Canonical modules of Rees algebras

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Let R be a normal Noetherian domain, and I an ideal in R for which the Rees algebra

$$\mathscr{R} = \mathscr{R}(I) = \mathscr{R}(R,I) = \bigoplus_{k=0}^{\infty} I^k T^k \subset R[T]$$

is a normal Cohen-Macaulay domain. The normality of the Rees algebra is equivalent to the integral closedness of all powers I^k. We will assume in the following that $\operatorname{ht} I \geq 2$.

The results can be generalized to the case in which $\mathscr{R}$ is not normal. Then $\mathscr{R}$ must be replaced by its normalization. Moreover, it would be possible to avoid the hypothesis $\operatorname{ht} I \geq 2$.

Main result: $$[\omega_{\mathscr{R}}] = [I\mathscr{R}] + \sum_{i=1}^{t} (1 - \operatorname{ht} \mathfrak{p}_i)[P_i]$$ where

- $[\ldots]$ class in divisor class group,
- R regular domain, ess. of finite type over a field K,
- $\omega_{\mathscr{R}}$ is the canonical module of $\mathscr{R}$,
- $P_1, \ldots, P_t$ are the minimal prime ideals of $I\mathscr{R}$,
- $\mathfrak{p}_i = P_i \cap R$,
- crucial hypothesis: $\mathfrak{p}_i^{(k)} = P_i^{(k)} \cap R$ for $i = 1, \ldots, t$ and all $k \in \mathbb{N}$.

Herzog-Vasconcelos [HV]

$I = \mathfrak{p}$ prime and $\mathfrak{p}^{(k)} = \mathfrak{p}^k$

$I\mathscr{R}$ is prime

Bruns-Conca [BC]

$I = I_t(X)$ ideal of minors,

"initial methods"

1 Divisor class group of a Rees algebra

Simis-Trung [ST]:

Theorem 1.1. *There is an exact sequence*

$$0 \to \mathbb{Z}^t \to \mathrm{Cl}(\mathscr{R}) \to \mathrm{Cl}(R) \to 0 \tag{1}$$

where a basis of $\mathbb{Z}^t$ is given by the classes $[P_i]$ of the minimal prime ideals $P_1, \ldots, P_t$ of the divisorial ideal $I\mathscr{R}$.

In the following: $\mathbb{Z}^t$ denotes $\mathbb{Z}[P_1] + \cdots + \mathbb{Z}[P_t]$.

For the proof of the theorem we use the following lemma, in which Spec^1 denotes the set of divisorial prime ideals.

Lemma 1.2. *Let I be an ideal of height at least 2, and, with the notation introduced, $V_i = \mathscr{R}_{P_i}$; then*

$$\mathscr{R} = R[T] \cap V_1 \cap \cdots \cap V_t, \tag{2}$$

and, moreover, $R[T]$ is a subintersection of $\mathscr{R}$,

$$R[T] = \bigcap \{\mathscr{R}_P : P \in \mathrm{Spec}^1(\mathscr{R}),\ P \neq P_1, \ldots, P_t\}. \tag{3}$$

Proof. $\mathscr{R} = \bigcap \{\mathscr{R}_P : P \in \mathrm{Spec}^1(\mathscr{R})\}$ (by normality).

(2) $\mathscr{R}_P = R[T]_Q$ with $Q \in \mathrm{Spec}^1(R[T])$ if $P \not\supset I\mathscr{R}$. (Easy localization argument.)

(3) One uses $\mathrm{ht}\, P_i R[T] \geq 2$ ([BC, Lemma 2.1]). $\square$

Proof of Theorem 1.1. By the lemma and Nagata's theorem one has an exact sequence

$$0 \to U \to \mathrm{Cl}(\mathscr{R}) \to \mathrm{Cl}(R[T]) \to 0$$

where $U = \mathbb{Z}P[P_1] + \cdots + \mathbb{Z}[P_t]$.

For linear independence of $[P_1], \ldots, [P_t]$ see [ST]).

Finally, by Gauß' theorem, $\mathrm{Cl}(R) = \mathrm{Cl}(R[T])$. $\square$

In order to control the global choices for modules that are defined only up to local isomorphism one needs the Picard group:

$\mathrm{Pic}(R)$ is the subgroup of $\mathrm{Cl}(R)$ formed by the isomorphism classes of projective rank 1 modules ($=$ invertible ideals).

Proposition 1.3. *The natural map* $\mathrm{Pic}(R) \to \mathrm{Pic}(\mathscr{R})$ *is an isomorphism.*

If R is locally factorial, then the natural map $\mathrm{Cl}(R) = \mathrm{Pic}(R) \to \mathrm{Pic}(\mathscr{R}) \subset \mathrm{Cl}(\mathscr{R})$ *splits the exact sequence* (1). *In particular* $\mathrm{Cl}(\mathscr{R}) = \mathbb{Z}^t \oplus \mathrm{Pic}(\mathscr{R})$.

Proof. $\mathrm{Pic}(R) \to \mathrm{Pic}(R[T])$ isomorphism by normality, factors through $\mathrm{Pic}(\mathscr{R})$, and $\mathrm{Pic}(\mathscr{R}) \cap \mathbb{Z}^t = 0$. □

2 Rees valuations and primary decomposition

As above $P_1, \ldots, P_t$ are the minimal prime ideals of $I\mathscr{R}$.

$V_i = \mathscr{R}_{P_i}$ is a discrete valuation domain, with associated Rees valuation v_i. It induces a (Rees) valuation on R.

If R is regular and $\mathfrak{p}$ a prime ideal, then the function

$$v_{\mathfrak{p}}(x) = \max\{k : x \in \mathfrak{p}^{(k)}\}$$

induces the $\mathfrak{p}$-adic valuation on R. Crucial hypothesis of the main result can be expressed in terms of the v_i:

$$\mathfrak{p}_i^{(k)} = P_i^{(k)} \cap R \quad \Longleftrightarrow \quad v_{\mathfrak{p}_i} = v_i | R.$$

Proposition 2.1. *Set* $J_i(j) = \{x \in R : v_i(x) \geq j\}$. *Then*

$$I^k = \bigcap_{i=1}^{t} J_i(kd_i), \qquad d_i = -v_i(T). \tag{4}$$

The intersection is irredundant for $k \gg 0$. *Moreover,*

$$I\mathscr{R} = \bigcap_{i=1}^{t} P_i^{(d_i)}.$$

Proof. Consider $\mathscr{R} = R[T] \cap V_1 \cap \cdots \cap V_t$ in each T-degree:

$$I^k T^k = \{aT^k : a \in R,\ v_i(a) \geq -v_i(T^k),\ i = 1, \ldots, t\},$$

and this is evidently equivalent to equation (4).

One has $I\mathscr{R} = \bigcap_{i=1}^{t} P_i^{(v_i(I\mathscr{R}))}$ and $0 = v_i(IT\mathscr{R}) = v_i(T) + v_i(I\mathscr{R})$ since $IT\mathscr{R} \not\subset P_i$. Therefore $v_i(I\mathscr{R}) = -v_i(T)$. $\square$

Proposition 2.2. *Let R be a regular ring and I an ideal of height ≥ 2.*

(a) Then the following are equivalent:

(i) $\mathscr{R}(I)$ is normal, and for each minimal prime ideal P of $I\mathscr{R}$ the Rees valuation v_P restricts on R to the valuation $v_{\mathfrak{p}}$, $\mathfrak{p} = P \cap R$;

(ii) there exist prime ideals $\mathfrak{p}_i, \dots, \mathfrak{p}_u$ in R and $d_1, \dots, d_u \in \mathbb{N}$ such that $I^k = \bigcap_{i=1}^{u} \mathfrak{p}_i^{(d_i k)}$ for all k.

(b) Moreover, if (i) *holds, and $P_1, \dots, P_t$ are the minimal prime ideals of $I\mathscr{R}$, then one can choose $\mathfrak{p}_i = P_i \cap R$, $d_i = -v_i(T)$, and the intersection in* (ii) *is irredundant for $k \gg 0$.*

(c) Conversely, if there exists k_i for each $i = 1, \dots, u$ such that $\mathfrak{p}_i^{(d_i k)}$ cannot be omitted in the representation of I^{k_i} in (ii)*, then the graded extensions of the v_i to $R[T]$ with $v_i(T) = -d_i$ are the Rees valuations of I on $Q(R[T])$.*

3 The canonical module of a subintersection

Strategy for the proof of the main result: Suppose R is factorial so that $\mathrm{Cl}(\mathscr{R}) = \mathbb{Z}^t$. Then

$$[\omega_{\mathscr{R}}] = w_1[P_1] + \cdots + w_t[P_t].$$

We have to determine the coefficients w_i. We have to find w_i.

First step: localize R at $\mathfrak{p}_i = P_i \cap R$. This preserves w_i, but also those components $\mathbb{Z}[P_j]$ of $\mathbb{Z}^t$ for which $\mathfrak{p}_j \subset \mathfrak{p}_i$.

Finer instrument: Analyze $\mathscr{R}_i = R[T] \cap V_i$. Then $[\omega_{\mathscr{R}} \otimes \mathscr{R}_i] = w_i[P_i\mathscr{R}_i]$, and the strategy works if

- $[\omega_{\mathscr{R}} \otimes \mathscr{R}_i] = [\omega_{\mathscr{R}_i}]$
- $[\omega_{\mathscr{R}_i}]$ can be computed.

So we have to analyze the behaviour of ω under subintersections. Main difficulty: ω is characterized by homological conditions, but subintersections are usually not flat extensions.

Lemma 3.1. *Let R be a normal Cohen-Macaulay domain, essentially of finite type over a perfect field K, and $r = \dim R$. Then*

$$\omega_R \cong \Big(\bigwedge^r \Omega_{R/K}\Big)^{**}.$$

Proof. It has been proved by Kunz [Ku1] that ω_R is given by the regular differential r-forms $\mathrm{R}_K^r(R)$, $r = \dim R$, and Platte and Storch [PS] have noticed that $\mathrm{R}_K^r(R) = \big(\bigwedge^r \Omega_{R/K}\big)^{**}$. Sketch:

Write $R = A/I$, A a regular domain.

$$0 \to I/I^2 \to \Omega_{A/K} \otimes_A R \to \Omega_{R/K} \to 0$$

is exact in codimension 1, since R is normal. Hence $[I/I^2] = -[\Omega_{R/K}]$ where $[M] = [(\wedge^m M)^{**}]$ for a module M of rank m over R. On the other hand (See [HV] for the details),

$$[I/I^2] = -[H_1(I)] = -[\mathrm{Ext}_A^c(R.A)] = -[\omega_R], \qquad c = \operatorname{codim} R. \quad \square$$

Theorem 3.2. *Let K be a field, R a normal Cohen–Macaulay K-algebra of essentially finite type over K, and $Y \subset \operatorname{Spec}^1(R)$. Suppose that the subintersection $S = \bigcap_{\mathfrak{p} \in Y} R_{\mathfrak{p}}$ is again of essentially finite type over K and Cohen–Macaulay.*

Then the canonical module of S is $(\omega_R \otimes_R S)^{\dagger\dagger}$, where † denotes the functor $\operatorname{Hom}_S(\_, S)$. In other words, the canonical class of S is the image of ω_R under the natural map $\operatorname{Cl}(R) \to \operatorname{Cl}(S)$.

Proof. If K is not perfect one replaces K by a subfield K_0 with $[K : K_0] < \infty$ that is admissible in the sense of [Ku2, 6.23] for R, S and regular K-algebras A and B of essentially finite type for which there exist presentations $R = A/I$ and $S = B/J$. Then Lemma 3.1 holds accordingly.

The embedding $\phi : R \to S$ gives rise to an R-linear map $d\phi : \Omega_{R/K} \to \Omega_{S/K}$, and $d\phi$ induces a natural S-linear map

$$\psi : \left(\bigwedge^r \Omega_{R/K} \right) \otimes_R S \to \bigwedge^r \Omega_{S/K}.$$

Let $\mathfrak{q}$ be a height 1 prime ideal of S. Then $S_{\mathfrak{q}} = R_{\mathfrak{q} \cap R}$, and therefore $\psi \otimes_S S_{\mathfrak{q}}$ is an isomorphism. It follows that the S-bidual extension $\psi^{\dagger\dagger}$ is an isomorphism at all height 1 prime ideals $\mathfrak{q}$ of S. Since the S-biduals are reflexive, $\psi^{\dagger\dagger}$ is an isomorphism itself.

The second statement about the divisor classes follows immediately, since $(J \otimes S)^{\dagger\dagger}$ is exactly the divisorial ideal of S to which a divisorial ideal J of R extends $\square$

4 Main result and proof

Proposition 4.1. *Let R be a regular local ring with maximal ideal $\mathfrak{m}$. Then the Rees algebra $\mathscr{R}_k = \mathscr{R}(\mathfrak{m}^k)$ is normal and Cohen–Macaulay. Its canonical module is unique up to isomorphism and has class $(k - \dim R + 1)[P_k] = [\mathfrak{m}^k \mathscr{R}_k] - (\dim R - 1)[P_k]$ where $P_k = \mathfrak{m}\mathscr{R}_k$ is the only divisorial ideal of $\mathscr{R}_k$ containing $\mathfrak{m}^k \mathscr{R}_k$.*

Proof. Only the essential point. Let $x_1, \ldots, x_r$, $r = \dim R$, be a regular system of parameters and set

$$J_k = (x_1 \cdots x_r \cdot TR[T]) \cap P_k.$$

Then J_k has class $-r[P_k] + [\mathfrak{m}^k \mathscr{R}_k] + [P_k] = [\mathfrak{m}^k \mathscr{R}_k] - (\dim R - 1)[P_k]$.

It remains that $J_k = \omega_{\mathscr{R}_k}$. $k = 1$: Herzog and Vasconcelos, $k > 1$: $\mathscr{R}_k$ Veronese subalgebra of $\mathscr{R}_1$. ☐

Theorem 4.2.

$$\sum_{i=1}^{t}(d_i + 1 - \operatorname{ht}\mathfrak{p}_i)[P_i] = [I\mathscr{R}] + \sum_{i=1}^{t}(1 - \operatorname{ht}\mathfrak{p}_i)[P_i]$$

class of a canonical module of $\mathscr{R}$ if

- *R regular domain, ess. of finite type over a field K,*
- *$P_1, \ldots, P_t$ are the minimal prime ideals of $I\mathscr{R}$,*
- $I\mathscr{R} = \bigcap_{i=1}^{t} P_i^{(d_i)}$
- $\mathfrak{p}_i = P_i \cap R$,
- *crucial hypothesis: $\mathfrak{p}_i^{(k)} = P_i^{(k)} \cap R$ for $i = 1, \ldots, t$ and all $k \in \mathbb{N}$.*

Moreover, $\mathscr{R}$ is Gorenstein if and only if $d_i = \operatorname{ht}\mathfrak{p}_i - 1$ for all $i = 1, \ldots, t$.

Proof. Let C be a module of the class given in the theorem. It is enough to show that each of its localizations $C_{\mathfrak{M}}$ with respect to maximal ideals $\mathfrak{M}$ of $\mathscr{R}$ is a canonical module of $\mathscr{R}_{\mathfrak{M}}$. Such a localization $\mathscr{R}_{\mathfrak{M}}$ is a localization of $\mathrm{R}_{\mathfrak{m}}$ with $\mathfrak{m} = R \cap \mathfrak{M}$. Since the definition of $[C]$ commutes with localization in R (in fact, primary decomposition commutes with such localizations), we may assume that R is regular local, and therefore factorial.

Then $\mathrm{Cl}(\mathscr{R}) = \mathbb{Z}^t$, $\mathrm{Pic}(\mathscr{R}) = 0$ (by Proposition 1.3), and we have a unique isomorphism class

$$[\omega_{\mathscr{R}}] = w_1[P_1] + \cdots + w_t[P_t].$$

for the canonical module of $\mathscr{R}$.

It is enough to determine, say, w_1. We localize R with respect to $\mathfrak{p}_1$, and may then assume that R is regular local with maximal ideal $\mathfrak{m} = \mathfrak{p}_1$.

In the next step we pass to the subintersection $S = R[T] \cap V_1$. But this subintersection is exactly $\mathscr{R}(\mathfrak{m}^{d_1})$, as follows from Propositions 2.1 and 2.2. (Since $\mathfrak{m}$ is a maximal ideal, ordinary and symbolic powers coincide.)

According to Theorem 3.2 the formation of the canonical class commutes with subintersection, and so w_1 is the coefficient of the canonical module of $\mathscr{R}(\mathfrak{m}^{d_1})$ with respect to the extension of P_1. By Proposition 4.1 this coefficient is $d_1 + 1 - \dim R = d_1 + 1 - \operatorname{ht} \mathfrak{p}_1$, as desired.

Gorenstein property: Because of the splitting $\mathrm{Cl}(\mathscr{R}) = \mathbb{Z}^t \oplus \mathrm{Pic}(\mathscr{R})$, the class

$$\sum_{i=1}^{t} (d_i + 1 - \operatorname{ht} \mathfrak{p}_i)[P_i]$$

is the $\mathbb{Z}^t$-component of any canonical module of $\mathscr{R}$. Therefore $\mathscr{R}$ is Gorenstein if and only if it vanishes. □

Corollary 4.3. *With the hypotheses of Theorem 4.2 suppose we can find an element* $x \in R$ *such that* $v_{P_i}(x) = \operatorname{ht} \mathfrak{p}_i$ *for all* i.

Then

$$\omega_{\mathscr{R}} = xTR[T] \cap P_1 \cap \cdots \cap P_t$$

is the graded canonical module of $\mathscr{R}$ *(with respect to the grading by* T*).*

Proof.

$$[TR[T] \cap \mathscr{R}] = [IT\mathscr{R}] = [I\mathscr{R}],$$

$$x\mathscr{R} = (xR[T] \cap \mathscr{R}) \cap P_1^{(d_1)} \cap \cdots \cap P_t^{(d_t)}$$

$$\implies \qquad [xR[T] \cap \mathscr{R}] = -(d_1[P_1] + \cdots + d_t[P_t])$$

□

Example 4.4. Condition on Rees valuations in Theorem 4.2 is crucial.

$I =$ the integral closure of the ideal $(X^2, Y^3, Z^5) \subset K[X, Y, Z]$.

Then $\mathscr{R} = \mathscr{R}(I)$ is normal.

A K-basis of $\mathscr{R}$ is given by all monomials $X^a Y^b Z^c T^d$ where $15a + 10b + 6c - 30d \geq 0$.

$$\Longrightarrow \quad \mathscr{R} = R[T] \cap V_1,$$

Valuation defining V_1 is the multigraded extension of function with $v_1(X) = 15$, $v_1(Y) = 10$, $v_1(Z) = 6$ and $v_1(T) = -30$.

$$\Longrightarrow \quad [I\mathscr{R}] = 30[P_1].$$

By toric calculation $\quad \omega_{\mathscr{R}} = XYZTR[T] \cap P_1$

$$\Longrightarrow \quad [\omega_{\mathscr{R}}] = (-15 - 10 - 6 + 30)[P_1] + [P_1] = 0.$$

So $\mathscr{R}$ is a Gorenstein ring.

Remark 4.5. (a) The hypotheses of the theorem can be weakened. If we *define* the canonical module via Kaehler differentials (the description used in the proof of Theorem 3.2), then the hypothesis that the Rees algebra is Cohen–Macaulay is no longer necessary. However, one must modify the statement as follows: up to a summand in $\operatorname{Pic}(\mathscr{R})$, the canonical module has class

$$\sum_{i=1}^{t}(d_i+1-\operatorname{ht}\mathfrak{p}_i)[P_i]+[\Omega_K(R)\otimes\mathscr{R}]$$

(b) One can generalize Theorem 4.2 in such a way that Example 4.4 is covered.

Isolation of each w_i does not use the hypothesis on the Rees valuations.

Therefore, as soon as one can compute the canonical module of $R[T] \cap V_i$ for each i, a generalization is possible.

A suitable hypothesis generalizing the condition $v_{P_i}|R = v_{\mathfrak{p}_i}$ is the following: there exists a regular system of parameters $x_1, \dots, x_m$ of $R_{\mathfrak{p}_i}$ such that each of the ideals $\{x \in R_{\mathfrak{p}_i} : v_{P_i}(x) \geq k\}$ is generated by monomials in $x_1, \dots, x_m$.

Then one can replace $\operatorname{ht} \mathfrak{p}_i$ in the theorem by $v_{P_i}(x_1 \cdots x_m) = \sum_{j=1}^{m} v_{P_i}(x_j)$.

However, there exist valuations that do not allow such a “monomialization”. A counterexample was communicated by D. Cutkosky.

Corollary 4.6. *Suppose that R and I satisfy the hypothesis of Theorem 4.2. The extended Rees algebra $\widehat{\mathscr{R}} = \mathscr{R}[T^{-1}]$ (or, the associated ring $\mathrm{gr}_I(R) = \mathscr{R}/I\mathscr{R}$) is Gorenstein*

$\Longleftrightarrow$ *there exist $c_i \in \mathbb{N}$, $i = 1, \ldots, t$, such that*

(i) $c_i d_i = \mathrm{ht}\,\mathfrak{p}_i - 1$ *for all $i = 1, \ldots, t$, and*

(ii) $c_i = c_j$ *whenever there exists a maximal ideal $\mathfrak{m}$ of R with $\mathfrak{p}_i, \mathfrak{p}_j \subset \mathfrak{m}$.*

Proof. $\widehat{\mathscr{R}}$ is a subintersection of $\mathscr{R}$, namely

$$\widehat{\mathscr{R}} = \bigcap \{\mathscr{R}_Q : Q \in \mathrm{Spec}^1(\mathscr{R}), Q \neq IT\mathscr{R}\}.$$

$\Longrightarrow$ $\mathrm{Cl}(\widehat{\mathscr{R}}) = \mathrm{Cl}(\mathscr{R})/\mathbb{Z}[IT\mathscr{R}]$

Attention: $\mathrm{Pic}(\widehat{\mathscr{R}})$ may be $\neq 0$!

Therefore $[\omega_{\mathscr{R}}] \in \mathbb{Z}[IT\mathscr{R}]$ must be tested locally w.r.t. R. □

5 Applications

$R = K[X_1, \ldots, X_n]$, $f_1, \ldots, f_m \in R$ forms of degree g

$$\Longrightarrow \quad S = K[f_1, \ldots, f_m] \cong \mathscr{R}/\mathfrak{m}\mathscr{R},$$

$\mathscr{R} = \mathscr{R}(I)$, $I = (f_1, \ldots, f_m)$, $\mathfrak{m} = (X_1, \ldots, X_m)$.

Furthermore, if $\dim S = n$, then $\mathfrak{m}\mathscr{R}$ is a minimal prime ideal of $I\mathscr{R}$, and $\mathscr{R}$ and S are divisorially "close" to each other.

This may allow the computation of ω_S.

Approach is successful for algebras generated by minors.

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