Polytope volumes in high dimension

Normaliz meets Lawrence

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Polytope volume in Noramliz

In version 3.9.0 (July 2021) Normaliz has three main algorithms for polytope volumes (years: implementation in Normaliz),

1. lex (placing) triangulation (2001), substantially improved (2012),
2. descent in the face lattice (2018), based on an implicit revlex (pulling) triangulation,
3. Lawrence's algorithm using a signed decomposition into simplices (2021).

Variants:

1. (1) and (3) can be combined with symmetrization (Schürmann): converts volume into an integral over a projection,
2. (2) can exploit isomorphism classes of faces.

Basic fact: simplex volume can be computed as a determinant.

Voume = lattice (normalized) volume
Euclidean volumes derived from lattice volume.

What algorithm should I use?

Each of the algorithms has its strength. Rules of thumb:

1. polytope has few vertices, but many facets: lex triangulation,
2. number of vertices $\approx$ number of facets: descent,
3. if there are *very* few facets: signed decomposition.

For algebraic, nonrational polytopes (in Normaliz since 2018) only lex triangulation is available, but Lawrence could be implemented for them.

Normaliz tries to be smart and follows these rules if the user does not specify an algorithm.

Remark. For Ehrhart series Normaliz has only lex triangulation in combination with symmetrization.

A challenge: social choice

Social choice = election of a leader (chairperson, mayor, president) from n candidates. Basic assumption: each voter has a preference ranking of these candidates. For n candidates there are $N = n!$ such rankings, say $R_1, \dots, R_N$.

The outcome of an election is the N-tuple $(x_1, \dots, x_N)$ where $x_i = \#$number of voters with ranking R_i. $\implies$ the voting outcomes correspond to the lattice points in $\mathbb{R}_+^N$.

The impartial anonymous culture (IAC) assumes that for a fixed number of voters all outcomes have the same probability.

Certain paradoxa and types of election results can be described by homogeneous linear inequalities for the outcomes. At this point Ehrhart theory enters the scenery: Ehrhart series count outcomes that satisfy the inequalities, and polytope volumes represent probabilities for these outcomes.

An evolution

For n candidates in an election, the polytopes of social choice have dimension $N - 1$, $N = n!$.

Before Normaliz, $n = 4$ was difficult. With lex triangulation, symmetrization and, later on, descent, it became easy.

An example: $n = 4$, Condorcet efficiency of plurality voting

lex tri	41 h
symm with lex tri	6:28 m
symm with signed dec	31.3 s
descent	0.9 s
descent with iso	2.5 s
signed dec	0.3 s

Our goal: $n = 5$, $N = 120$.

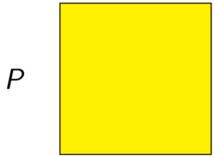
Duality and signed decomposition I

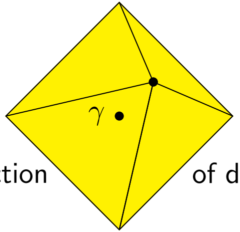
Let the polytope P be given as $P = C \cap H$ with a pointed cone $C \subset \mathbb{R}^d$, $\dim C = d$, and a hyperplane $H = \{x : \gamma(x) = 1\}$ where $\gamma \in (\mathbb{R}^d)^*$ is a "grading".

Consider the dual cone

$$C^* = \{\lambda \in (\mathbb{R}^d)^* : \lambda(x) \geq 0 \text{ for } x \in C\}.$$

Then $\gamma \in \operatorname{relint}(C^*)$. Let$\Gamma$ be a "generic" triangulation of C^*: $\gamma \notin G$ where G is any hyperplane intersecting any simplicial cone $\delta \in \Gamma$ in a hyperplane.





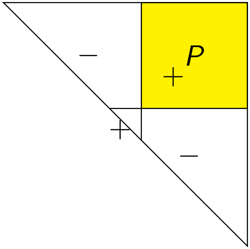
Duality and signed decomposition II

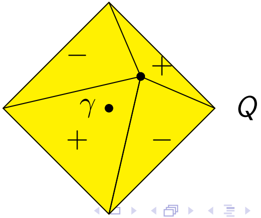
For fixed $\delta \in \Gamma$ the hyperplanes G as above decompose $(\mathbb{R}^d)^*$ into 2^d "orthants" and γ lies in the interior of exactly one, say D_δ.
$\implies D_\delta^* \subset (\mathbb{R}^d)^{**} = \mathbb{R}^d$ intersects H in a bounded polytope Q_δ.

Theorem (Lawrence)

$$\operatorname{vol} P = \sum_\delta (-1)^{e_\delta} \operatorname{vol} Q_\delta$$

where e_δ counts the number of hyperplanes G through facets of δ with $\gamma \in G^-$, $\delta \subset G^+$.





Is there a catch?

In applications, like our loved social choice computations, the polytopes are defined by very few inequalities (relative to their dimensions), and triangulations are not so difficult to compute, not even in dimension 120. However, there is a catch, hidden in the innocent property "generic".

There seems to be no better method than the following. We take an arbitrary, say lex, triangulation Δ_0 of C^*. It induces a "hollow " triangulation Γ_0 on the boundary of C^*. Then we search a "generic" vector $g \in C^*$ and take the "star triangulation" Γ by simplicial pyramids with apex g and bases in the hollow triangulation.

Inevitable: g has large coordinates $\implies$ the Q_δ have terrible rational coordinates $\implies$ volumes are horrible rational numbers.

Adding rational numbers and fixed precision

Normaliz uses a cascade type of summation that is rather good for only moderately complicated dases – one can spend 99% of the computation time on this addition if one does it the naive way.

But for really large computations, the fractions can take gigabytes, and addition becomes impossible. For these cases there is a fixed precision mode, in which vol Q_δ is computed precisely, but then rounded to, say, 100 decimal digits: error $\le \#\Gamma * 10^{-100}$.

Remark. We learnt the existence of the Lawrence algorithm from a paper about vinci (Büeler and Enge, `https://www.math.u-bordeaux.fr/~aenge/`). vinci uses only floating point arithmetic, and cannot cope with the numerical instability of the algorithm in the range where we want to apply it. The fractions in the alternating sum can easily reach an absolute value of 10^{100} and the sum is perhaps only 10^{-5}.

Overview of the algorithm

The algorithm proceeds in 4 steps:

1. Find a triangulation Δ_0 of C^*.
2. Find the hollow triangulation Γ_0 induced by Δ_0.
3. Find a generic point g.
4. Compute the volume.

Each step needs a sophisticated implementation to reach the order of magnitude presented by the Condorcet efficiency of plurality voting of 5 candidates:

- $\dim C = 120$,
- 128 inequalities,
- $\#\Delta_0 \approx 2.4 * 10^9$,
- $\#\Gamma \approx 39 * 10^9$,
- $|\text{entries of } g| \approx 10^{10}$.

Only doable in reasonable time since step (4) can be distributed to the nodes in a high performance cluster.

Computation time and memory usage

We compare 3 computations for elections with 5 candidates:

- CP: probability of Condorcet paradox,
- CW2nd: probability of Condorcet winner and second,
- CEP: Condorcet efficiency of plurality voting.

dim $C = 120$, 32 parallel threads, fixed precision only for CEP.

	CP	CW2nd	CEP
#ineq	124	126	128
$\#\Delta_0$	137,105	$16 * 10^6$	$2.4 * 10^9$
$\#\Gamma_0$	$6.7 * 10^6$	$609 * 10^6$	$39.4 * 10^9$
time Δ_0	0.5 s	834 s	6 h
time Γ	6.8 s	1633 s	105 h
time g	10.7 s	1277 s	35 h
time vol	52.5 S	37,296 s	$<$ 12 h HPC
RAM in GB	1.7	57	640

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