An algorithm for volumes of polytopes with applications to social choice

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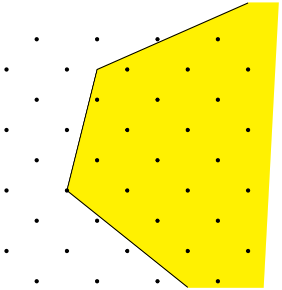
The tasks of Normaliz

Normaliz computes the set N of lattice points in a rational polyhedron.
Main computation goals:

- Generation: Describe N by generators.
- Enumeration Given a grading, count the elements of degree k (for all k).



A rational polyhedron is defined by (inhom)

- linear inequalities with coefficients from $\mathbb{Z}$.

An affine lattice is defined by (inhom)

- diophantine linear equations and
- linear congruences.

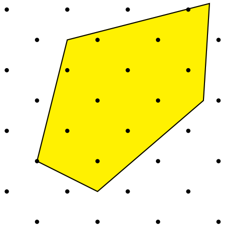
I.e., Normaliz solves linear diophantine systems.

Special tasks for polytopes

A polytope P is a bounded polyhedron.

There are obvious special tasks:

- Lattice points: find the lattice points in P.
- Volume: compute the volume of P.

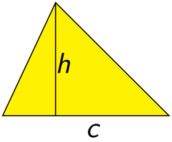


Subtasks of Generation and Enumeration:
lattice points $\subset$ Hilbert basis($C(P)$),
volume $\sim$ lead coeff of Ehrhart quasipolynomial.

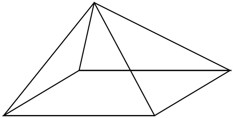
Since 3.4.0 and 3.5.2 (just released) Normaliz has special algorithms for lattice points and volumes that are *not* "truncations" of the Hilbert basis or Ehrhart series algorithms.

The recursive nature of volume formulas

In elementary geometry we learn two volume formulas:



$$\text{area}(\Delta) = \ell(c)\ell(h)/2,$$
$$\text{vol}(\Pi) = \text{ht}(\Pi)\,\text{area}(B)/3.$$



Both are recursive in nature; they defer the volume computation in dimension n to volume computations in dimensions $n-1$ and 1.

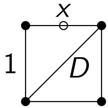
What if the base of Π is not a triangle? Archimedes would triangulate, and we do the same, but, in the new algorithm, only implicitly.

Lattice volume and lattice height

Want exact volumes of rational polytopes $P \implies$ must replace euclidean volume and length by lattice volume and lattice height.

The lattice volume Vol is the Lebesgue measure in $A = \operatorname{aff}(P)$ scaled in such a way that the smallest lattice simplex in A has lattice volume 1. (If $A \cap \mathbb{Z}^n = \emptyset$, replace A by $A - y$ for some $y \in A$.)

$$\implies \quad \operatorname{Vol}(P) \in \mathbb{Q} \quad \text{for all rational polytopes } P.$$



$P \subset \mathbb{R}^n, \dim P = n \implies \operatorname{Vol}(P) = n! \operatorname{vol}(P)$. But if $\dim P < n$, $\operatorname{vol}(P) \notin \mathbb{Q}$ in general: $\operatorname{vol}(D) = \sqrt{2}$, $\operatorname{Vol}(D) = 1$.

The lattice height $\operatorname{Ht}_H(x)$ of $x \in \mathbb{Q}^n$ over a rational affine subspace H counts the number of lattice parallels of H between x and H if $x \in \mathbb{Z}^n$ and $H \cap \mathbb{Z}^n \neq \emptyset$. Otherwise scale: $\operatorname{Ht}_H(x) = \operatorname{Ht}_{kH}(kx)/k$.

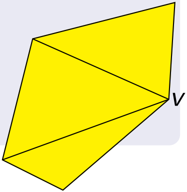
Example: $\operatorname{Ht}_D(x) = 1/2$.

Recursion and pyramid decomposition

Theorem

Let $P \subset \mathbb{R}^n$ be a rational polytope, and $v \in P$. Then

$$\mathrm{Vol}(P) = \sum_{F \text{ facet of } P} \mathrm{Ht}_F(v)\,\mathrm{Vol}(F).$$

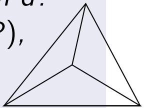


For simplices we compute the volume directly:

Theorem

Let $S = \mathrm{conv}(v_0, \ldots, v_d)$ be a rational simplex of dimension d. Then, with respect to coordinates in a lattice basis of $\mathrm{aff}(P)$,

$$\mathrm{Vol}(S) = |\det(v_1 - v_0, \ldots, v_d - v_0)|.$$

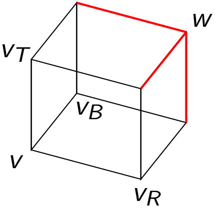


The volume of a cube

We start with the vertex v and its opposite facets T (top), B (back), R (right):

$$\mathrm{Vol}(P) = \mathrm{Ht}_T(v)\,\mathrm{Vol}(T) + \mathrm{Ht}_B(v)\,\mathrm{Vol}(B) + \mathrm{Ht}_R(v)\,\mathrm{Vol}(R).$$

Bad choice: memoryless depth-first recursion. Would compute volumes of lower-dimensional faces over and over again.



Best choice for the cube: with v_T, v_B, v_R as vertices for T, B, R we get

$$\begin{aligned}\mathrm{Vol}(P) = {} & \big(\mathrm{Ht}_T)(v)\,\mathrm{Ht}_{T\cap B}(v_T) + \mathrm{Ht}_B(v)\,\mathrm{Ht}_{T\cap B}(v_B)\big)\,\mathrm{Vol}(T\cap B)\\ & + \big(\mathrm{Ht}_R)(v)\,\mathrm{Ht}_{T\cap R}(v_R) + \mathrm{Ht}_T(v)\,\mathrm{Ht}_{T\cap R}(v_T)\big)\,\mathrm{Vol}(T\cap R)\\ & + \big(\mathrm{Ht}_B)(v)\,\mathrm{Ht}_{B\cap R}(v_B) + \mathrm{Ht}_R(v)\,\mathrm{Ht}_{B\cap R}(v_R)\big)\,\mathrm{Vol}(B\cap R)\end{aligned}$$

Only 3 simplex volumes must be computed.

Descent systems

A descent system $\mathcal{D} = (\mathcal{D}_0, \dots, \mathcal{D}_{d-1})$ allows the recursive computation of $\operatorname{Vol}(P)$: Each layer $\mathcal{D}_i$ is a collection of $d - i$-dimensional faces of P such that

1. $\mathcal{D}_0 = \{P\}$,
2. for each nonsimplex $F \in \mathcal{D}_i$ there exists a vertex $v \in F$ such that $G \in \mathcal{D}_{i+1}$ for all facets G of F not containing v.

In the implementation of

$$\operatorname{Vol}(P) = \sum_{F \text{ facet of } P} \operatorname{Ht}_F(v) \operatorname{Vol}(F)$$

should we pull $\operatorname{Vol}(F)$ up to P or push $\operatorname{Ht}_F(v)$ down to F? Better: *push down* $\implies$ *no backtracking*, need only store two consecutive layers.

Therefore each face F needs a weight $w(F)$. We start with $w(P) = 1$.

The algorithm

Starting with $\mathcal{D}_0$, each layer $\mathcal{D}_i$ is processed in a parallelized loop. If $\mathcal{D}_i = \emptyset$, the computation is finished. Otherwise each $F \in \mathcal{D}_i$ is treated as follows, creating $\mathcal{D}_{i+1}$ successively:

1. Decide whether F is a simplex; if so, $w(F)\operatorname{Vol}(F)$ is added to the accumulated total volume V, and we are done with F.
2. Otherwise we must find the facets G of F,
3. select the vertex v,
4. for each facet G not containing v
    1. compute $\operatorname{Ht}_G(v)$,
    2. insert G with $w(G) = 0$ into $\mathcal{D}_{i+1}$ if it has not yet been found by an already processed face $F' \in \mathcal{D}_i$, or retrieve it otherwise,
    3. increase $w(G)$ by $w(F)\operatorname{Ht}_G(v)$.

Critical magnitude: $\sum_i \#\mathcal{D}_i \implies$ the algorithm can only be used for P with a small, at most moderate number of nonsimplex facets.

A baby monster calculation (4 rules)

A poytope P, $\dim P = 23$, with 36 facets and $233,644$ vertices:

```
Descent from dim 24, size 1
Descent from dim 23, size 4
...
Descent from dim 7, size 146534149
...
Descent from dim 3, size 18701975
...
Descent from dim 2, size 3318748
...
Mult 15434...86375185706443451991 7747 /1973489199416...319602031820800000000000
Mult (float) 0.0782082
Full tree size 17,872,168,126,827
Number of descent steps 2,228,384,724
Number of simplicial Faces 59,223,693
Total number of faces 711,676,197
Multiplicity by descent done
387966.73user 1621.90system 5:36:24elapsed 1930%CPU
```

Memory usage $\approx$ 37 GB

Comparison with vinci (very preliminary)

vinci (B. Büeler, A. Enge, K. Fukuda) is an established package for the computation of polytope volumes:

1. specialized for volume computation,
2. several algorithms ("hot" similar to ours)
3. floating point arithmetic,
4. m often very fast, but extreme usage of memory.

Normaliz:

1. embedded into a polyhedral geometry package,
2. acceptable speed by parallelization,
3. integral/rational arithmetic,
4. memory friendly.

Example "3 rules" (on a Dell R901 with 128 GB RAM): vinci "rlass" 138 GB/6m 22s/2m 42 s, Normaliz (20 threads) 700 MB/3m 12 s/60m 14.

Mathematics in elections

Mathematical models can be applied to two types of elections.

1. Elections of persons: president, mayor, dean etc. Goal: election of the candidate with the highest degree of general approval.
2. Elections of parliaments: parliament of state, city council, university senate etc. Goal: fair representation of "parties".

The theory of fair representation has mainly been driven by the a priori distribution of seats in the US house of representatives to the states.

We will be concerned with the election of persons.

William V. Gehrlein and Dominique Lepelley represent the aspects of social choice that we discuss today.

Preference rankings and the election result

The basic assumption in the mathematics of social choice is the existence of individual preference rankings: every voter ranks the candidates in linear order. (It would be possible to allow indifferences.)

We use capital letters for the candidates. Examples for three candidates:

$$A \succ B \succ C$$
$$C \succ A \succ B$$

For n candidates there exist $N = n!$ preference rankings, usually numbered in lexicographic order.

The result of the election is the N-tuple

$$(v_1, \ldots, v_N), \qquad v_i = \#\{\text{voters of preference ranking } i\}.$$

$n = 3, N = 6,\ n = 4, N = 24,\ n = 5, N = 120 \ldots$

Impartial Anonymous Culture

In the following we want to compute probabilities of certain events related to election schemes. This requires a probability distribution on the set of election results. We fix the number k of voters.

The Impartial Anonymous Culture (IAC) is the equidistribution on the set of election results: every election result is assumed to have equal probability.

This model does *not* treat the voters as independent individuals. If every voter rolls a dice to choose his/her preference ranking, then the resulting probability distribution of election results is the multinomial distribution!

The IAC lacks certain properties that one would intuitively expect. For example, ignoring one candidate does not map IAC(n) to IAC($n-1$).

The Condorcet paradox

The *Marquis de Condorcet* (1743–1794) was a leading intellectual in France before and during the revolution. He already observed that there is no ideal election scheme and suggested solutions.

We say that candidate A beats candidate B in majority, $A >_M B$, if

$$\#\{\text{voters with } A \succ B\} > \#\{\text{voters with } B \succ A\}.$$

The Condorcet winner (CW) beats all other candidates in majority.

If a single person is to be elected, then there is general agreement that the CW is the person with the largest common approval.

Condorcet realized that a CW need not exist: the relation $>_M$ is not transitive. This phenomenon is called the Condorcet paradox.

It is our guiding example.

Inequalities for the Condorcet winner

An election result for 3 candidates in tabular form:

number of voters	x_{ABC}	x_{ACB}	x_{BAC}	x_{BCA}	x_{CAB}	x_{CBA}
	A	A	B	B	C	C
ranking	B	C	A	C	A	B
	C	B	C	A	B	A

A is the CW if

$$\begin{aligned} A >_M B: &\quad x_{ABC} + x_{ACB} + x_{CAB} > x_{BAC} + x_{BCA} + x_{CBA}, \\ A >_M C: &\quad x_{ABC} + x_{ACB} + x_{BAC} > x_{BCA} + x_{CAB} + x_{CBA}. \end{aligned}$$

If we are only interested in probabilities for $k \to \infty$, we can allow ties and replace $>$ by $\geq$. Recall: k is always the number of voters.

Important observation; the election results with "A as CW" are the lattice points in a rational cone C defined by the inequalities above.

The asymptotic behavior

For large numbers of voters we want to find the probability of a certain event E, given by the lattice points in a cone $C \subset \mathbb{R}^d$. Because of IAC we can define it by

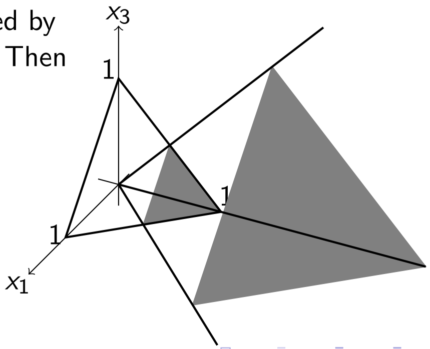
$$\operatorname{prob}(E) = \lim_{k\to\infty} \frac{\#\{\text{degree } k \text{ latt points in } C\}}{\#\{\text{degree } k \text{ latt points in } \mathbb{R}^d_+\}} = \lim_{k\to\infty} \frac{H(C,k)}{\binom{N+k-1}{N-1}}.$$

Let $\mathcal{U}$ be the unit simplex spanned by the unit vectors and $P = \mathcal{U} \cap C$. Then

$$\operatorname{prob}(E) = \frac{\operatorname{vol} P}{\operatorname{vol} \mathcal{U}}.$$

In terms of lattice volume:

$$\operatorname{prob}(E) = \operatorname{Vol}(P).$$



Condorcet efficiency of plurality voting

The election of the Condorcet winner lacks universality, and the probability of failure is significantly > 0. (Whether an election that counts all preference rankings is feasible, is another question.) The easiest way out: plurality voting. That is: the person with the largest number of first places is elected, or the candidates are even ranked by the number of first places.

A measure for the quality of an election scheme is its Condorcet efficiency:

$$\mathrm{CE(scheme)} = \frac{\mathrm{prob(scheme\ elects\ CW)}}{\mathrm{prob(CW\ exists)}} = \frac{\mathrm{prob(scheme\ elects\ CW)}}{1717/2048}$$

For plurality one gets:

$$\mathrm{CE(plurality)} = \frac{10658098255011916449318509}{14352135440302080000000000} \approx 0.7426$$

Runoff

Many election schemes use two rounds: a second ballot if the plurality winner has $\leq 50\%$ first places. In the runoff the two candidates with the highest numbers of first places run against each other. This clearly improves the Condorcet efficiency since the CW wins the second round, provided he/she is at least 2nd in the first round:

$$\mathrm{CE(runoff)} = \frac{19627224002877404784030049}{21528203160453120000000000} \approx 0.9117$$

This is a significant increase, justifying the effort of the runoff.

On the other hand, one can ask for the probability that the winner of the first round also wins the runoff:

$$\mathrm{prob(1st\ round\ winner\ wins\ runoff)} = \frac{9185069468583833}{12173449145352192} \approx 0.7545$$

3 rounds

Now we want to improve further by a 3-round election scheme:

1. in the first round the plurality loser is discarded;
2. in the second round the plurality loser among the 3 remaining candidates is discarded;
3. the winner is chosen by plurality among the surviving 2 candidates.

Caution: in the 2-round system the CW wins as soon as he/she reaches the second round. This is no longer true for 3 rounds, because the CW may now drop out in the second round. Therefore *no naive monotonicity*! But

$$\mathrm{CE}(\text{3-round system}) \approx 0.929184$$
$$= \frac{129178312275188795293522359266689257253407234828397}{139023462671726486558162887377734860800000000000000}$$

The gain over 2 rounds is very small and does not justify the extra effort.

3 rules

In social choice also the negative plirality rule (NPR) is discussed: voters cat a vote against their least preferred candidate, and the winner is the person with the least number of negative votes.

In the example "3 rules" mentioned above we have computed the probability that the CW wins both PR and negative plurality (NPR) under the comndition that a CW exists:

$$p(\text{CW wins PR and NPR} \mid \text{CW exists}) \approx 0.37965$$

If we go to "4 rules" by adding the Borda rule, it decreases to

$$p(\text{CW wins PR, NPR abd BR} \mid \text{CW exists}) \approx 0.31283.$$

References

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