Lattice polytopes

Algebraic, geometric and combinatorial aspects

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Sources

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Toric varieties

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Lecture 1

Basic notions

Polyhedral geometry

Definition

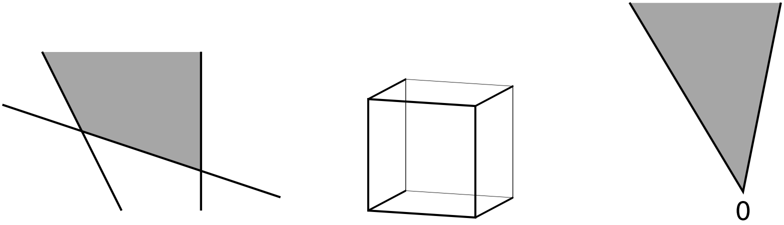
- A polyhedron is the intersection of finitely many closed affine halfspaces.
- A polytope is a bounded polyhedron.
- A cone is the intersection of finitely many *linear* halfspaces.

A closed affine halfspace is a set

$$H_\alpha^+ = \{x \in \mathbb{R}^d : \alpha(x) \geq 0\}$$

where α is an affine form, i. e. a polynomial function of degree 1. It is linear if α is a linear form. Its bounding hyperplane is

$$H_\alpha = \{x \in \mathbb{R}^d : \alpha(x) = 0\}.$$



A polyhedron, a polytope and a cone

The dimension of P is $\dim \operatorname{aff}(P)$.

Definition

A face of P is the intersection $P \cap H$ where H is a *support hyperplane*, i. e. $P \subset H^+$ and $P \not\subset H$.

A facet is a maximal face. A vertex is a face of dim 0.

$P, \emptyset$ *improper faces*.

For full-dimensional cones the (essential) support hyperplanes $H_i = \{x : \lambda_i(x) = 0\}$ are unique:

Proposition

Let $P \subset \mathbb{R}^d$, $\dim P = d$. If the representation $P = H_1^+ \cap \cdots \cap H_s^+$ is irredundant, then the hyperplanes H_i are uniquely determined (up to enumeration). Equivalently, the affine forms α_i are unique up to positive scalar factors.

Finite generation of cones

Theorem (Minkowski-Weyl)

Let $C \neq \emptyset$ be a subset of $\mathbb{R}^m$. Then the following are equivalent:

- *there exist finitely many elements $y_1, \dots, y_n \in \mathbb{R}^m$ such that $C = \mathbb{R}_+ y_1 + \dots + \mathbb{R}_+ y_n$;*
- *there exist finitely many linear forms $\lambda_1, \dots, \lambda_s$ such that C is the intersection of the half-spaces $H_i^+ = \{x : \lambda_i(x) \geq 0\}$.*

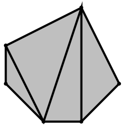
A *simplicial cone* is generated by linearly independent vectors.

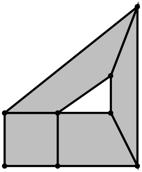
Corollary

P is a polytope $\iff$ $P = \operatorname{conv}(x_1, \dots, x_m)$

One considers the cone over P (see below) and applies the theorem.

A *simplex* is the convex hull of a affinely independent set.





Definition

An (embedded) *polyhedral complex* Π is a finite collection of polyhedra $P \subset \mathbb{R}^d$ such that

- with P each face of P belongs to Π,
- $P, Q \in \Pi \Rightarrow P \cap Q$ is a face of P (and of Q)

Π is a *fan* if it consists of cones.

A *polytopal complex* consists of polytopes.

A *simplicial complex* consists of simplices.
A *simplicial fan* consists of simplicial cones.

A *subdivision* of Π is a polyhedral complex Π' such that each face of Π is the union of faces of Π'.

A *triangulation* of a polytopal complex is a subdivision into a simplicial complex.

A *triangulation* of a fan is a subdivision into a simplicial fan.

There are always enough triangulations:

Theorem

Let Π be a polytopal complex and X a finite subset of $|\Pi| = \bigcup_{P\in\Pi} P$ such that $\operatorname{vert}(\Pi) \subset X$. Then there there exists a triangulation Δ of Π such that $X = \operatorname{vert}(\Delta)$.

An analogous theorem holds for fans.

Rationality

Proposition

Let C be a cone. The generating elements $y_1, \dots, y_n$ can be chosen in $\mathbb{Q}^m$ (or $\mathbb{Z}^m$) if and only if the α_i can be chosen as linear forms with rational (or integral) coefficients.

Such cones are called *rational*. For them there is a unique choice of the α_i satisfying

- $\alpha_i(\mathbb{Z}^d) \subset \mathbb{Z}$,
- $1 \in \alpha_i(\mathbb{Z}^d)$.

With this standardization, we denote them by σ_i and call them *support forms*.

Affine monoids and their algebras

Definition

An *affine monoid* M is (isomorphic to) a finitely generated submonoid of $\mathbb{Z}^d$ for some $d \geq 0$, i. e.

- $M + M \subset M \quad$ (M is a semigroup);
- $0 \in M$ (now M is a monoid);
- there exist $x_1, \dots, x_n \in M$ such that $M = \mathbb{Z}_+ x_1 + \dots + \mathbb{Z}_+ x_n$.

Often affine monoids are called affine semigroups.

$\operatorname{gp}(M) = \mathbb{Z}M \quad$ is the group generated by M.

$\operatorname{gp}(M) \cong \mathbb{Z}^r \quad$ for some $r = \operatorname{rank} M = \operatorname{rank} \operatorname{gp}(M)$.

Let K be a field (can often be replaced by a commutative ring). Then we can form the monoid algebra

$$K[M] = \bigoplus_{a \in M} KX^a, \qquad X^a X^b = X^{a+b}$$

$X^a =$ the basis element representing $a \in M$.

$M \subset \mathbb{Z}^d$ (affine) $\Rightarrow K[M] \subset K[\mathbb{Z}^d] = K[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ is a (finitely generated) monomial subalgebra.

Sometimes a problem: additive notation in M versus multiplicative notation in $K[M]$ (and exponential notation is often cumbersome).

Proposition

Let M be a monoid.

1. *M is finitely generated $\iff K[M]$ is a finitely generated K-algebra.*
2. *M is an affine monoid $\iff K[M]$ is an affine domain.*

Proposition

The Krull dimension of $K[M]$ is given by

$$\dim K[M] = \operatorname{rank} M.$$

Proof. $K[M]$ is an affine domain over K. Therefore

$$\begin{aligned}\dim K[M] &= \operatorname{trdeg} \operatorname{QF}(K[M])\\ &= \operatorname{trdeg} \operatorname{QF}(K[\operatorname{gp}(M)])\\ &= \operatorname{trdeg} \operatorname{QF}(K[\mathbb{Z}^r])\\ &= r\end{aligned}$$

where $r = \operatorname{rank} M$.

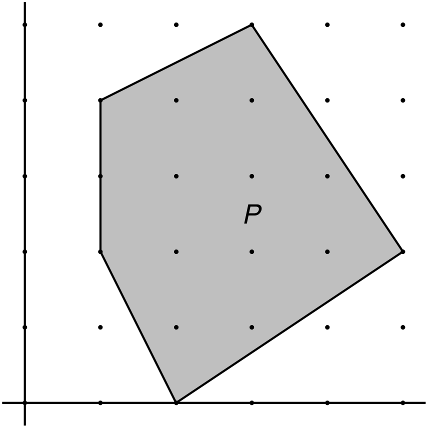
Sources for affine monoids (and their algebras) are

- monoid theory,
- ring theory,
- invariant theory of torus actions,
- enumerative theory of linear diophantine systems,
- lattice polytopes and rational polyhedral cones,
- coordinate rings of toric varieties,
- initial algebras with respect to monomial orders.

Lattice polytopes

Definition

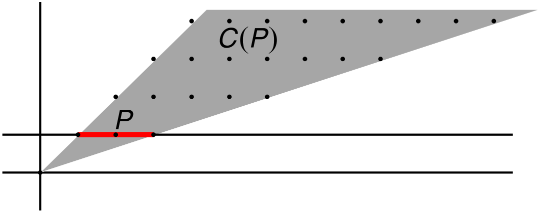
The convex hull $\operatorname{conv}(x_1, \dots, x_m)$ of points $x_i \in \mathbb{Z}^d$ is called a *lattice polytope*.



Definition

The polytopal monoid associated with P is

$$M(P) = \mathbb{Z}_+\{(x,1) : x \in P \cap \mathbb{Z}^d\}.$$



Definition

The cone over (an arbitrary polytope) P is

$$C(P) = \mathbb{R}_+\{(x,1) \in \mathbb{R}^{d+1} : x \in P\}.$$

Definition

The polytopal algebra associated with P is the monoid algebra

$$K[P] = K[M(P)].$$

$K[P]$ has a natural $\mathbb{Z}_+$-grading in which the generators have degree 1.

Objects of algebraic geometry:

- M affine monoid: Spec $K[M]$ is an affine toric variety
- P lattice polytope: Proj $K[P]$ is a projective toric variety

Toric varieties are not necessarily normal (Oberwolfach convention January 2006).

Affine charts for projective toric varieties

Proj $K[P]$ is covered by the affine varieties $K[M_v]$ where M_v is a *corner monoid* of P: one has $v \in \operatorname{vert}(P)$ and

$$M_v = \mathbb{Z}_+\{x - v : x \in P \cap \mathbb{Z}^d\}.$$

Then

$$\operatorname{Proj} K[P] \text{ smooth} \iff K[M_v] \cong K[X_1, \dots, X_d] \iff M_v \cong \mathbb{Z}_+^d$$

for all $v \in \operatorname{vert}(P)$.

Especially each vertex of P must be contained in exactly d edges.

Toric ideals

Presentation of affine monoid algebras:

Let $R = K[x_1, \dots, x_n]$. Then we have a presentation

$$\pi : K[X] = K[X_1, \dots, X_n] \to K[x_1, \dots, x_n], \qquad X_n \mapsto x_n.$$

Let $I = \operatorname{Ker} \pi$ and $M = \{\pi(X^a) : a \in \mathbb{Z}_+^n\}$

Theorem

The following are equivalent:

1. *M is an affine monoid and $R = K[M]$;*
2. *I is prime, generated by binomials $X^a - X^b$, $a, b \in \mathbb{Z}_+^n$;*
3. *$I = IK[X^{\pm 1}] \cap K[X]$, I is generated by binomials $X^a - X^b$, and $U = \{a - b : X^a - X^b \in I\}$ is a direct summand of $\mathbb{Z}^n$.*

Definition

A prime ideal I as above is called a *toric ideal*.

$X^a - X^b \in I \iff a_1x_1 + \cdots + a_nx_n = b_1x_1 + \cdots + b_nx_n$ (using additive notation in the monoid): The binomials in I represent the linear dependencies of the vectors generating M, and, in the polytopal case, the affine dependencies of the lattice points of P ($\sum a_i = \sum b_i$ in this case).

Algorithmic approach for the computation of the toric ideal I ($M = \mathbb{Z}_+x_1 + \cdots + \mathbb{Z}_+x_n$):

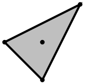
- Compute the kernel $\mathbb{Z}c_1 + \cdots + \mathbb{Z}c_{n-r}$ of $\pi : \mathbb{Z}^n \mapsto \operatorname{gp}(M)$, $e_i \mapsto x_i$, $r = \operatorname{rank} M$
- Saturate the ideal generated by $X^{c_i^+} - X^{c_i^-}$, $i = 1, \dots, n-r$ with respect to $X_1, \dots, X_n$.

$$c_i = c_i^+ - c_i^-, \quad c_i^+, c_i^- \mathbb{Z}^+.$$

Examples

$$K[P] = K[Y^0Z, Y^1Z, Y^2Z] \cong K[X_1, X_2, X_3]/(X_1X_3 - X_2^2)$$

$$K[P] = K[Y_1^0Y_2^0Z, Y_1^0Y_2^1Z, Y_1^1Y_2^0Z, Y_1^1Y_2^1Z]$$
$$\cong K[X_1, X_2, X_3, X_4]/(X_1X_4 - X_2X_3)$$



$$K[P] = K[Y_1^{-1}Y_2^0Z, Y_1^0Y_2^{-1}Z, Y_1^1Y_2^1Z, , Y_1^0Y_2^0Z]$$
$$\cong K[X_1, X_2, X_3, X_4]/(X_1X_2X_3 - X_4^3)$$

An affine monoid M generates the cone

$$\mathbb{R}_+ M = \left\{ \sum a_i x_i : x_i \in M,\ a_i \in \mathbb{R}_+ \right\}$$

Since $M = \sum_{i=1}^{n} \mathbb{Z}_+ x_i$ is finitely generated, $\mathbb{R}_+ M$ is finitely generated (and therefore a cone)

The structures of M and $\mathbb{R}_+ M$ are connected in many ways.

Gordan's lemma and normality

As seen above, affine monoids define rational cones. The converse is also true.

Lemma (Gordan's lemma)

Let $L \subset \mathbb{Z}^d$ be a subgroup and $C \subset \mathbb{R}^d$ a rational cone. Then $L \cap C$ is an affine monoid.

Proof. Let $V = \mathbb{R}L \subset \mathbb{R}^d$. Then $V \cap \mathbb{Q}^d = \mathbb{Q}L$ and $C \cap V$ is a rational cone in V

$\Rightarrow$ We may assume that $L = \mathbb{Z}^d$.

C is generated by elements $y_1, \dots, y_n \in M = C \cap \mathbb{Z}^d$.

$$x \in C \Rightarrow x = a_1 y_1 + \cdots + a_n y_n \qquad a_i \in \mathbb{R}_+.$$

$$x = x' + x'', \qquad x' = \lfloor a_1 \rfloor y_1 + \cdots + \lfloor a_n \rfloor y_n.$$

Clearly $x' \in M$. But

$$x \in M \Rightarrow x'' \in \mathrm{gp}(M) \cap C \Rightarrow x'' \in M.$$

x'' lies in a bounded set $B \Rightarrow$

M generated by $y_1, \dots, y_n$ and the finite set $M \cap B$.

More precisely we have shown:

Theorem

Suppose $y_1, \dots, y_n \in M$ generate C. Then M is a finitely generated module over $N = \mathbb{Z}_+ y_1 + \cdots + \mathbb{Z}_+ y_n$:

$$M = \bigcup_{x \in B \cap M} N + x.$$

The monoid $M = L \cap C$ has a special property: it is integrally closed in L:

Integral closure

Definition

A monoid M is *integhrally closed in an overmonoid* $N \iff$

$$x \in N,\ \ kx \in M \text{ for some } k \in \mathbb{Z}, k > 0 \quad \Rightarrow \quad x \in M.$$

Two types of integral closedness are interesting for us:

- $M \subset \mathbb{Z}^n$ integrally closed in $\mathbb{Z}^n$,
- M integrally closed in $\mathrm{gp}(M)$. In this case M is called *normal*.

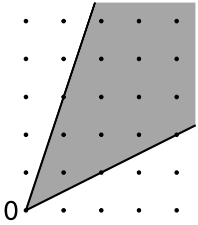
Being normal is necessary for integral closedness in any group containing M.

Theorem

- $M \subset \mathbb{Z}^d$ *integrally closed affine monoid* $\iff$ *there exists a rational cone* C *such that* $M = C \cap \mathbb{Z}^d$;
- $M \subset \mathbb{Z}^d$ *affine monoid* $\Rightarrow$ *the integral closure* $\widehat{M} = \mathbb{R}_+ M \cap \mathbb{Z}^d$ *is a finitely generated* M*-module and therefore an affine monoid.*

This applies especially to the choice $\mathbb{Z}^d \cong \operatorname{gp}(M)$. In this case $\bar{M} = \widehat{M}$ is the *normalization*.

Briefly: **Normal affine monoids are discrete cones.**



Normality of $K[M]$

An integral domain is *normal* if it is integrally closed in its field of fractions Q: if $x \in Q$ satisfies an equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad a_i \in R,$$

then $x \in R$. Examples: all factorial domains.

Theorem

Let M be an affine monoid, K a field. Then $K[M]$ is normal if and only if M is normal.

Positivity and Hilbert bases

Definition

A monoid M is *positive* if $x, -x \in M \Rightarrow x = 0$.

Definition

A *grading* on M is a homomorphism $\deg : M \to \mathbb{Z}$. It is *positive* if $\deg x > 0$ for $x \neq 0$.

Proposition

For M affine the following are equivalent:

1. *M is positive;*
2. *$\mathbb{R}_+ M$ is pointed (i. e. contains no full line);*
3. *M is isomorphic to a submonoid of $\mathbb{Z}^s$ for some s;*
4. *M has a positive grading.*

Proof. (c) $\Rightarrow$ (d) $\Rightarrow$ (a) trivial.

(a) $\Rightarrow$ (b) Set $C = \mathbb{R}_+ M$. One shows:
$\{x \in C : -x \in C\} = \mathbb{R}\{x \in M : -x \in M\}$.
Therefore: M positive $\Rightarrow$ C pointed.

(b) $\Rightarrow$ (c) Let C be positive. Recall: for each facet F of C there exists a unique linear form $\sigma_F : \mathbb{R}^d \to \mathbb{R}$ with the following properties:

- $F = \{x \in C : \sigma_F(x) = 0\}, \quad \sigma_F(x) \geq 0$ for all $x \in C$;
- σ has integral coefficients, $\quad \sigma(\mathbb{Z}^d) = \mathbb{Z}$.

We have called them *support forms* of C.

Let $s = \#\,\text{facets}(C)$ and define

$$\sigma : \mathbb{R}^d \to \mathbb{R}^s, \qquad \sigma(x) = \big(\sigma_F(x) : F \text{ facet}\big).$$

Then $\sigma(M) \subset \sigma(\bar{M}) \subset \mathbb{Z}_+^s$. Since C is positive, σ is injective!
We call σ the *standard embedding*.

$x \in M$ is *irreducible* if $x = y + z \Rightarrow y = 0$ or $z = 0$.

Proposition

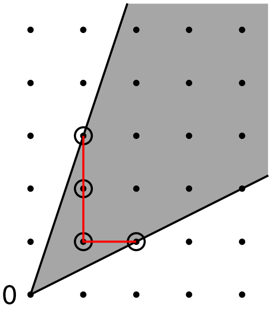
The irreducible elements of a positive affine monoid form its unique minimal system of generators.

Proof. Every system of generators contains all the irreducible elements. Enough to show: every element is the sum of irreducibles. Can assume: $M \subset \mathbb{Z}_+^n$. Set $\deg x = s_1 + \cdots + x_n$. Now induction on $\deg x$. Either x is irreducible or it decomposes as $x = y + z$ with $\deg y, \deg z < \deg x$.

Definition

The set of irreducible elements is called the *Hilbert basis* of M.

Typical picture for normal monoids of rank 2:



Hilb(M) is the set of lattice points in the bottom of M.

In higher dimension is situation is complicated.

Given a positive affine monoid $M \subset \mathbb{Z}^d$ by a set of generators, one can compute Hilbert bases of the normalization $\bar{M}$ and and the integral closure $\widehat{M}$ in $\mathbb{Z}^d$:

W.B., R. Koch, normaliz, available from

`ftp.math.uos.de/pub/osm/kommalg/software/`

normaliz computes various other data. It is accessible from Singular via a library.

Lecture 2

Unimodular covers and triangulations

A nonnormal polytope

Let P be a lattice polytope. In general $M(P)$ is not normal.
Example:

$$P = \{x \in \mathbb{R}^3 : x_i \geq 0,\ \ 15x_1 + 10x_2 + 6x_3 \leq 30\}.$$

One has

$$P = \operatorname{conv}\big(\ (0,0,0),\ (5,0,0),\ (0,3,0),\ (0,0,2)\ \big)$$

Evidently, $\operatorname{gp}(M(P)) = \mathbb{Z}^4$, $(1,2,4,2) \in C(P)$, but
$(1,2,4,2) \notin M(P) \Rightarrow M(P)$ not integrally closed in $\mathbb{Z}^4$

BUT $\operatorname{gp}(M(P)) = \mathbb{Z}^4 \Rightarrow M(P)$ is not normal.

In other words: $M(P) \neq \bar{M}(P)$, $\operatorname{Hilb}(M(P)) \neq \operatorname{Hilb}(\bar{M}(P))$.

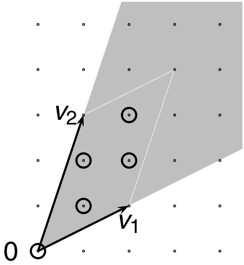
Where can we find the remaining elements of $\operatorname{Hilb}(\widehat{M}(P))$?

Strategy

- Triangulate polytope (or cone)
- investigate simplices (or simplicial cones)

Simplicial cones

Recall: A cone is simplicial if it is generated by linearly independent vectors.



$$\text{par}(v_1, \dots, v_n) = \{q_1 v_1 + \cdots + q_n v_n : 0 \le q_i < 1,\ i = 1, \dots, n\}$$

$$E = \mathbb{Z}^n \cap \text{par}(v_1, \dots, v_n)$$

$v_1, \dots, v_n \in \mathbb{Z}^n$ linearly independent. Set

$$M = \mathbb{Z}_+ v_1 + \cdots + \mathbb{Z}_+ v_n, \qquad U = \operatorname{gp}(M), \qquad C = \mathbb{R}_+ M$$

Proposition

Then

1. *E is a system of generators of the M-module $C \cap \mathbb{Z}^n$;*
2. $(x + M) \cap (y + M) = \emptyset$ *for $x, y \in E$, $x \neq y$;*
3. $\#E = [\mathbb{Z}^n : U]$*;*
4. $\operatorname{Hilb}(C \cap \mathbb{Z}^n) \subset \{v_1, \dots, v_n\} \cup E$*.*

Especially $C \cap \mathbb{Z}^n$ is the disjoint union of the M-modules $x + M$, $x \in M$.

Note: If $\deg v_i = 1$ for all i, then $\deg x < r$ for $x \in E$.

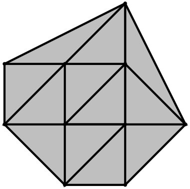
Via triangulation this helps to bound the Hilbert basis is general.

Bounding the Hilbert basis

Theorem (Ewald-Wessels, B.-Gubeladze-Trung)

Let P be a lattice polytope, Then $\widehat{M}(P)$ *is generated by elements of degree* $\leq \dim P - 1$.
More precisely, as a module over $M(P)$ *its is generated by elements of degree* $\leq \dim P - 1$.

Proof. Triangulate P such that each simplex contains only its vertices as lattice points:



Enough to consider a simplex P whose only lattice points are the vertices.

Proposition on simplicial cones implies: Can generate $\widehat{M}(P)$ by elements with $\deg y \leq= \dim P$. (The simplex has $\dim P + 1$ vertices!)

Cute trick: assume $x \in E$ (notation as above) has degree d. Then $v_1 + \cdots + v_{d+1} - y$ has degree 1 and lies in P. But then $y = v_1 + \cdots + v_{i-1} + v_{i+1} + \cdots + v_{d+1}$ for some i. Contradiction!

The bound in the theorem is the best possible in all dimensions.

In the corollary we apply attributes to P that, strictly speaking, are defined for $M(P)$:

Corollary

Let P be a lattice polytope.

- *If $\dim P = 2$, then P is integrally closed.*
- *More generally, cP is integrally closed for $c \geq \dim P - 1$.*

Proof. If we cut $C(P)$ by a hyperplane at height c, we obtain cP. Therefore

$$\widehat{M}(cP) = \{x \in \widehat{M}(P) : \deg y \equiv 0 \quad (c)\}.$$

If $\deg y = mc$, then $y = x + z_1 + \cdots + z_u$ with $\deg x \leq \dim P - 1$ and $\deg z_j = 1$ ($u = mc - \deg x$). Cut this sum into subsums of degree c.

This result follows the principle that a graded rings “improves” by the passage to Veronese subrings, or – in projective algebraic geometry – a divisor “improves” it is replaced by a high multiple.

Quadratic generation of toric ideals

Let me mention another result that can be proved rather easily by combinatorial methods:

Theorem (B.-Gubeladze-Trung)

Let $c \geq \dim P$*. Then*

- $K[cP]$ *has a presentation as a residue class ring of a polynomial ring by an ideal with a quadratic Gröbner basis.*
- *The toric ideal defining* $K[cP]$ *is generated in degree* 2*.*
- $K[cP]$ *is a Koszul algebra.*

A recent result improves this in characteristic 0:

Theorem (Hering-Schenck-Smith)

$\mathbb{C}[cP]$ has the Green-Lazarsfeld property N_p for $c \geq \min(\dim P + c - 1, c)$.

In ring-theoretic terms: a graded ring R has

- N_0 if it is generated in degree 1;
- N_1 if it has N_0 and the defining ideal is generated in degree 2;
- N_p, $p \geq 2$, if it has N_1 and the syzygies up to step p are linear.

The theorem can be refined if one takes into account the degrees of the elements ion the interior of $C(P)$.

Open questions:

Suppose P lattice polytope such that Proj $K[P]$ is smooth.

- Is $k[P]$ Koszul or at least defined by a toric ideal generated in degree 2?
- Is $K[P]$ normal?

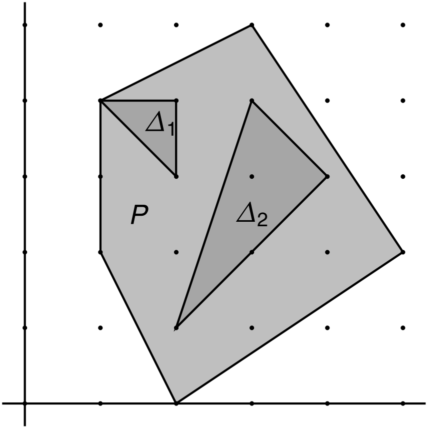
The second question has a good chance for a positive answer since the corner monoids are free. However, normality of the corner monoids does not imply normality of P (the other direction holds).

Can one explain normality of lattice polytopes?

– in the sense that it is equivalent to simpler, formally stronger properties

I think the answer is no, but this is not the last word.

Recall: $P = \text{conv}(x_1, \dots, x_n) \subset \mathbb{R}^d$, $x_i \in \mathbb{Z}^d$, is called a lattice polytope.



$\Delta = \text{conv}(v_0, \dots, v_d)$, $v_0, \dots, v_d$ affinely independent, is a simplex.

Set $U_\Delta = \sum_{i=0}^{d} \mathbb{Z}(v_i - v_0)$.

$$\mu(\Delta) = [\mathbb{Z}^d : U_\Delta] = \text{multiplicity of } \Delta$$

Δ is **unimodular** if $\mu(\Delta) = 1$.

Δ is **empty** if $\operatorname{vert}(\Delta) = \Delta \cap \mathbb{Z}^d$.

Lemma

$$\mu(\Delta) = d!\operatorname{vol}(\Delta) = \pm\det\begin{pmatrix} v_1 - v_0 \\ \vdots \\ v_d - v_0 \end{pmatrix}$$

When is P covered by its unimodular subsimplices?

For short: P has UC.

Multiplicity for simplicial cones:

Let $v_1, \dots, v_d \in \mathbb{Z}^d$ be linearly independent elements, and suppose that each v_i has coprime entries. Set $C = \mathbb{R}_+ v_1 + \dots + \mathbb{R}_+ v_d$. Then

$$\mu(C) = |\det(v_1, \dots, v_d)| = \mu(\operatorname{conv}(0, v_1, \dots, v_d)).$$

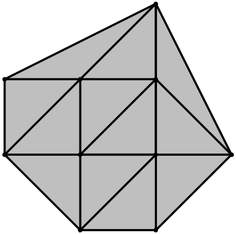
C is *unimodular* if $\mu(C) = 1$.

Low Dimensions

$d = 1$: 

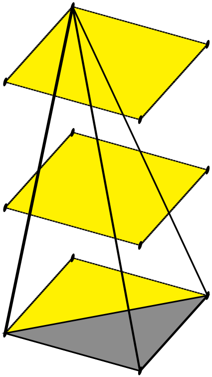
P has a unique unimodular triangulation.

$d = 2$:



Recall: lattice polytopes of dim 2 are integrally closed
Therefore every empty lattice triangle is unimodular $\Rightarrow$ every 2-polytope has a unimodular triangulation.

$d = 3$: There exist empty simplices of arbitrary multiplicity!



Already clear from the existence of a nonnormal 3-polytope. Namely:

Proposition

P has UC $\Rightarrow M(P) = \widehat{M}(P)$ (P is integrally closed).

Recall: P is integrally closed $\iff$

- $\mathrm{gp}(M(P)) = \mathbb{Z}^{d+1}$ and
- $M(P)$ is a normal monoid ($M(P) = C_P \cap \mathrm{gp}(M(P))$)

Does every integrally closed polytope have UC?

Generalization to rational cones

Let C be a rational cone. Then $M = C \cap \mathbb{Z}^d$ is automatically an integrally closed affine monoid.

Generalization of the condition UC is UHC: C the union of its unimodular subcones that are generated by elements of $\operatorname{Hilb}(M)$

Does every C have UHC?

Theorem (Sebő)

Cones of dimension 3 have even a unimdodular Hilbert triangulation.

This generalizes the theorem that polytopes of dimension 2 have unimodular triangulations.

The integral Carathéodory property

The condition UHC can be weakened in an algebraic way. Recall

Theorem (Carathéodory)

Let $C = \mathbb{R}_+ x_1 + \cdots + \mathbb{R}_+ x_n$ be a cone of dimension d, and $y \in C$. Then there exists $i_1, \dots, i_d$ and $a_{i_j} \in \mathbb{R}_+$ such that $y = a_{i_1} x_{i_1} + \cdots + a_{i_d} x_{i_d}$.

This can be viewed as a consequence of the existence of triangulations, but is more elementary.

Discrete version is the integral Carathéodory property ICP:

An positive affine monoid M has ICP if every $y \in M$ contained in a submonoid generated by d elements of $\mathrm{Hilb}(M)$, $d = \operatorname{rank} M$.

Theorem (B.-Gubeladze)

If M has ICP, then

(a) *M is normal*

(b) *every element of M is contained in a submonoid generated by linearly independent elements of $\operatorname{Hilb}(M)$.*

Also UHC can be formulated for affine monoids:
replave "linearly independent elements of $\operatorname{Hilb}(M)$" in (b)
by "basis of $\operatorname{gp}(M)$ contained in $\operatorname{Hilb}(M)$".
Thus UHC $\Rightarrow$ ICP.

Since UHC and ICP imply normality, it is enough to consider monoids of type $C \cap \mathbb{Z}^d$.

Does every rational cone C have ICP? (We assign properties to C that are defined for $C \cap \mathbb{Z}^d$).

Counterexamples

- (Bouvier and Gonzalez-Springberg, 1994) There exists a cone of dimension 4 without a unimodular triangulation into subcones generated by elements of Hilb(M), $M = C \cap \mathbb{Z}^5$.
- (Kantor-Sarkaria, 2003) There exists a normal lattice polytope of dimension 3 without a unimodular triangulation.
- (B.-Gubeladze-Henk-Martin-Weismantel, 1998) There exists a normal lattice polytope of dimension 5 without ICP.
- (2006) There exists a normal lattice polytope of dimension 5 with ICP that violates UHC.

$C_6 \cap \mathbb{Z}^6$ with Hilbert basis $z_1, \dots, z_{10}$, is of form $C(P_5)$, $\dim P_5 = 5$, P_5 integrally closed, and violates UHC and ICP

$$
\begin{aligned}
z_1 &= (0, 1, 0, 0, 0, 0), & z_6 &= (1, 0, 2, 1, 1, 2),\\
z_2 &= (0, 0, 1, 0, 0, 0), & z_7 &= (1, 2, 0, 2, 1, 1),\\
z_3 &= (0, 0, 0, 1, 0, 0), & z_8 &= (1, 1, 2, 0, 2, 1),\\
z_4 &= (0, 0, 0, 0, 1, 0), & z_9 &= (1, 1, 1, 2, 0, 2),\\
z_5 &= (0, 0, 0, 0, 0, 1), & z_{10} &= (1, 2, 1, 1, 2, 0).
\end{aligned}
$$

If we add

$$
z'_{11} = (0, -1, 2, -1, -1, 2) \qquad z'_{12} = (1, 0, 3, 0, 0, 3)
$$

to the Hilbert basis, then we obtain a cone C'_6 of type $C(P'_5)$ that satisfies ICP, but violates UHC.

Interesting fact: both monoids $C_6 \cap \mathbb{Z}^6$ and $C_6' \cap \mathbb{Z}^6$ have the Frobenius group F_{20} as their automorphism group.

The results of very long searches for counterexamples in dimensions 6 and 7 suggest the following:

- C_6 is the minimal counterexample to ICP and UHC.
- All other counterexamples "contain" it.

Open problem: Do all cones of dimensions 4 and 5 have UHC?

See W.B., *On the integral Carathéodory property*, Exp. Math., to appear

Carathéodory rank

Definition

Let M be a positive affine monoid, $y \in M$. Set

$$\rho(y) = \min\{m : y = a_1x_1 + \cdots + a_mx_m,\ a_i \in \mathbb{Z}_+,\ x_i \in \operatorname{Hilb}(M)\},$$

and

$$\operatorname{CR}(M) = \max\{\rho(x) : x \in M\}.$$

Then $\operatorname{CR}(M)$ is the *Carathéodory rank* of M.

Theorem (Cook-Vonlupt-Schrijver, Sebő)

Suppose $\operatorname{rank} M \geq 3$. *If M is normal, then* $\operatorname{CR}(M) \leq 2\operatorname{rank}(M) - 2$.

Proof uses ideas very similar to the proof of the theorem of Ewald-Wessels.
If M is not normal, then there is no bound for CR in terms of rank.

Clearly, M has ICP $\iff \mathrm{CR}(M) = \operatorname{rank} M$.

C_6 shows: in general $\mathrm{CR}(M) > \operatorname{rank}(M)$, even if M is normal. But it is open to what extent the bound in the theorem is sharp, or close to being sharp.

Using C_6 and direct sums:

Proposition

For all $n \geq 6$ there exist normal affine monoids with $\mathrm{CR}(M) \geq \lfloor \frac{7}{6} \operatorname{rank} M \rfloor$.

Strategy of the search for counterexamples to UHC and ICP

M always a normal monoid, $M = C \cap \mathbb{Z}^d$.

Definition

Let $x \in \operatorname{Hilb}(M)$ and $M' = \mathbb{Z}_+ \operatorname{Hilb}(M) \setminus \{x\}$. We call x destructive, if $\operatorname{rank} M' < \operatorname{rank} M$ or $\operatorname{Hilb}(M) \setminus \{x\}$ is not the Hilbert basis of $\mathbb{R}_+ M' \cap \operatorname{gp}(M)$

Roughly speaking, $\operatorname{Hilb}(M) \setminus \{x\}$ is not the Hilbert basis of a cone of full dimension. A non-destructive element must span an extreme ray of $\mathbb{R}_+ M$.

Definition

M (or $\mathbb{R}_+ M$) is tight if every element of $\operatorname{Hilb}(M)$ is destructive.

Suppose x generates an extreme ray of $\mathbb{R}_+ M$. Then

$$M[-x] \cong \mathbb{Z} \oplus M'', \qquad M'' \text{ normal}, \ \operatorname{rank} M'' = \operatorname{rank} M - 1.$$

Lemma

Suppose x is non-destructive. Then

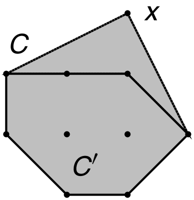
- *$M'.M''$ have UHC $\Rightarrow$ M has UHC*
- $\mathrm{CR}(M) \leq \min(\mathrm{CR}(M'), \mathrm{CR}(M'') + 1)$.

Corollary

A minimal counterexample to UHC or ICP is tight.

Strategy for finding counterexamples:

- generate a “random” normal monoid M;
- shrink it to a tight normal M_0;
- check M_0 for the property in question.

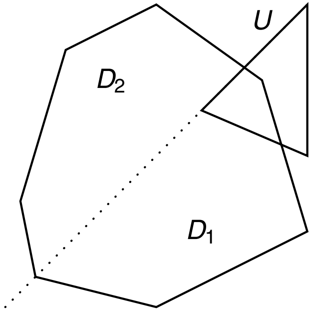


Tightening a cone

Deciding UHC

Suppose D is a subcone of C. Then

- D is contained in a unimodular subcone: discard D;
- D does not properly intersect any unimodular subcone: UHC violated;
- otherwise split D along support hyperplane of a unimodular subcone, and apply recursion.



```
UNICOVER(D, n)
for i ← n to N
do
    if D ⊂ U_i
       then return
    if int(D) ∩ int(U_i) ≠ ∅
       then (D_1, D_2) ← SPLIT(D, U_i)
                UNICOVER(D_1, i)
                UNICOVER(D_2, i)
                return
OUTPUT( D not u-covered )
return
```

```
MAIN()
Create the list U_1, . . . , U_N of u-subcones of C
UNICOVER(C, 1)
```

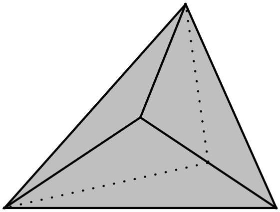
Unimodular triangulations of cones

If we omit the H in UHC for cones, then no condition remains:

Theorem

A rational cone C has a triangulation into unimodular subcones (spanned by integral vectors).

Proof. Start with arbitrary triangulation. Refine by iterated stellar subdivision to reduce multiplicities.



Problem: bound the size of the vectors spanning the unimodular subcones!

Multiples of polytopes again

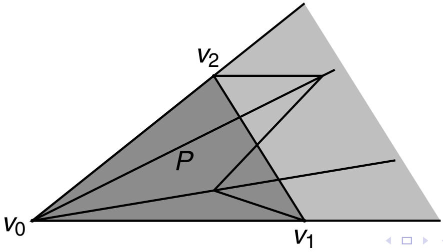
Idea: find a unimodular cover of a lattice polytope by

- triangulate each corner cone into unimodular subcones
- extend the “basic simplices” to a tiling of $\mathbb{R}^d$;
- hopefully the tiles contained in P cover P.

u vertex of P; then the *corner cone* at u is

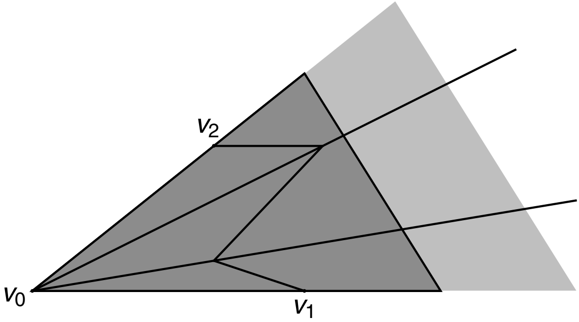
$$\mathbb{R}_+(P-u)$$

First step:

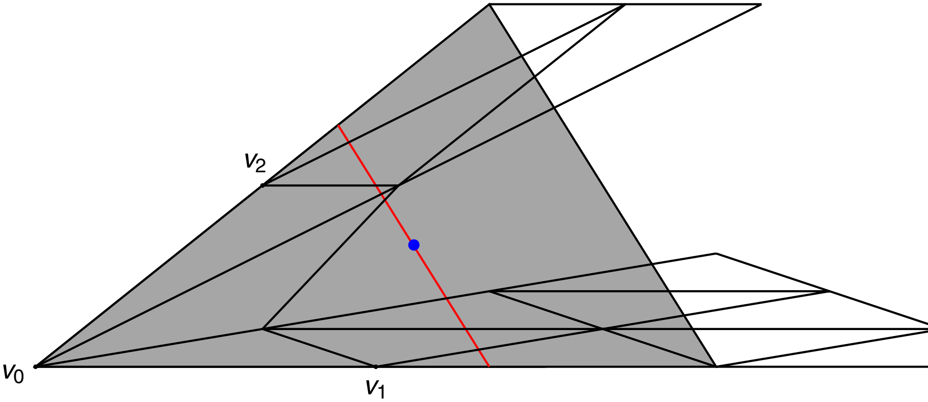


cP has the same corner cones as P, and if $c \gg 0$, then the tiles become small, enough tiles should lie in cP and cover it.

In the next step the "basic" tiles get into $c'P$



Tile corner cones:



Then we increase again so that tiles in $c''P$ cover area between vertex and *red* line through barycenter • (from all vertices) $\Rightarrow$ $c''P$ is covered.

Consequence: for each P there exists $c > 0$ such that cP has UC.
But much more is true: c can be bounded in terms of the dimension:

Theorem (B.-Gubeladze, v. Thaden)

Let P be a d-polytope. cP has UC for all $c \geq \mathfrak{c}_d^{\mathrm{pol}}$,

$$\mathfrak{c}_d^{\mathrm{pol}} = O\left(d^{16.5}\right)\left(\frac{9}{4}\right)^{(\mathrm{ld}\,\gamma(d))^2},$$

$$\gamma(d) = (d-1)\lceil\sqrt{d-1}\rceil.$$

The step from corner cones to P is rather easy and "polynomial":

Lemma

Suppose the basic simplices of the corner cones lie in P. Then cP has a unimodular cover for all $c > d\sqrt{d}$.

So one must find a good bound for the unimodular triangulation of rational cones! Enough to do simplicial cones.

Theorem

Let C be a rational simplicial d-cone and Δ_C the simplex spanned by O and the extreme integral generators. Then

1. (M. v. Thaden) *C has a triangulation into unimodular simplicial cones D_i such that* $\operatorname{Hilb}(D_i) \subset c\Delta_C$ *for some*

$$c \le \frac{d^2}{4}(\mu(C))^7 \left(\frac{9}{4}\right)^{(\operatorname{ld}(\mu(C)))^2}.$$

2. *C has a cover by unimodular simplicial cones D_i such that* $\operatorname{Hilb}(D_i) \subset c\Delta_C$ *for some*

$$c \le \frac{d^2}{4}(d+1)\big(\gamma(d)\big)^8 \left(\frac{9}{4}\right)^{(\operatorname{ld}(\gamma(d)))^2},$$

$$\gamma(d) = \left\lceil \sqrt{d-1} \right\rceil (d-1)$$

von Thaden claims that he can improve the bounds to polynomial size, at least for covers. (Should be the main result of his thesis.)

Best possible value: $\mathfrak{c}_d^{\mathrm{pol}} = \dim P - 1$.
We know this for $d = 2$. It also holds for $d = 3$.

Results of Lagarias & Ziegler , Kantor & Sarkaria:

Proposition

$\dim P = 3 \Rightarrow$ *cP has UC for $c \geq 2$.*

Theorem

$4P$ has a unimodular triangulation for all P.

Wrong for $2P$ in general! $3P$??.

Knudsen-Mumford triangulations

Theorem (Knudsen-Mumford, 1973)

Let P be a lattice polytope. Then there exists $c > 0$ such that cP and $c'CP$ for all $c' > 0$ have unimodular triangulations.

Open problem: Does cP have a unimodular triangulation for all $C \gg 0$? Does there exist a uniform bound only in terms of dimension?

Lecture 3

Hilbert functions and homological properties

The ADG conjectures

In 1966 H. Anand, V. C. Dumir, and H. Gupta investigated a combinatorial problem:

Suppose that n distinct objects, each available in k identical copies, are distributed among n persons in such a way that each person receives exactly k objects.

What can be said about the number $H(n, k)$ of such distributions?

They formulated some conjectures:

(ADG-1) there exists a polynomial $P_n(k)$ of degree $(n-1)^2$ such that $H(n,k) = P_n(k)$ for all $k \gg 0$;

(ADG-2) $H(n,k) = P_n(k)$ for all $k > -n$; in particular $P_n(-k) = 0$, $k = 1, \dots, n-1$;

(ADG-3) $P_n(-k) = (-1)^{(n-1)^2} P_n(k-n)$ for all $k \in \mathbb{Z}$.

These conjectures were proved and extended by R. P. Stanley using methods of commutative algebra. Basic idea: interpret $H(n,k)$ (n fixed) as the Hilbert function of some graded ring.

The weaker version (ADG-1) of (ADG-2) has been included for didactical purposes. A compact account has been given in

W.B., *Commutative algebra arising from the Anand-Dumir-Gupta conjectures*;
http://www.math.uos.de/staff/phpages/brunsw/Allahabad.pdf

Reformulation:

a_{ij} = number of copies of object i that person j receives

$\Rightarrow A = (a_{ij}) \in \mathbb{Z}_+^{n\times n}$ such that

$$\sum_{l=1}^{n} a_{il} = \sum_{m=1}^{n} a_{mj} = k, \qquad i, j = 1, \dots, n.$$

$H(n, k)$ is the number of such matrices A.

The system of equations is part of the definition of *magic squares*. Stanley calls the matrices A magic squares, though the usually requires further properties for those.

Two famous magic squares:

8	1	6
3	5	7
4	9	2

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

The 3×3 square can be found in ancient Chinese sources and the 4×4 appears in Albrecht Dürer's engraving *Melancholia* (1514). It has remarkable symmetries and shows the year of its creation.

A step towards algebra

Let $\mathscr{M}_n$ be the set of all matrices $A = (a_{ij}) \in \mathbb{Z}_+^{n\times n}$ such that

$$\sum_{l=1}^{n} a_{il} = \sum_{m=1}^{n} a_{mj}, \qquad i,j = 1,\dots,n.$$

By Gordan's lemma $\mathscr{M}_n$ is an affine, normal monoid, and $A \mapsto$ magic sum $k = \sum_{k=1}^{n} a_{1k}$ is a positive grading on $\mathscr{M}$.

$\operatorname{rank} \mathscr{M}_n = (n-1)^2 + 1$

Theorem (Birkhoff-von Neumann)

$\mathscr{M}_n$ is generated by the its degree 1 elements, namely the permutation matrices.

Translation into commutative algebra

We choose a field K and form the algebra

$$R = K[\mathscr{M}_n].$$

It is a normal affine monoid algebra, graded by the "magic sum", and generated in degree 1. $\dim R = \operatorname{rank} \mathscr{M}_n = (n-1)^2 + 1$.

$\Rightarrow$ $H(n,k) = \dim_K R_k = H(R,k)$ is the Hilbert function of R!

$\Rightarrow$ (ADG-1): there exists a polynomial P_n of degree $(n-1)^2$ such that $H(n.k) = P_n(k)$ for $k \gg 0$.

In fact, take P_n as the Hilbert polynomial of R.

A recap of Hilbert functions

Let K be a field, and $R = \bigoplus_{k=0}^{\infty} R_k$ a graded K-algebra generated by homogeneous elements $x_1, \dots, x_n$ of degrees $g_1, \dots, g_n > 0$.

Let M be a non-zero, finitely generated graded R-module.

Then $\quad H(M, k) = \dim_K M_k < \infty \quad$ for all $k \in \mathbb{Z}$.

$H(M, \_) : \mathbb{Z} \to \mathbb{Z}$ is the *Hilbert function* of M.

We form the *Hilbert* (or *Poincaré*) series

$$H_M(t) = \sum_{k \in \mathbb{Z}} H(M.k) t^k.$$

Fundamental fact:

Theorem (Hilbert-Serre)

Let K be a field, and $R = \oplus_{k=0}^{\infty} R_k$ a graded K-algebra generated by homogeneous elements $x_1, \dots, x_n$ of degrees $g_1, \dots, g_n > 0$.

Let M be a non-zero, finitely generated graded R-module Then there exists a Laurent polynomial $Q \in \mathbb{Z}[t, t^{-1}]$ such that

$$H_M(t) = \frac{Q(t)}{\prod_{i=1}^{n}(1 - t^{g_i})}.$$

More precisely: $H_M(t)$is the Laurent expansion at 0 of the rational function on the right hand side.

Refinement: M is finitely generated over a graded Noether normalization $K[x_1, \dots, x_d]$:

$S \subset R$ is a *graded Noether normalization* of M if

- $S = K[x_1, \dots, x_d]$ with algebraically independent elements $x_1, \dots, x_d$, $d = \dim M = \dim R/\operatorname{Ann} M$;
- M is a finitely generated S-module.

In other words, a Noether normalization allows us to study M as a graded module over a polynomial ring S such that $\operatorname{Ann}_S M = 0$. We also say that $x_1, \dots, x_d$ is a *homogeneous system of parameters* (hsop).

Special case: let us say that M is essentially standard graded (est) if M is finitely generated over $K[R_1]$. For this it is enough that R is est, and this certainly holds if $R = K[R_1]$.

If M is est, then we can choose a Noether normalization in degree 1: $\deg x_1 = \cdots = \deg x_d = 1$, at least after an extension of K.

Theorem

Suppose that M is est and let $d = \dim M$. Then

$$H_M(t) = \frac{Q(t)}{(1-t)^d}.$$

Moreover:

There exists a polynomial $P_M \in \mathbb{Q}[X]$ such that

$$H(M,i) = P_M(i), \qquad i > \deg H_M,$$
$$H(M,i) \neq P_M(i), \qquad i = \deg H_M.$$

$e(M) = Q(1) > 0$, *and if $d \geq 1$, then*

$$P_M = \frac{e(M)}{(d-1)!}X^{d-1} + \textit{terms of lower degree.}$$

Recall

(ADG-1) there exists a polynomial $P_n(k)$ of degree $(n-1)^2$ such that $H(n,k) = P_n(k)$ for all $k \gg 0$;

With $R = K[\mathscr{M}_n]$ this is equivalent to

$$H_R(t) = \frac{1 + h_1 t + \dots h_u t^u}{(1-t)^d}, \qquad d = \operatorname{rank} \mathscr{M}_n.$$

and so (ADG-1) has been proved. Furthermore

(ADG-2) $H(n,k) = P_r(n)$ for all $k > -n$; in particular $P_n(-k) = 0$, $k = 1, \dots, n-1$;

is equivalent to $u - d = -n$ ($h_u \neq 0$). How can we translate

(ADG-3) $P_n(-k) = (-1)^{(n-1)^2} P_n(k-n)$ for all $k \in \mathbb{Z}$

into a property of the rational function $H_R(t)$? Note that the shift $-n$ in (ADG-3) is the conjectured degree of $H_R(t)$.

Lemma

Let R be an est graded K-algebra, $\dim R = d$, with Hilbert polynomial P. Suppose $\deg H_R(t) = g < 0$. Then the following are equivalent:

- $P(-k) = (-1)^{d-1} P(k+g)$ *for all* $k \in \mathbb{Z}$;
- $(-1)^d H_R(t^{-1}) = t^{-g} H_R(t)$;
- $h_i = h_{u-i}$ *for all* i*: the* h*-vector is palindromic.*

The equivalence of the last two assertions is very easy, but the equivalence with the first is tricky. (Goes back to Polya, explicitly stated by Popoviciu.)

For (ADG-2) we have to compute the degree of the Hilbert series as a rational function.

For (ADG-3) it is best to work with the most compact form

$$(-1)^d H_R(t^{-1}) = t^{-g} H_R(t).$$

We will see that the proofs of (ADG-2) and (ADG-3) can be based on homological properties of normal affine monoid algebras.

Hochster's theorem

Definition

Let M be a graded module over a positively graded K-algebra R. Then M is called *Cohen-Macaulay* if it is a free module over a graded Noether normalization S of M.

An intrinsic definition shows that the choice of S is irrelevant. Note that M always has a finite free resolution over S, and it is Cohen-Macaulay if and only if the resolution has length 0.

Theorem (Hochster)

Let M be an affine normal monoid. Then $K[M]$ is Cohen-Macaulay for every field K.

There is no easy proof of this powerful theorem. For example, it can be derived from the Hochster-Roberts theorem, using that $K[M] \subset K[X_1, \dots, X_s]$ can be chosen as a direct summand.

A special case is rather simple:

Proposition

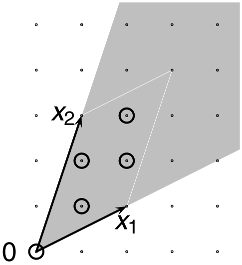
Let M be a simplicial affine normal monoid. Then $K[M]$ is Cohen-Macaulay for every field K.

Let $d = \operatorname{rank} M$. Choose elements $x_1, \dots, x_d \in M$ generating $\mathbb{R}_+ M$, and set

$$\operatorname{par}(x_1, \dots, x_d) = \left\{ \sum_{i=1}^{d} q_i x_i : q_i \in [0, 1) \right\}.$$

With $N = \mathbb{Z}_+ x_1 + \cdots + \mathbb{Z}_+ x_d$: we know from the previous lecture:

$M = \bigcup_{z \in E} z + N$,
N is free,
The union is disjoint.



In commutative algebra terms:
$K[M]$ is finite over $K[N]$.
$K[N] \cong K[X_1, \ldots, X_d]$.
$K[M]$ is a free module over $K[N]$.

$\Rightarrow K[M]$ is Cohen-Macaulay.

A combinatorial consequence of the Cohen-Macaulay property

Theorem

Let M be a graded Cohen-Macaulay module over the positively graded K-algebra R and $x_1, \dots, x_d$ a hsop. for M, $e_i = \deg x_i$. Let

$$H_M(t) = \frac{h_a t^a + \cdots + h_b t^b}{\prod_{i=1}^d (1 - t^{e_i})}, \qquad h_a, h_b \neq 0.$$

Then $h_i \geq 0$ for all i.

If $M = R = K[R_1]$, then $h_i > 0$ for all $i = 0, \dots, b$.

Proof. $h_a t^a + \cdots + h_b t^b$ is the Hilbert series of

$$M/(x_1 M + \cdots + x_d M).$$

So each h_i is the dimension of a vector space.

For (ADG-3) we have to show $(-1)^{(n-1)^2+1} H_R(t^{-1}) = t^n H_R(t)$ for $R = K[\mathscr{M}_n]$.

Strategy:

- Find an R-module ω with $H_\omega(t) = (-1)^d H_R(t^{-1})$, $d = \dim R$

Possible for R Cohen-Macaulay: ω is the canonical module of R.

- Compute ω for $R = K[\mathscr{M}_n]$ and show that $\omega \cong R(g)$, $g = \deg H_R(t)$.

$R(g)$ free module of rank 1 with generator in degree $-g$. Thus $H_{R(g)}(t) = t^{-g} H_R(t)$.

More generally:

- Compute ω for $R = K[M]$ with M affine normal.

The canonical module

In the following: R positively graded Cohen-Macaulay K-algebra, $x_1,\dots,x_d$ hsop, $\deg x_i = g_i$

First $R = S = K[X_1,\dots,X_d]$:

$$(-1)^d H_S(t^{-1}) = \frac{(-1)^d}{\prod_{i=1}^d (1-t^{-g_i})} = \frac{t^{g_1+\cdots+g_d}}{\prod_{i=1}^d (1-t^{-g_i})} = H_\omega(t)$$

with $\omega = \omega_S = S(-(g_1+\cdots+g_d))$

Now R free over $S = K[x_1, \dots, x_d]$, with homogeneous basis $y_1, \dots, y_m$:

$$R \cong \bigoplus_{j=1}^{m} Sy_j \cong \bigoplus_{i=1}^{u} S(-i)^{h_i}, \qquad h_i = \#\{j : \deg y_j = i\},$$

$$H_R(t) = (h_0 + h_1 t + \cdots + h_u t^u) H_S(t) = Q(t) H_S(t)$$

Set $\omega_R = \operatorname{Hom}_S(R, \omega_S)$. Then, with $s = g_1 + \cdots + g_d$

$$\omega_R \cong \bigoplus_{i=1}^{u} \operatorname{Hom}_S(S(-i)^{h_i}, S(-s)) \cong \bigoplus_{i=1}^{u} S(-i+s)^{h_i},$$

$$H_{\omega_R}(t) = (h_0 t^s + \cdots + h_u t^{s-u}) H_S(t) = Q(t^{-1}) t^s H_S(t)$$
$$= (-1)^d Q(t^{-1}) H_S(t^{-1}) = (-1)^d H_R(t^{-1}).$$

Let us rewrite $H_\omega(t)$:

$$H_\omega(t) = \frac{h_u t^{s-u} + \cdots + h_0 t^s}{\prod_{i=1}^d (1 - t^{-g_i})}$$

Since $s = g_1 + \cdots + g_d$, one has $s - u = -\deg H_R(t)$. Therefore

Corollary

$$\deg H_R(t) = -\min\{k : \omega_k \neq 0\}.$$

R-module structure of ω_R

Multiplication in the first component makes $\omega_R = \operatorname{Hom}_S(R, S)$ an R-module:

$$a \cdot \varphi(\_) = \varphi(a \cdot \_).$$

But: Is ω_R independent of S ? Indeed

Theorem

ω_R depends only on R (up to isomorphism of graded modules).

The proof requires homological algebra, after reduction from the graded to the local case.

The canonical module of $K[M]$

In the following M affine, normal, positive monoid. We want to find the canonical module of $R = K[M]$ (Cohen-Macaulay by Hochster's theorem).

Theorem (Danilov, Stanley)

The ideal I generated by the monomials in the interior of $\mathbb{R}_+ M$ is the canonical module of $K[M]$ (with respect to every positive grading of M or even multigraded).

Various proofs via

- differentials (Danilov),
- combinatorics (Stanley),
- local cohomology (Stanley; see *Cohen-Macaulay rings*),
- divisors

The case of a simplicial normal monoid is again elementary.

After all these preparations, (ADG-2) and (ADG-3) follow easily. Consider a magic square $A = (a_{ij})$. Then

$$A \in \operatorname{int}(\mathscr{M}_n) \iff a_{ij} > 0 \text{ for all } i, j \iff A - \mathbf{1} \in \mathscr{M}_n$$

where $\mathbf{1}$ is the magic square with all entries 1: Thus $\operatorname{int}(\mathscr{M}_n) = \mathbf{1} + \mathscr{M}_n$.

The magic sum of $\mathbf{1}$ is n. So $\omega_R = R(-n)$. With $d = (n-1)^2 + 1$ this implies

$$H_R(t^{-1}) = (-1)^d t^n H_R(t) : \quad \text{(ADG-3)}$$

and

$$\deg H_R(t) = -n : \quad \text{(ADG-2)}$$

Recall that we have written

$$H_R(t) = \frac{1 + h_1 t + \cdots + h_u t^u}{(1-t)^d}.$$

Hochster's theorem $\Rightarrow$

Theorem

If R is a normal affine monoid algebra, then

$$h_i \geq 0 \qquad \textit{for all } i$$

(ADG-2) $\Longleftrightarrow$ $h_i = h_{u-i}$ for all i. What else can be said about the h-vector?

Stanley conjectured: the h-vector for $R = K[\mathscr{M}_n]$ is unimodal:

$$1 = h_0 \leq h_1 \leq \cdots \leq h_{\lfloor u/2 \rfloor}$$

This will be our next topic. But we should not forget to introduce a class of rings to which $K[\mathscr{M}_n]$ belongs.

Gorenstein rings

Definition

A positively graded Cohen-Macaulay K-algebra is *Gorenstein* if $\omega_R \cong R(h)$ for some $h \in \mathbb{Z}$.

Actually, there is no choice for h:

Theorem (Stanley)

Let R be Gorenstein of Krull dimension d. Then

- $\omega_R \cong R(g)$, $g = \deg H_R(t)$;
- $h_0 = h_{u-i}$ *for* $i = 0, \dots, u$*: the h-vector is palindromic;*
- $H_R(t^{-1}) = (-1)^d t^{-g} H_R(t)$.

Conversely, if R is a Cohen-Macaulay integral domain such that $H_R(t^{-1}) = (-1)^d t^{-h} H_R(t)$ for some $h \in \mathbb{Z}$, then R is Gorenstein.

Counting lattice points in polytopes

This area was pioneered by E. Ehrhart. As we will see, we have already proved central theorems about lattice point counting.

Let $P \subset \mathbb{R}^d$ be a lattice polytope. We set

$$E(P,k) = \#(P \cap \frac{1}{k}\mathbb{Z}^d) = \#(kP \cap \mathbb{Z}^d).$$

This is the *Ehrhart function* of P.

the corresponding power series is the *Ehrhart series*

$$E_P(t) = \sum_{k=0} E(P,k)t^k.$$

We know

$$kP \cap \mathbb{Z}^d \leftrightarrow \{x \in \widehat{M}(P) : \deg x = k\}.$$

Therefore, with $R = K[\widehat{M}(P)]$,

$$E(P, k) = H(R, k)$$

where H is again the Hilbert function.

In general R is not generated in degree 1 as an algebra. But: $\widehat{M}(P)$ is a finitely generated $M(P)$-module, in other words:

P lattice polytope $\Rightarrow$ $R = K[\widehat{M}(P)]$ is an est graded algebra.

Theorem

Let P be a lattice polytope of dimension d. Then

$$E_P(t) = H_R(t) = \frac{1 + h_1 t + \dots h_u t^u}{(1-t)^{d+1}}$$

with

$$\deg H_R(t) = u - (d+1) = -\min\{k : \operatorname{int}(P) \cap \mathbb{Z}^d \neq \emptyset\} < 0.$$

Thus, with the Hilbert polynomial Q of R,

$$E(P,k) = Q(k) \text{ for all } k \geq 0.$$

One calls Q the Ehrhart polynomial of P. In the theorem we have omitted the reciprocity of $H_R \leftrightarrow P$ and $H_\omega \leftrightarrow \operatorname{int}(P)$ which in this context is called *Ehrhart reciprocity*.

Lecture 4

Unimodality of *h*-vectors

Questions on unimodality

Let R be an est (essentially standard) graded K-algebra. Recall that the Hilbert series can be written

$$H_R(t) = \frac{1 + h_1 t + \dots h_u t^u}{(1-t)^d}, \qquad d = \dim R,\ h_u \neq 0.$$

If R is Cohen-Macaulay, then $h_i \geq 0$ for all i.

If R is Gorenstein, then $h_{u-i} = h_i$ for all i, and if R is a Cohen-Macaulay domain, then this symmetry of the h-vector implies that R is Gorenstein.

For $R = K[\mathscr{M}_n]$, the (Gorenstein) algebra over the monoid of magic squares, Stanley conjectured that the h-vector is unimodal:

$$1 = h_0 \leq h_1 \leq \cdots \leq h_{\lfloor u/2 \rfloor}$$

This conjecture was proved by Athanasiadis (2003), and then extended to a larger class of Gorenstein monoid algebras by B. and Römer (2005). See

W.B., T. Römer, *h-vectors of Gorenstein polytopes*, J. Comb. Th. Ser. A, 114 (2007), 65–76.

As counterexamples show, the property est is not sufficient for the unimodality of the h-vector: one needs that $R = K[R_1]$ is standard graded (as $K[\mathscr{M}_n]$ is).

As far as I know, there is no example of a standard graded Gorenstein domain with a non-unimodal h-vector, and to prove or disprove this property is the greatest challenge in combinatorial commutative algebra.

In the characterization of Gorenstein rings by the symmetry condition $h_{u-i} = h_i$ the hypothesis domain cannot be omitted.

If $R = K[X_1, \ldots, X_n]/I$ is a Gorenstein ring, then one often finds monomial orders on the polynomial ring such that $R' = K[X_1, \ldots, X_n]/\operatorname{in}(I)$ is Cohen-Macaulay, but not Gorenstein. Nevertheless, $H_R(t) = H_{R'}(t)$.

Here $\operatorname{in}(I)$ is the *initial ideal* of I, i. e. the ideal generated by the leading monomials of the elements of I.

Question: suppose $R = K[X_1, \ldots, X_n]/I$ is Gorenstein. Can one find a monomial order on $K[X_1, \ldots, X_n]$ such that $R' = K[X_1, \ldots, X_n]/\operatorname{in}(I)$ is Gorenstein as well?

Actually, the proof of our result on h-vectors results from the (successful) attempt to answer this question for a class of monoid domains. The connection between unimodality of the h-vector and initial ideals is given by certain triangulations.

The g-theorem

There is a famous instance where unimodality of the h-vectors and more is known.

Let Δ be a simplicial complex of dimension d. (The dimension of Δ is the maximal dimension of a face (= simplex) of Δ). It is useful to set $d = \dim \Delta + 1$.

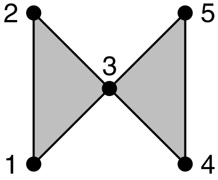
The i-th component of the f-vector counts the number of faces of dimension i:

$$f(\Delta) = (f_0, \dots, f_{d-1}), \qquad f_i = \#\{\delta \in \Delta : \dim \delta = i\}$$

The f-vectors of simplicial complexes are characterized by the *Kruskal-Katona theorem*.

Example:

Δ



$\dim \Delta = 2$

$f(\Delta) = (5, 6, 2)$

$h(\Delta) = (1, 2, -1)$

$$I(\Delta) = \big(X_1X_4,\ X_1X_5,\ X_2X_4,\ X_2X_5\big)$$

$h(\Delta)$ and $I(\Delta)$ will be explained below.

Next one defines the h-vector $h(\Delta)$ through the polynomial identity

$$\sum_{i=0}^{d} h_i s^i (1+s)^{d-i} = \sum_{i=0}^{d} f_{i-1} s^i \qquad (f_{-1}=1).$$

Let $v_1, \dots, v_n$ be the vertices of Δ. A subset $N = \{v_{i_1}, \dots, v_{i_m}\}$ is a nonface of Δ is N is *not* the vertex set of a face of Δ.

In the polynomial ring $K[X_1, \dots, X_n]$ we set

$$X^N = X_{i_1} \cdots X_{i_m},$$

and define the Stanley-Reisner ring or face ring of Δ by

$$K[\Delta] = K[X_1, \dots, X_n]/I(\Delta), \qquad I(\Delta) = (X^N : N \text{ nonface of } \Delta).$$

It is enough to take the minimal nonfaces.

Proposition

Let Δ be a simplicial complex of dimension d, $R = K[\Delta]$, and $h(\Delta) = (1 = h_0, \dots, h_u)$. Then

$$H_R(t) = \frac{1 + h_1 t + \cdots + h_d t^u}{(1-t)^u}.$$

Thus we can understand $h(\Delta)$ as the h-vector of a graded ring.

The most interesting simplicial complexes are the *boundary complexes of simplicial polytopes*. (A polytope is *simplicial* if all its facets are simplices.) They satisfy the *Dehn-Sommerville* equations

$$h_i = h_{d-i}, \qquad d = \dim P.$$

a condition that we have encountered already. (Note that $h_d = 1$ and that $\dim \partial P = d - 1$.)

A famous theorem of McMullen is the *upper bound theorem* (conjectured by Motzkin):

Theorem

The h-vector of (the boundary of) a simplicial polytope is bounded above by the the h-vector of the cyclic polytope of the same dimension and the same number of vertices.

The upper bound theorem was later on generalized by Stanley to *simplicial spheres*, i, e. simplicial complexes for which $|\Delta| \cong S^{d-1}$. The main argument:

Theorem (Stanley)

Let Δ be a simplicial sphere. Then $K[\Delta]$ is a Gorenstein ring.

Note that a simplicial sphere need not be the boundary of a simplicial polytope. Boundaries of polytopes are *shellable* (Brugesser-Mani), and this was used by McMullen. However, there exist non-shellable simplicial spheres.

For a simplicial complex satisfying the Dehn-Sommerville equations one can define the g-vector

$$g_i = h_i - h_{i-1}, \qquad i = 0, \dots, \lfloor d/2 \rfloor.$$

McMullen conjectured and Billera and Lee (sufficiency) and Stanley (necessity) proved that the g-vectors of simplicial polytopes can be characterized as follows:

Theorem (g-theorem)

The h-vectors $h(\partial P) = (1 = h_0, \dots, h_d)$ of the boundaries of simplicial polytopes P are characterized by the conditions

- $h_i = h_{d-i}$, $i = 0, \dots, d$;
- *the corresponding g-vector is the h-vector of a standard graded K-algebra.*

In particular, $g_i \geq 0$ for all i, and so $h_{i-1} \leq h_i$, $i = 0, \dots, \lfloor d/2 \rfloor$.

There is a theorem of Macaulay that characterizes the h-vectors of standard graded K-algebras in terms of explicit inequalities (involving binomial expansions). Therefore we say that the g-vector is a Macaulay sequence if it is the h-vector of a standard graded K-algebra.

Open problem: does the g-theorem hold for simplicial spheres?

The advantage of simplicial polytopes is that one can assign a toric variety to them, and Stanley used topological properties of this variety.

Strategy: In order to prove unimodality (or even the Macaulay condition) for an h-vector, show it is the h-vector of the boundary of a simplicial polytope (or at least a simplicial sphere, and hope for the solution of the open problem).

Unimodality for Gorenstein polytopes

Let us say that P is a Gorenstein polytope if

- $R = K[\widehat{M}(P)]$ is standard graded ($\iff M(P) = \widehat{M}(P)$),
- R is Gorenstein.

Stanley's conjecture on the unimodality of the h-vector for magic squares can now be generalized as follows:

Question: suppose P is a Gorenstein polytope. Is the h-vector $h(P)$ of R unimodal?

The answer is *no* if we do not require that R is standard graded. Counterexample by Mustaţă and Payne (connection with stringy Hodge numbers).

Towards unimodality for Gorenstein polytopes

We have already observed that the canonical module of the algebra $K[M]$ for an affine normal monoid M has the interior $\operatorname{int}(M) = M \cap \operatorname{int}(\mathbb{R}_+ M)$ of M as its monomial basis. This implies the equivalence of (1) and (2) in

Lemma

Let M be a normal affine monoid. Then the following are equivalent:

1. *$K[M]$ is Gorenstein;*
2. *$\operatorname{int}(M) = x + M$ for some $x \in M$;*
3. *there exists $x \in M$ with $\sigma_F(x) = 1$ for all facets F of $\mathbb{R}_+ M$ and the corresponding support form σ_F.*

the equivalence of (3) is an old observation, but crucial for the following.

Let x be as in the lemma, and write $x = y_1 + \cdots + y_v$ with $y_i \in M$. Then the $\sigma_F(y_i)$ are pairwise disjoint 0-1-vectors. Roughly speaking, this is a strong unimodularity condition.

Theorem

Let M be a normal affine monoid with $R = K[M]$ Gorenstein and choose $x = y_1 + \cdots + y_v$ as above. Then

- *the elements $X^{y_1} - X^{y_2}, \dots, X^{y_{v-1}} - X^{y_v}$ form a regular R-sequence;*
- *the residue class ring $R/$(this sequence) is again a nor mal affine monoid Gorenstein algebra.*

The proof is based on the construction of a unimodular triangulation that can be projected along the vector subspace generated by the differences $y_i - y_{i+1}$.

Now suppose that $M = M(P)$ for a Gorenstein polytope. Then x can be decomposed into a sum of degree 1 elements. Modulo a regular sequence of 1-forms the h-vector does not change, and the property of being standard graded is preserved:

Corollary

Suppose P is a Gorenstein polytope. Then there exists a Gorenstein polytope P' with the following properties:

- *P' has exactly one interior lattice polytope (namely the projection of x);*
- *$h(P') = h(P)$.*

This reduces the problem of unimodality to the integrally closed reflexive polytopes. (P reflexive $\iff$ P contains exactly one interior lattice polytope and $K[\widehat{M}(P)]$ is Gorenstein.)

In order to get closer to the g-theorem we need a unimodular triangulation ($\Rightarrow K[\widehat{M}(P)]$ standard graded).

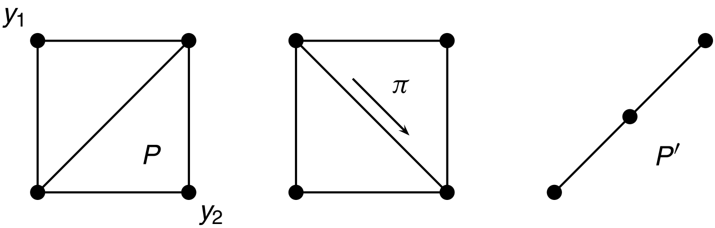
Theorem

Let P be a Gorenstein polytope with a unimodular triangulation. Then there exists a Gorenstein polytope P' with the following properties:

- *P' has exactly one interior lattice polytope;*
- *∂P has a triangulation Δ such that $h(P) = h(\Delta)$.*

For the proof one has to start with the given unimodular triangulation and to modify it to one that can be projected.

The g-theorem appears at the horizon: at least Δ is a simplicial sphere (and somewhat more than that).



Change of the triangulation and projection

In order to apply the g-theorem (as it is now), the hypothesis on the triangulation must be strengthened:

Corollary

Let P be a Gorenstein polytope with a regular unimodular triangulation. Then there exists a Gorenstein polytope P' with the following properties:

- *P' has exactly one interior lattice polytope;*
- *∂P has a triangulation Δ such that $h(P) = h(\Delta)$;*
- *Δ can be deformed to the boundary complex of a simplicial polytope.*

Therefore $h(P)$ is a Macaulay sequence and, in particular, is unimodal.

A regular subdivision arises as the decomposition of P into the domains of linearity of a piecewise *convex* affine function.

A regular subdivision and a nonregular refinement

Regularity is also a keyword in the Gröbner basis theory of toric ideals ($=$ deformation to monomial ideals). See B. Sturmfels, *Gröbner bases and convex polytopes*, AMS 1996.

Corollary

Let P be a Gorenstein polytope with a regular unimodular triangulation, and let I_P be the toric ideal defining $K[P] = K[X_1, \dots, X_n]/I_P$. Then there exists a monomial order on $K[X_1, \dots, X_n]$ such that $K[X_1, \dots, X_n]/\operatorname{in}(I_P)$ is also Gorenstein.

Here regularity of the triangulation is essential.