Hilbert depth of powers of the maximal ideal

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The framework

We consider

- $R = K[X_1, \dots, X_n]$, K a field, with the
  - standard grading, $\deg X_i = 1$,
- a finitely generated graded R-module.

The Hilbert function of M is denoted by

$$H(M,k) = \dim_K M_k, \qquad k \in \mathbb{Z}.$$

The Hilbert series is defined by

$$H_M(T) = \sum_{k\in\mathbb{Z}} H(M,k)T^k.$$

The Hilbert series is the Laurent expansion of a rational function at 0 of the form

$$H_M(T) = \frac{Q_M(T)}{(1-T)^n}, \qquad Q_M(T) \in \mathbb{Z}[T, T^{-1}].$$

In the following it is important to use the fixed denominator $(1-T)^n$, even if $(1-T)^d$, $d = \dim M$, would be sufficient.

Let us say that $\sum_{k=0}^{\infty} a_k T^k$ is positive if $a_k \geq 0$ for all k.

Suppose depth $M \geq t$. Then (after an extension of K) there exists an M-sequence $\mathbf{x} = x_1, \dots, x_t$ of 1-forms, and

$$H_{M/\mathbf{x}M}(T) = (1-T)^t H_M(T).$$

In particular: $(1-T)^t H_M(T)$ is positive.

Interpretation: if $(1-T)^t H_M(T)$ is positive, there might exist a module N such that $H_N(T) = H_M(T)$ and depth $N \geq t$.

Hilbert depth

Theorem (Uliczka)

Then the following numbers coincide:

1. $\max\{\operatorname{depth} N : H_M(T) = H_N(T)\}$,
2. $\max\{p : (1-T)^p H_M(T)$ *positive*$\}$,
3. $n - \min\{q : Q_M(T)/(1-T)^q$ *positive*$\}$,
4. *the maximum d such that*

$$H_M(T) = \sum_{e=d}^{n} \frac{Q_e(T)}{(1-T)^e}, \qquad Q_e(T) \in \mathbb{Z}_+[T, T^{-1}],$$

Definition

The Hdepth M, the Hilbert depth of M, is the common value of the numbers in the theorem.

The naive bound

Hilbert depth is very hard to compute in general, and even for such "harmless"modules as the syzygies of the maximal ideal $\mathfrak{m} = (X_1, \dots, X_n)$ it is not known in general.

There is a naive upper bound: Suppose the Hilbert series starts as

$$H_M(T) = a_k T^k + a_{k+1} T^{k+1} + \dots$$

Then

$$(1-T)^t H_M(T) = a_k T^k + (a_{k+1} - t a_k) T^{k+1} + \dots$$

Therefore

$$\operatorname{Hdepth} M \le \left\lfloor \frac{a_{k+1}}{a_k} \right\rfloor .$$

Powers of the maximal ideal

Let us consider the powers $\mathfrak{m}^s$. One has

$$H_{\mathfrak{m}^s}(T) = \binom{n+s-1}{s} T^s + \binom{n+s}{s+1} T^{s+1} + \cdots$$

Therefore

$$\operatorname{Hdepth} \mathfrak{m}^s \leq \left\lfloor \frac{n+s}{s+1} \right\rfloor = \left\lceil \frac{n}{s+1} \right\rceil$$

Theorem

$$\operatorname{Hdepth} \mathfrak{m}^s = \left\lceil \frac{n}{s+1} \right\rceil.$$

$s = 1$: $\operatorname{Hdepth} \mathfrak{m} = \lceil n/2 \rceil$ (not so hard). Actually: $\mathfrak{m}$ has Stanley depth $\lceil n/2 \rceil$ (Biró et al.).

Proposition

For any integer $0 < r < n$, we have

$$(1-T)^r H_{\mathfrak{m}^s}(T) = \binom{n+s-1}{s} T^s + \sum_{k=s+1}^{r+s-1} \left[\binom{n+k-1-r}{k} + (-1)^{k-1} \sum_{j=0}^{s-1} (-1)^j \binom{r}{k-j} \binom{n+j-1}{j} \right] T^k + \sum_{k=r+s}^{\infty} \binom{n+k-1-r}{k} T^k.$$

In fact, $(1-T)^r H_{\mathfrak{m}^s}(T) = (1-T)^r (H_R(T) - H_{R/\mathfrak{m}^s}(T))$.

The critical expression can be rewritten in two ways:

Lemma

For all positive integers n, s, r, k, we have

$$\sum_{j=s}^{k}(-1)^{k-j}\binom{n+j-1}{j}\binom{r}{k-j}$$
$$=\binom{n+k-r-1}{k}+(-1)^{k-1}\sum_{j=0}^{s-1}(-1)^{j}\binom{r}{k-j}\binom{n+j-1}{j}$$
$$=\binom{n+k-r-1}{k}+(-1)^{k+s}\sum_{t=1}^{r}\binom{r-t}{k-s}\binom{n-t+s-1}{s-1}.$$

Break the Koszul complex for $X_1, \dots, X_r$ into two parts:

$$0 \to M \to \bigwedge^{u-1} F(-u+1) \to \cdots \to F \to R \to S \to 0, \qquad F = R^r,$$

$$0 \to \bigwedge^{r} F(-r) \to \bigwedge^{r-1} F(-r+1) \to \cdots \to \bigwedge^{u} F(-r) \to M \to 0.$$

Computing the Hilbert function of M from both exact sequences and equating the expressions gives the first identity.

For the second identity we use that we had computed the Hilbert function in a different way:

$$H(M,k) = \sum_{t=1}^{r} \binom{r-t}{u-1} \binom{n-t+k-u}{k-u}.$$

Finally one has to prove:

Proposition

Let n and s be positive integers, and let $r = \lceil n/(s+1) \rceil$. Then, for all $k = s+1, s+2, \dots, s+r-1$, we have

$$\binom{n+k-r-1}{k} \geq \sum_{t=1}^{r} \binom{r-t}{k-s} \binom{n-t+s-1}{s-1}.$$

Important tool: $\psi(x) = \Gamma'(x)/\Gamma(x)$.

Correction of our arguments by Jiayun Lin (Canton).