Castelnuovo-Mumford regularity over general base rings

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Gröbner bases, determinants and cohomology

with Aldo, Matteo and Claudiu Raicu

The environment

Castelnuovo–Mumford regularity (regularity for short) was introduced by Mumford for coherent sheaves on projective space (1966) and transferred to modules over standard graded algebras over fields by Eisenbud and Goto (1984) and by Ooishi (1982).

Our main goal: replace the field as the base ring by an arbitrary commutative Noetherian ring:

$R = \bigoplus_{i \in \mathbb{N}} R_i$ $\mathbb{N}$-graded ring with R_0 commutative and Noetherian. We assume that R is standard graded, i.e., it is generated as an R_0-algebra by finitely many elements $x_1, \dots, x_n$ of degree 1.

Useful additional object: polynomial ring $S = R_0[X_1, \dots, X_n]$ with $\mathbb{N}$-graded structure induced by $\deg X_i = 1$. The graded R_0-algebra map $S \to R$ sending X_i to x_i induces an S-module structure on R and hence on every R-module.

The definition

Let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a finitely generated graded R-module. The regularity of M is defined in terms of local cohomology modules $H^i_{Q_R}(M)$ with support in

$$Q_R = R_+ = (x_1, \dots, x_n).$$

The modules $H^i_{Q_R}(M)$ are $\mathbb{Z}$-graded and $H^i_{Q_R}(M)_j = 0$ for $j \gg 0$. We set

$$\operatorname{reg}(M) = \max\{i + j : H^i_{Q_R}(M)_j \neq 0\}.$$

We may as well consider M as an S-module via the map $S \to R$ and local cohomology supported in

$$Q_S = (X_1, \dots, X_n).$$

Since $H^i_{Q_S}(M) = H^i_{Q_R}(M)$, the resulting regularity is the same.

Basic observations

1. $\operatorname{reg}(M(-a)) = \operatorname{reg}(M) + a$.
2. $\operatorname{reg}(S) = 0$ because $H^i_{Q_S}(S) = 0$ for $i \neq n$ and $H^n_{Q_S}(S) = (X_1 \cdots X_n)^{-1} R_0[X_1^{-1}, \dots, X_n^{-1}]$.
3. If $0 \to N \to M \to L \to 0$ is a short exact sequence of finitely generated graded R-modules with maps of degree 0 then :

$$\begin{aligned} \operatorname{reg}(N) &\leq \max\{\operatorname{reg}(M), \operatorname{reg}(L) + 1\} \\ \operatorname{reg}(M) &\leq \max\{\operatorname{reg}(L), \operatorname{reg}(N)\} \\ \operatorname{reg}(L) &\leq \max\{\operatorname{reg}(M), \operatorname{reg}(N) - 1\}. \end{aligned}$$

Minimal set of generators of M = minimal w.r.t. inclusion. Number of elements not uniquely determined, but the set of degrees is uniquely determined: given by $i \in \mathbb{Z}$ such that $[M/Q_R M]_i \neq 0$.
$\Longrightarrow$ Well defined largest degree of a minimal generator

$$t_0(M) = \max\{i \in \mathbb{Z} : [M/Q_R M]_i \neq 0\} \quad \text{if } M \neq 0.$$

We use t_0 because $M/Q_R M \simeq \operatorname{Tor}_0^R(M, R_0) = \operatorname{Tor}_0^S(M, R_0)$.

Regularity and t_0

Lemma (Ooishi)

$t_0(M) \le \operatorname{reg}(M)$.

Proof.

Can assume R_0 local with maximal ideal $\mathfrak{m}$ and infinite residue field. $M = H^0_{Q_S}(M)$ is trivial. Otherwise $M' = M/H^0_{Q_S}(M)$.

$t_0(H^0_{Q_S}(M)) \le \operatorname{reg}(M)$ and $\operatorname{reg}(M') \le \operatorname{reg}(M)$
$t_0(M) = \max\{t_0(M'), t_0(H^0_{Q_S}(M))\}$

$\implies$ can assume $M = M'$. Then Q_R contains a degree 1 nonzerodivisor x of M. Exact sequence

$$0 \to M(-1) \to M \to \overline{M} = M/xM \to 0$$

implies that $\operatorname{reg}(\overline{M}) \le \operatorname{reg}(M)$. Since $t_0(M) = t_0(\overline{M})$, done by induction on Krull dim of M.

Koszul homology and minimal free resolutions

We define variants of regularity, the first in terms of Koszul homology:

$$\operatorname{reg}_1(M) = \max\{j - i : H_i(Q_R, M)_j \neq 0\}.$$

The next two in terms of a minimal graded free S-resolution

$$\mathbb{F} : \cdots \to F_c \to F_{c-1} \to \cdots \to F_0 \to 0$$

Minimality means: F_{i+1} is mapped to a minimal system of generators of $\operatorname{Ker}(F_i \to F_{i-1})$. We set

$$\operatorname{reg}_2(\mathbb{F}) = \max\{t_0(F_i) - i \ : \ i = 0, \dots, n - \operatorname{grade}(Q_S, M)\},$$
$$\operatorname{reg}_3(\mathbb{F}) = \max\{t_0(F_i) - i : i \in \mathbb{N}\}.$$

A priori it is not clear that $\operatorname{reg}_2(\mathbb{F})$ and $\operatorname{reg}_3(\mathbb{F})$ depend only on M.

The main theorem

Theorem

For every minimal graded free S-resolution $\mathbb{F}$ of M one has

$$\operatorname{reg}(M) = \operatorname{reg}_1(M) = \operatorname{reg}_2(\mathbb{F}) = \operatorname{reg}_3(\mathbb{F}).$$

We skip the most complicated inequality $\operatorname{reg}(M) \le \operatorname{reg}_1(M)$.

Proof of $\operatorname{reg}_1(M) \le \operatorname{reg}_2(\mathbb{F})$.

One has

$$H_i(Q, M) = \operatorname{Tor}_i^S(M, R_0) = H_i(\mathbb{F} \otimes R_0)$$

$\implies$ $H_i(Q, M)$ subquotient of $F_i \otimes R_0$ $\implies$

$$\max\{j : H_i(Q, M)_j \ne 0\} \le t_0(F_i).$$

Furthermore, $H_i(Q, M) = 0$ if $i > n - \operatorname{grade}(Q_S, M)$. Therefore $\operatorname{reg}_1(M) \le \operatorname{reg}_2(\mathbb{F})$. □

The main theorem – continuation of proof

$\operatorname{reg}_2(\mathbb{F}) \leq \operatorname{reg}_3(\mathbb{F})$ is trivial.

Proof of $\operatorname{reg}_3(\mathbb{F}) \leq \operatorname{reg}(M)$.

consider

$$0 \to M_{i+1} \to F_i \to M_i \to 0.$$

By the minimality of $\mathbb{F}$ we have $t_0(F_i) = t_0(M_i) \leq \operatorname{reg}(M_i)$. Hence

$$\operatorname{reg}(M_{i+1}) \leq \max\{t_0(F_i), \operatorname{reg}(M_i) + 1\} = \operatorname{reg}(M_i) + 1$$

for all $i \geq 0$. It follows that

$$t_0(F_i) = t_0(M_i) \leq \operatorname{reg}(M_i) \leq \operatorname{reg}(M) + i$$

for every i, that is, $t_0(F_i) - i \leq \operatorname{reg}(M)$. Hence

$$\operatorname{reg}_3(\mathbb{F}) \leq \operatorname{reg}(M). \quad \square$$

Further developments

Marc Chardin has discussed several aspects of regularity and introduced a variant that in our hierarchy could be called $\operatorname{reg}_4(M)$. It coincides with $\operatorname{reg}(M)$.

Marc attributes the equality $\operatorname{reg}(M) = \operatorname{reg}_1(M)$ to Jouanolou (see paper in references).

The following example shows that the comparison of Koszul homology and minim al free resolutions is not straightforward. Let $R_0 = K[t]/(t^2)$ local, next best case after a field for R_0.

$S = R = R_0[X]$. Set $N = R/(t)$ and $M = N(-2) \oplus R/(X)$.
R_0 local $\implies$ minimal free resolution $\mathbb{F}$ is uniquely determined.We have

$$t_0(F_1) = 2, \quad \text{but} \quad t_1(M) = t_1(R/(X)) = 1.$$

Standard bigraded algebras

$R = \bigoplus_{(i,j)\in\mathbb{N}^2} R_{(i,j)}$ is $\mathbb{N}^2$-graded with $R_{(0,0)}$ commutative Noetherian, R generated by $x_1, \dots, x_n$, $\deg x_i = (1,0)$ and $y_1, \dots, y_m$, $\deg y_j = (0,1)$. Choices making R standard $\mathbb{N}$-graded:

1. $(1,0)$-grading: $R^{(*,0)} = R_{(0,0)}[R_{(1,0)}]$, R standard graded over $R^{(*,0)}$
2. $(0,1)$-grading: $R^{(0,*)} = R_{(0,0)}[R_{(0,1)}]$, R standard graded over $R^{(0,*)}$
3. R with total degree.

Every finitely generated $\mathbb{Z}^2$-graded R-module is similarly $(1,0)$graded, $(0,1)$-graded (or totally graded).
Consequently one can consider $\operatorname{reg}_{(1,0)}(M)$ or $\operatorname{reg}_{(0,1)}(M)$.

$M^{(*,j)} = \bigoplus_v M_{(v,j)}$ finitely generated $\mathbb{Z}$-graded $R^{(*,0)}$-module.
$M^{(i,*)} = \bigoplus_v M_{(i,v)}$ finitely generated $\mathbb{Z}$-graded $R^{(0,*)}$-module.

$(1,0)$- and $(0,1)$-regularity

In the bigraded case $S = R_{(0,0)}[X_1, \dots, X_n, Y_1, \dots, Y_m]$ with $\deg X_i = (1,0)$ and $\deg(Y_j) = (0,1)$.

Theorem

Let M be a finitely generated $\mathbb{Z}^2$-graded R-module, $\mathbb{F}$ a bigraded free minimal S-resolution of M. Set

1. v_i *= max v such that F_i has minimal generator in deg $(v, *)$,*
2. w_i *= max w such that F_i has minimal generator in deg $(*, w)$,*
3. $\operatorname{reg} M^{(*,j)}$ *= regularity of $M^{(*,j)}$ as $R^{(*,0)}$-module,*
4. $\operatorname{reg} M^{(i,*)}$ *= regularity of $M^{(i,*)}$ as $R^{(0,*)}$-module.*

Then

$$\max\{\operatorname{reg} M^{(*,j)} : j \in \mathbb{Z}\} = \operatorname{reg}_{(1,0)} M = \max\{v_i - i : i = 0, \dots, n\},$$
$$\max\{\operatorname{reg} M^{(i,*)} : i \in \mathbb{Z}\} = \operatorname{reg}_{(0,1)} M = \max\{w_i - i : i = 0, \dots, m\}.$$

Application: linear powers

R standard graded over R_0, $I \subset R$ ideal, generated by $f_1, \dots, f_g$ in constant degree d. I has linear powers if $\operatorname{reg}(I^v) = dv$ for all v. Consider the Rees algebra $\mathcal{R}(I) = \bigoplus_v I^v T^v \subset R[T]$ with $\mathbb{Z}^2$-graded structure

$$\mathcal{R}(I)_{(i,v)} = (I^v)_{vd+i}$$

In $\mathcal{R}(I)$ $f_1, \dots, f_g$ have degree $(0,1)$. With notation from above, $\mathcal{R}(I)^{(*,v)} = I^v(vd)$. Theorem implies

$$\operatorname{reg}_{(1,0)} \mathcal{R}(I) = \max\{\operatorname{reg} \mathcal{R}(I)^{(*,v)} : v \in \mathbb{N}\} = \max\{\operatorname{reg} I^v - vd : v \in \mathbb{N}\}$$

Theorem

1. $\operatorname{reg} I^v \le vd + \operatorname{reg}_{(1,0)} \mathcal{R}(I)$ *for all* v, $=$ *for at least one* v.
2. I *has linear powers* $\iff$ $\operatorname{reg}_{(1,0)} \mathcal{R}(I) = 0$.

Generalizations possible for family of ideals and products of powers and/ or relative to an R-module.

Nonstandard bigradings

For assertions on reg I^v in case I is not generated in a single degree we must also control nonstandard bigradings. Consider polynomial ring

$$A = A_0[Y_1, \dots, Y_g]$$

with a (non-standard) $\mathbb{Z}^2$-graded structure given by

$$\deg Y_j = (d_j, 1), \qquad d_1, \dots, d_g \in \mathbb{N},$$

and A_0 in degree $(0, 0)$. For a $\mathbb{Z}^2$-graded A-module $N = \bigoplus N_{(i,v)}$ and $v \in \mathbb{Z}$ set

$$\rho_N(v) = \sup\{i \in \mathbb{Z} : N_{(i,v)} \neq 0\} \in \mathbb{Z} \cup \{\pm\infty\}.$$

Goal: understand behavior of ρ_N.

Nature of ρ_N

Theorem

Let N be a $\mathbb{Z}^2$-graded and finitely generated A-module. Then $\rho_N(v)$ is eventually either a linear function of v with leading coefficient in $\{d_1, \dots, d_g\}$ or $-\infty$.

Steps in the proof:

Lemma

$0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_p = N$ *chain of $\mathbb{Z}^2$-graded A-modules. Then* $\rho_N(v) = \max\{\rho_{N_i/N_{i-1}}(v) : i = 1, \dots, p\}$ *for all v.*

Lemma

F bigraded free module, U bigraded submodule, $<$ term order on F. Then $\rho_{F/U}(v) = \rho_{F/\operatorname{in}_<(U)}(v)$ for all v.

Second lemma brings us into a monomial situation, the first to shifted copies of A/P, P generated by subset of the Y_j and a prime ideal of A_0.

Regularity of powers of ideals

We can now generalize the theroem of Cutkosky-Herzog-Trung and Kodiyalam. Again $R = R_0[R_1]$.

Theorem

I homogeneous ideal of R generated by homogeneous elements $f_1, \dots, f_g$ of degree $d_1, \dots, d_g$. Then there exist $\delta \in \{d_1, \dots, d_g\}$ and $c \in \mathbb{Z}$ such that

$$reg(I^v) = \delta v + c \quad \text{for } v \gg 0.$$

Can be generalized to $\operatorname{reg} I^v M$ for an A-module M and to $\operatorname{reg} I_1^{v_1} \cdots I_m^{v_m} M$. In the latter case we must take the maximum of linear functions.

The proof follows Cutkosky-Herzog-Trung, but generalizes and simplifies it somewhat.

The proof

Take the polynomial ring $R[Y_1, \dots, Y_g]$ and map it to the Rees algebra $\mathcal{R}(I) \subset R[T]$, $Y_i \mapsto f_i T$. $R[T]$ is $\mathbb{Z}^2$-graded if we extend the grading of R by $\deg T = (0, 1)$, and $\mathcal{R}(I)$ is a bigraded subalgebra.

So $R[Y_1, \dots, Y_g] \to \mathcal{R}(I)$ is a bigraded map for $\deg Y_j = (d_j, 1)$. We can compute regularity from Koszul homology, and

$$H(\mathcal{R}(I), Q_R) = \bigoplus_v H(I^v, Q_R).$$

It is annihilated by $Q_R = R \cdot R_1$. We can identifyR/Q_R with R_0.

Upshot: $N = H(\mathcal{R}(I), Q_R)$ is a bigraded module over $R_0[Y_1, \dots, Y_g]$, and $N_{(i,v)} = H_i(I^v, Q_R)$.

$\implies \operatorname{reg}(I^v) = \rho_{N^{(i,*)}}(v)$.

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