational cone if it satisfies the equivalent conditions.

C pointed $\iff$ $(x, -x \in C \implies x = 0)$. Then $x_1, \dots, x_n$ unique up to positive scalars if n is minimal, generating the extreme rays.

$\dim C = d \implies$ as $\lambda_1, \dots, \lambda_s$ are unique up to positive scalars if s are minimal, defining the support hyperplanes.

The (equivalent) conversions extreme rays $\leftrightarrow$ support hyperplanes are fundamental tasks in computational convex geometry.

Basic theorems: Gordan's lemma

Let $C \subset \mathbb{R}^d$ be a pointed rational cone.

Theorem

Then $M = C \cap \mathbb{Z}^d$ is an affine, i.e., finitely generated monoid (or semigroup).

$x \in M$ is irreducible if $x = y + z \implies x = 0$ or $y = 0$.

Theorem

- *every element of M is a sum of irreducible elements.*
- *M has only finitely many irreducible elements.*
- *The irreducible elements form the unique minimal system of generators $\mathrm{Hilb}(M)$ of M (or C), the Hilbert basis.*

Our computation goal Generation can be made precise now: compute $\mathrm{Hilb}(M)$.

Reduction

A chaharcteristic feature of Hilbert basis computations:

- Compute a set $E \supset \mathrm{Hilb}(M)$ of generators.
- ReduceCE to $\mathrm{Hilb}(M)$.

The reduction is in principle very simple if one knows the support hyperplanes (or rather the linear forms $\lambda_1, \dots, \lambda_s$).

Theorem

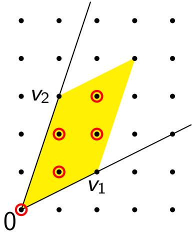
Let E be a system of generators of $M = C \cap \mathbb{Z}^d$. An element is $x \in M$ reducible if and only if there exists $y \in E$, $y \neq x$, such that $\lambda_i(x - y) \geq 0$ for $i = 1, \dots, s$.

Simplicial cones

Let $v_1, \dots, v_d \in \mathbb{Z}^d$ be linearly independent, generating the simplicial cone C. Set

$$\operatorname{par}(C) = \{x : x = \alpha_1 v_1 + \dots + \alpha_d v_d, \quad 0 \le \alpha_i < 1,\ i = 1, \dots, d\}.$$

The set $\operatorname{par}(C)$ is a fundamental domain for the action of $U = \mathbb{Z}v_1 + \dots \mathbb{Z}v_d$ on $\mathbb{R}^d$ by parallel translation.



Theorem

1. *The set* $E' = \operatorname{par}(C) \cap \mathbb{Z}^d$ *represents the classes of* $\mathbb{Z}^d/U$.
2. $\#E' = |\det(v_1, \dots, v_d)|$.
3. $E = \{v_1, \dots, v_d\} \cup E'$ *generates the monoid* $C \cap \mathbb{Z}^d$.

$\Longrightarrow$ E' by enumerating the residue classes in $\mathbb{Z}^d/U$ and division with remainder. Must be as fast as possible!

The Normaliz primalalgorithm

We start from a system of generators of the cone C. (All coordinate transformations done.)

1. Compute the support hyperplanes of C;
2. Compute a triangulation Δ;
3. evaluation of the simplicial cones in the triangulation:
   1. enumeration of the set of lattice points E_σ in the fundamental domain of a simplicial subcone σ,
   2. "local" reduction of E_σ to the Hilbert basis $\mathrm{Hilb}(\sigma)$,
4. Collection of the local data:
   1. "global" reduction of $\bigcup_{\sigma\in\Delta} \mathrm{Hilb}(\sigma)$ to $\mathrm{Hilb}(C)$,

Open questions:

1. How do we find the support hyperplanes?
2. How do we compute a triangulation?

Support hyperplanes: Fourier-Motzkin elimination

This is an incremental algorithm that builds a cone by successive extending the system of generators $x_1, \dots, x_n$ and determining the support hyperplanes in this process.

Start: We may assume that $x_1, \dots, x_d$ are linearly independent. The computation of the support hyperplanes is then simply the inversion of the matrix with rows $x_1, \dots, x_d$. (In principle superfluous.)

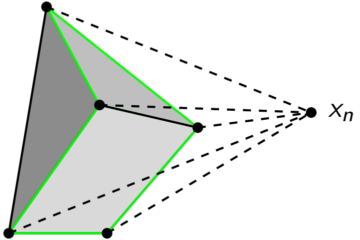
Extension: we add $x_{d+1}, \dots, x_n$ successively: from the support hyperplanes of $C' = \mathbb{R}_+ x_1 + \dots + \mathbb{R}_+ x_{n-1}$ we must compute the support hyperplanes of $C = C' + \mathbb{R}_+ x_n$.

We describe this process geometrically.

Geometry of Fourier-Motzkin elimination

We determine the boundary V of the part of C' that is visible from x_n and its decomposition into subfacets. Together with x_n these span the new facets of C. The facets of C' that are visible from x_n ($\lambda_i(x_n) < 0$), are discarded.

In the cross-section of a 4-dimensional cone:

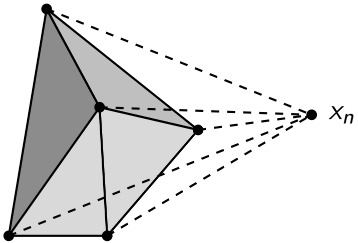


We have to find V: pair visible ($\lambda_i(x_n) < 0$) and invisible ($\lambda_j(x_n) > 0$) facets: Each segment (in this dimension) of V is the intersection of such a pair. Problem: Many pairs.

Incremental triangulation

It follows the same inductive scheme, interleaved with Fourier-Moatzkin elimination: we obtain a triangulation of C if we extend the triangulation of C' by the simplicial cones that are spanned by x_n and the facets visible from x_n.

In the cross-section of a 4-dimensional cone:



Main problem: triangulation may be very large. And we must pair the simplices with the visible facets . . .

Bottlenecks

Combinatorial complexity:

- large number of support hyperplanes and
- large triangulations

Countermeasure: localize their computation by pyramid decomposition, breaking the task into (almost) independent units.
$\implies$ efficient parallelization.

Arithmetical complexity:

- large determinants of simplicial cones

Countermeasure: Subdivision based on the IP solver SCIP and approximation of simplicial cones.

Challenges mastered

	Condorcet effic. of plurality voting	linear or,der polytope for S_6	$5 \times 5 \times 3$ contigency tables
Reference	Schürmann	Sturmfels-Welker	Ohsugi-Hibi
computation	Ehrhart series	volume	Hilbert basis
embed dimension	24	16	55
dimension	24	16	43
# extreme rays	3928	720	75
# Hilbert basis	(25,192)	(720)	75
# supp hyperplanes	30	910	306,955
# full triangulation	288,509,390,884	5,745,903,354	(9,248,527,905)
# eval simpl cones	288,509,390,884	383,889,521	383,961
computation time	84:26:19 h	16:57 min	2:07 min

Normaliz 3.1.3, 4 Xeon E5-2660 at 2.20GHz, using 30 parallel threads.