Experiments in lattice polytopes

Winfried Bruns

FB Mathematik/Informatik
Universität Osnabrück

wbruns@uos.de

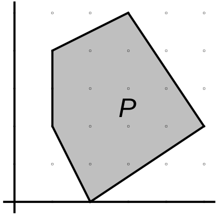
Sevilla, February 11, 2007

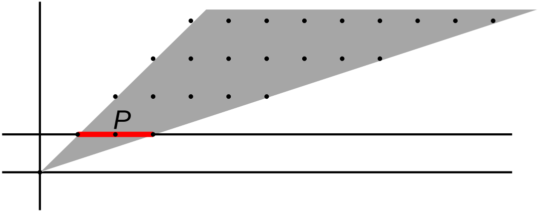
Lattice polytopes

Definition

A lattice polytope P is the convex hull of finitely many points $x_i \in \mathbb{Z}^d$. The monoid $M(P)$ defined by P is the submonoid of $\mathbb{Z}^{d+1}$ generated by the vectors

$$(x, 1), \qquad x \in P \cap \mathbb{Z}^d.$$





$L(P) = P \cap \mathbb{Z}^d$

Normality, very ampleness and smoothness

K is an algebraically closed field, for example $K = \mathbb{C}$. Then P defines a graded K-algebra $K[P] = K[M(P)]$: the subalgebra of $K[X_1^{\pm}1, \dots, X_{d+1}]$ generated by the monomials

$$X_1^{a_1} \cdots X_d^{a_d} X_{d+1}, \qquad (a_1, \dots, a_d) \in L(P).$$

This is a graded K-algebra in which the degree is the X_{d+1}-degree. We associate with it

- the affine variety $\operatorname{Spec} K[P]$
- the projective variety $\operatorname{Proj} K[P]$.

These are toric varieties:

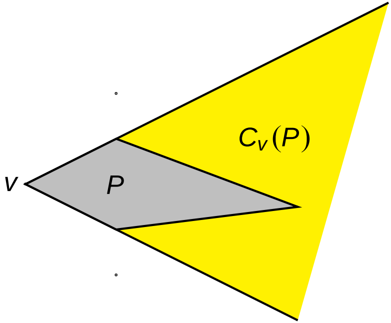
- $\mathbb{T}^{d+1} = (K^*)^{d+1}$ acts on $\operatorname{Spec} K[P]$
- $\mathbb{T}^d$ acts on $\operatorname{Proj} K[P]$.

For $v \in \operatorname{vert}(P)$ we call

$$M(P_v) = \mathbb{Z}_+\{x - v : x \in L(P)\} \qquad C(P_v) = \mathbb{R}_+\{x - v : x \in P\}$$

the corner monoid and corner cone of P at v.
Then $\operatorname{Proj} K[P]$ is covered by its standard affine charts

$$\operatorname{Spec} K[M(P_v)].$$



Algebraic-geometric attributes of polytopes

We can transfer attributes from the algebraic-geometric objects to P, provided they do not depend on K.

- P is normal if $K[P]$, equivalently $\operatorname{Spec} K[P]$ is normal;
- P is very ample if $\operatorname{Proj} K[P]$ is normal;
- P is very smooth if $\operatorname{Proj} K[P]$ is smooth.

Combinatorial description

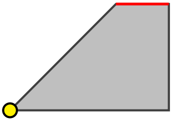
Suppose $\mathbb{Z}^d$ is affinely generated by $L(P)$ (standing assumption).

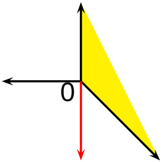
- P normal $\iff \overline{M}(P) = C(P) \cap \mathbb{Z}^{d+1} = M(P)$
- P very ample $\iff \overline{M}(P) \setminus M(P)$ finite
  $\iff$ the monoid of lattice points in the corner cone $\mathbb{R}_+(P - v)$ is generated by $\{x - v : x \in L(P)\}$ for all vertices v of P
- P smooth $\iff$ P each corner monoid is generated by a basis of $\mathbb{Z}^d$ (the corners are unimodular).

Classically, toric varieties are defined by fans.

$$\mathscr{V}(\mathscr{F}) = \bigcup_{D \in \mathscr{F}} \operatorname{Spec} K[D^* \cap \mathbb{Z}^d]$$

They are automatically normal.





Given P, we can consider its normal fan $\mathscr{N}(P)$:

$$\text{the maximal cones in } \mathscr{N}(P) \text{ are } C(P_v)^*,\ v \in \operatorname{vert}(P)$$

P very ample $\iff \mathscr{V}(\mathscr{N}(P)) = \operatorname{Proj} K[P]$.

Covering properties of normal lattice polytopes

Hierarchy of successively weaker (?) properties:

- (RUHT) P has a regular unimodular triangulation
- (UHT) P has a unimodular triangulation
- (UHC) P is covered by its unimodular subsimplices
- (FHC) $M(P)$ the union of free submonoids generated by elements in $E(P) = L(P) \times 1$
- (ICP) $M(P)$ is the union of its submonoids generated by $\leq d + 1$ elements of $E(P)$.
- P is normal
- P is very ample

B & G: (ICP) $\Rightarrow$ (FHC) $\Rightarrow$ normality, so (FHC) $\iff$ (ICP)

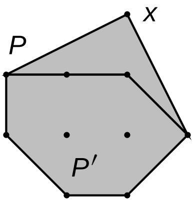
Counterexamples

- Ohsugi & Hibi: (UHT) $\not\Rightarrow$ (RUHT) ($\dim P = 9$, edge polytope of a graph)
- Kantor & Sarkaria: (UHC) $\not\Rightarrow$ (UHT) ($\dim P = 3$ sharp)
- B: (ICP) $\not\Rightarrow$ (UHC) ($\dim P = 5$)
- B &G with Henk, Martin, Weismantel: normal $\not\Rightarrow$ (UHC), normal $\not\Rightarrow$ (ICP) ($\dim P = 5$)
- B & G: very ample $\not\Rightarrow$ normal ($\dim P = 3$ sharp)

Shrinking

Search strategy:

- Produce a normal polytope P in a random way controlled by some parameters.
- Reduce P in size in order to increase the probability for a counterexample, at least heuristically.
- When P cannot be reduced any further, test the critical property, for example (UHC) or (ICP)

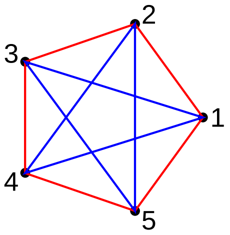


Definition

Let Q be a property of lattice polytopes. Then P is Q-tight if P has Q, but $\operatorname{conv}(L(P) \setminus \{x\})$ lacks it for all $x \in \operatorname{vert}(P)$.

Q is normality in the search for counterexamples to (UHC) or (ICP). First counterexample $P_{5,10}$ (dimension 5, 10 lattice points) to (UHC) and (ICP) found by B. and Gubeladze in May 1998 as a counterexample to (UHC), shown to fail (ICP) by Henk, Martin, and Weismantel.

The search for further counterexamples (1998 – 2001) yielded only one more polytope $P_{5,12}$ with 12 lattice points. It, too, violates (ICP). Actually $P_{5,12}$ appeared only once whereas (isomorphic copies of) $P_{5,10}$ returned over and over again.



$P_{5,10}$ spanned by the vectors

$(0, e_i + e_j) : \{i, j\}$ red edge

and

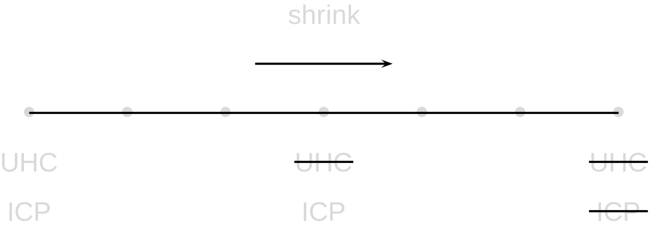
$(1, e_i + e_j) : \{i, j\}$blue edge

Open question at this point: (ICP) $\Rightarrow$ (UHC) ?

July 2006: Compare $P_{5,12}$ to $P_{5,10}$ . . .

Heureka: $L(P_{5,12}) \supset L(P_{5,10})$, there exist interesting objects in the vicinity of $P_{5,10}$.

Idea: along a shrink path ending in $P_{5,10}$ (or another counterexample), the stronger property (UHC) should be lost before the weaker property (ICP), at least sometimes:

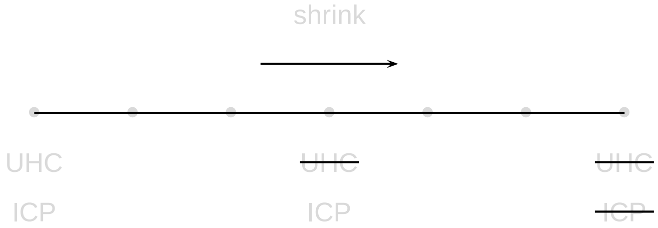


The idea works, hopefully . . .

July 2006: Compare $P_{5,12}$ to $P_{5,10}$ ...

Heureka: $L(P_{5,12}) \supset L(P_{5,10})$, there exist interesting objects in the vicinity of $P_{5,10}$.

Idea: along a shrink path ending in $P_{5,10}$ (or another counterexample), the stronger property (UHC) should be lost before the weaker property (ICP), at least sometimes:

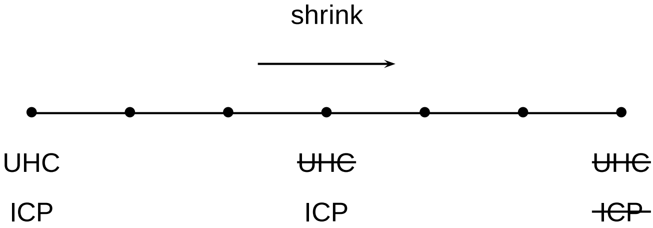


The idea works, hopefully ...

July 2006: Compare $P_{5,12}$ to $P_{5,10}$ ...

Heureka: $L(P_{5,12}) \supset L(P_{5,10})$, there exist interesting objects in the vicinity of $P_{5,10}$.

Idea: along a shrink path ending in $P_{5,10}$ (or another counterexample), the stronger property (UHC) should be lost before the weaker property (ICP), at least sometimes:



The idea works, hopefully ...

It does, and many polytopes with (ICP), but without (UHC) were found. Soon after, also tight such examples emerged.

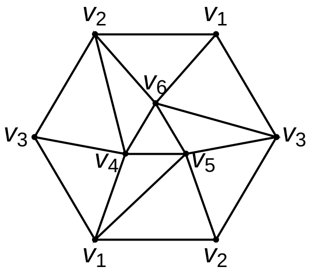
But: all counterexamples found in dimensions 5, 6, 7 (cone dimensions 6, 7, 8) contain $P_{5,10}$ in a well-defined sense.

Question 1. Is $P_{5,10}$ the unique minimal counterexample to (ICP) and (UHC) ?

Question 2. Does every polytope in dimensions 3 and 4 have (UHC) ?

Very ample, nonnormal polytopes

Consider minimal triangulation $\varDelta$ of $\mathbb{RP}^2$:

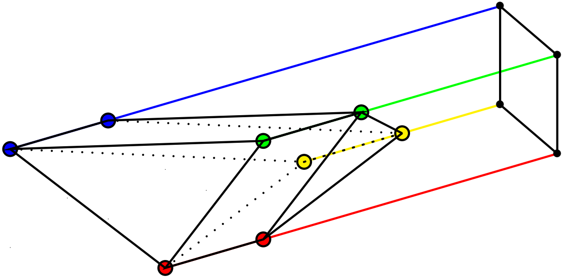


P spanned by $e_i + e_j + e_k$,

$\{v_i, v_jj, v_k\}$ a facet of $\varDelta$

$\dim P = 5$, $\overline{M}(P) \setminus M(P)$ is a single point $\Rightarrow$ P very ample, nonnormal

But even in dimension 3 very ample nonnormal polytopes exist (found by shrinking):



$$P = \operatorname{conv}\big((0,0) \times I_1\big) \cup \big((0,1) \times I_2\big) \cup \big((1,1) \times I_3\big) \cup \big((1,0) \times I_4\big).$$

$$I_1 = \{0,1\}, \quad I_2 = \{2,3\}, \quad I_3 = \{1,2\}, \quad I_4 = \{3,4\}$$

Smooth polytopes

Questions:

- Is every smooth polytope normal?
- Is $I(P) = \operatorname{Ker}\big(K[X_p : p \in L(P)] \to K[P]\big)$ generated by degree 2 monomials? Is $K[P]$ even a Koszul algebra?
- Is $I(P)$ generated by degree 2 monomials up to finite length?

Positive answers only in special cases, but no counterexamples found yet.

Proposition

P simple lattice polytope of dimension d.

- *P is smooth $\iff$ for all vertices v and all $x \in L(P)$, x not an edge neighbor of v, the polytope*

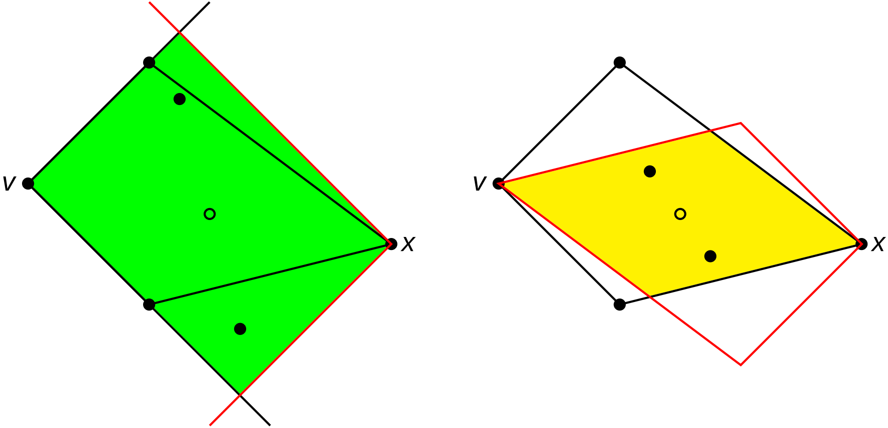
$$(v + C_v) \cap (x - C_v)$$

  contains a lattice point $\neq x, v$.

- *P is smooth and I_P generated in degree 2 up to finite length $\iff$ for all vertices v and all $x \in L(P)$, x not an edge neighbor of v, the polytope*

$$P \cap (-P + x + v)$$

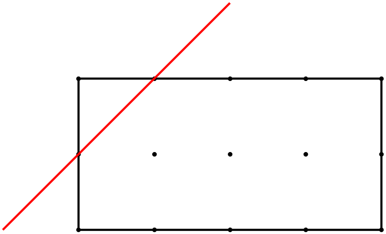
  contains a lattice point $\neq x, v$.


v
x
v
x

How to fund smooth polytopes:

1. Create many polytopes (for example by shrinking) and select the smooth ones
   Advantage: small polytopes
   Disadvantage: most likely harmless polytopes
2. Create random fans, refine them to smooth fans $\mathscr{F}$, and compute support polytopes P, i.e. $\mathscr{N}(P) = \mathscr{F}$
   Disadvantage: polytopes often gigantic

Often one can reduce smooth polytopes in size by cutting off faces



In the normal fan this corresponds to a stellar subdivision (preserving unimodularity).
Reduces number of lattice points, but increases the complexity of the face lattice.

How to compute support polytopes

Let $\mathscr{F}$ be a smooth fan, i.e. all maximal cones of $\mathscr{F}$ are unimodular. The searched for polytope P with $\mathscr{N}(P) = \mathscr{F}$ is given by inequalities

$$\lambda_\rho(x) \geq b_\rho, \quad \rho \text{ a ray of } \mathscr{F},\ b_\rho \in \mathbb{Z}.$$

Let D be such a maximal cone, with rays $\lambda_1, \dots, \lambda_d \in (\mathbb{R}^d)^*$. For each $(b_1, \dots, b_d) \in \mathbb{Z}^d$ the system

$$\lambda_i(x) = b_i, \qquad i = 1, \dots, d.$$

has a unique integral solution $v_D \in \mathbb{R}^d$ since $\det|\lambda_1, \dots, \lambda_d| = \pm 1$.

In order to have each v_D appear as a vertex of P, the values b_ρ assigned to each ray ρ of $\mathscr{F}$ must satisfy a system of linear inequalities in the strict sense.

Unfortunately these systems are rather huge in general, and this limits the range of this method.

In the experiments we start from the projective fan with rays $e_1^* \dots, e_d^*, -a_1e_1^* - \cdots - a_de_d^*$, $a_1, \dots, a_d > 0$, and refine it by stellar subdivision to a unimodular fan.