Toric rings and discrete convex geometry

Lectures for the School on

Commutative Algebra and Interactions

with Algebraic Geometry and Combinatorics

Trieste, May/June 2004

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Preface

This text contains the computer presentation of 4 lectures:

1. Affine monoids and their algebras
2. Homological properties and combinatorial applications
3. Unimodular covers and triangulations
4. From vector spaces to polytopal algebras

Lecture 1 introduces the affine monoids and relates them to the geometry of rational convex cones. Lecture 2 contains the homological theory of normal affine semigroup rings and their applications to enumerative combinatorics developed by Hochster and Stanley.

Lectures 3 and 4 are devoted to lines of research that have been pursued in joint work with Joseph Gubeladze (Tbilisi/San Francisco).

A rather complete expository treatment of Lectures 1 and 2 is contained in

W. BRUNS. *Commutative algebra arising from the Anand-Dumir-Gupta conjectures*. Preprint.

Most of Lecture 3 and much more – in particular basic notions and results of polyhedral convex geometry – is to be found in

W. BRUNS AND J. GUBELADZE. *K-theory, rings, and polytopes*. Draft version of Part 1 of a book in progress.

For Lecture 4 there exists no coherent expository treatment so far, but a brief overview is given in

W. BRUNS AND J. GUBELADZE. *Polytopes and K-theory*. Preprint.

A previous exposition, covering various aspects of these lectures is to be found in

W. BRUNS AND J. GUBELADZE. *Semigroup algebras and discrete geometry*. In L. Bonavero and M. Brion (eds.), Toric geometry. Séminaires et Congrès 6 (2002), 43–127

All these texts can be downloaded from (or via)

http://www.math.uos.de/staff/phpages/brunsw/course.htm

They contain extensive lists of references.

Osnabrück, May 2004 *Winfried Bruns*

Lecture 1

Affine monoids and their algebras

Affine monoids and their algebras

An affine monoid M is (isomorphic to) a finitely generated submonoid of $\mathbb{Z}^d$ for some $d \geq 0$, i. e.

- $M + M \subset M$ (M is a semigroup);
- $0 \in M$ (now M is a monoid);
- there exist $x_1, \dots, x_n \in M$ such that $M = \mathbb{Z}_+ x_1 + \dots \mathbb{Z}_+ x_n$.

Often affine monoids are called affine semigroups.

$\operatorname{gp}(M) = \mathbb{Z}M$ is the group generated by M.

$\operatorname{gp}(M) \cong \mathbb{Z}^r$ for some $r = \operatorname{rank} M = \operatorname{rank} \operatorname{gp}(M)$.

Proposition 1.1. *The Krull dimension of $K[M]$ is given by*

$$\dim K[M] = \operatorname{rank} M.$$

Proof. $K[M]$ is an affine domain over K. Therefore

$$\begin{aligned}\dim K[M] &= \operatorname{trdeg} \mathrm{QF}(K[M]) \\ &= \operatorname{trdeg} \mathrm{QF}(K[\mathrm{gp}(M)]) \\ &= \operatorname{trdeg} \mathrm{QF}(K[\mathbb{Z}^r]) \\ &= r\end{aligned}$$

where $r = \operatorname{rank} M$.

Let K be a field (or a commutative ring). Then we can form the monoid algebra

$$K[M] = \bigoplus_{a \in M} KX^a, \qquad X^a X^b = X^{a+b}$$

$X^a =$ the basis element representing $a \in M$.

$M \subset \mathbb{Z}^d \Rightarrow K[M] \subset K[\mathbb{Z}^d] = K[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ is a monomial subalgebra.

Proposition 1.2. *Let M be a monoid.*

(a) M is finitely generated $\Longleftrightarrow K[M]$ is a finitely generated K-algebra.

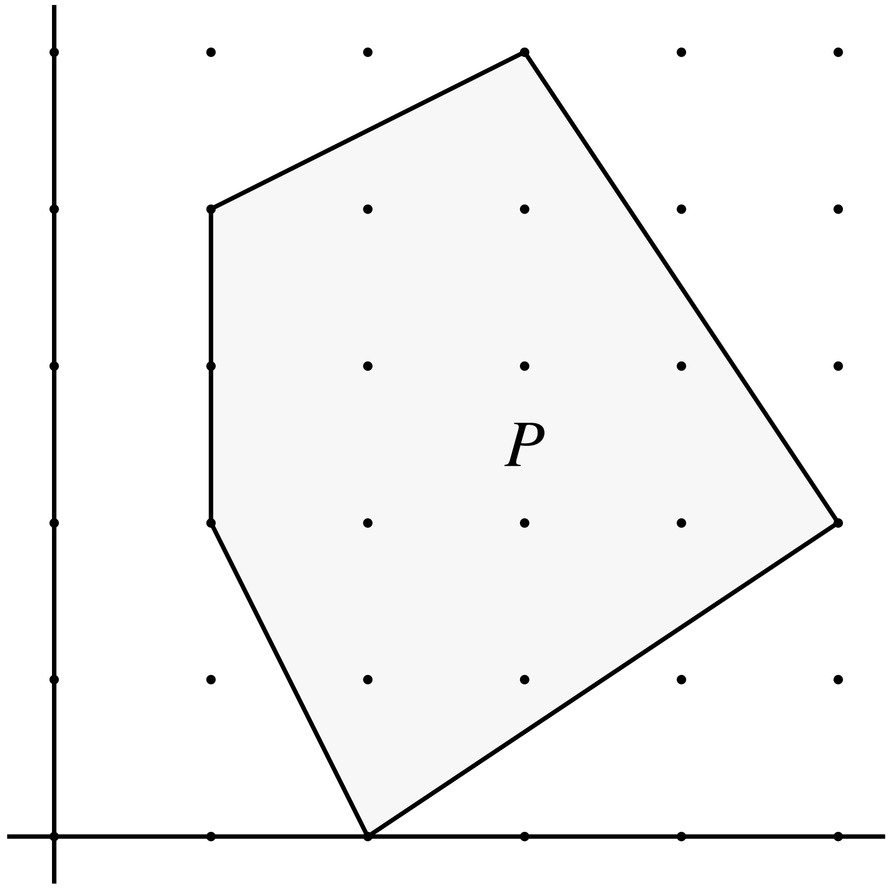
(b) M is an affine monoid $\Longleftrightarrow K[M]$ is an affine domain.

Sources for affine monoids (and their algebras) are

- monoid theory,
- ring theory,
- initial algebras with respect to monomial orders,
- invariant theory of torus actions,
- enumerative theory of linear diophantine systems,
- lattice polytopes and rational polyhedral cones,
- coordinate rings of toric varieties.

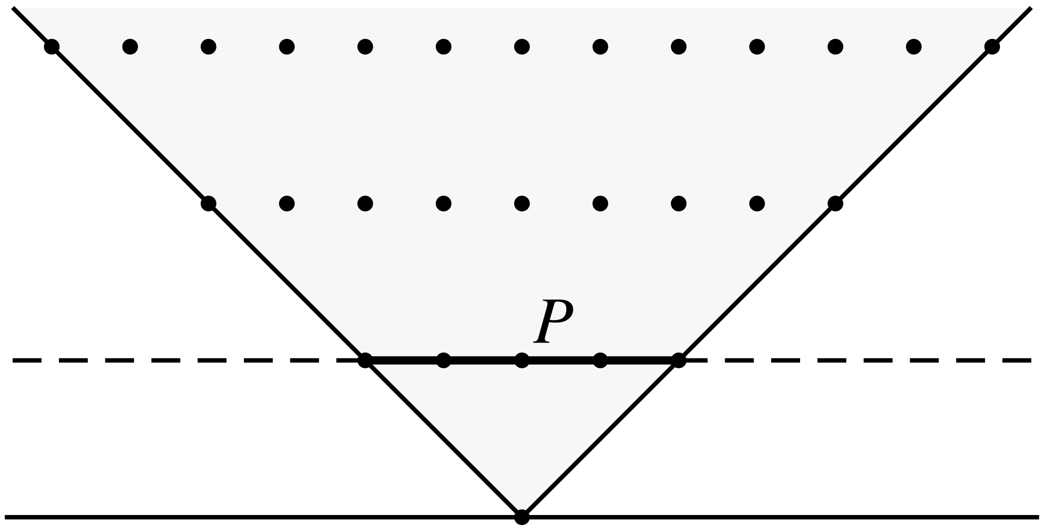
Polytopal monoids

Definition 1.3. The convex hull $\mathrm{conv}(x_1, \ldots, x_m)$ of points $x_i \in \mathbb{Z}^n$ is called a lattice polytope.



With a lattice polytope $P \subset \mathbb{R}^n$ we associate the polytopal monoid

$$M_P \subset \mathbb{Z}^{n+1} \quad \text{generated by} \quad (x,1), x \in P \cap \mathbb{Z}^n.$$



Vertical cross-section of a polytopal monoid

Such monoids will play an important role in Lectures 3 and 4.

Presentation of an affine monoid algebra

Let $R = K[x_1,\dots,x_n]$. Then we have a presentation

$$\pi : K[X] = K[X_1,\dots,X_n] \to K[x_1,\dots,x_n], \qquad X_n \mapsto x_n.$$

Let $I = \operatorname{Ker}\pi$ and $M = \{\pi(X^a) : a \in \mathbb{Z}_+^n\}$

Theorem 1.4. *The following are equivalent:*

(a) M *is an affine monoid and* $R = K[M]$*;*

(b) I *is prime, generated by binomials* $X^a - X^b$*,* $a, b \in \mathbb{Z}_+^n$*;*

(c) $I = I \cap K[X^{\pm 1}]$*,* I *is generated by binomials* $X^a - X^b$*, and* $U = \{a - b : X^a - X^b \in I\}$ *is a direct summand of* $\mathbb{Z}^n$*.*

Proof. (a) $\Rightarrow$ (b) Since R is a domain, I is prime. Let $f = c_1X^{a_1} + \cdots + c_mX^{a_m} \in I$, $c_i \in K$, $c_i \neq 0$, $a_1 >_{\text{lex}} \cdots >_{\text{lex}} a_m$. There exists $j > 1$ with $\pi(X^{a_1}) = \pi(X^{a_j})$, and so $X^{a_1} - X^{a_j} \in I$. Apply lexicographic induction to $f - c_1(X^{a_1} - X^{a_j})$.

(b) $\Rightarrow$ (c) Since I is an ideal, U is a subgroup. Since I is prime and $X_i \notin I$ for all i, $I = I \cap K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$. Let $u \in \mathbb{Z}^n$, $m > 0$ such that $mu \in U$, $u = v - w$ with $v, w \in \mathbb{Z}_+^n$. Clearly $X^{um} - X^{vm} \in I$. We can assume $\operatorname{char} K \nmid m$. Then

$$X^{um} - X^{vm} = (X^u - X^v)(X^{u(m-1)} + X^{u(m-2)v} + \cdots + X^{(m-1)}),$$

and the second term is not in $(X_1 - 1, \ldots, X_n - 1) \supset I$.

(c) $\Rightarrow$ (a) Consider $K[X] \to K[X^{\pm 1}] = K[\mathbb{Z}^n] \to K[\mathbb{Z}^n/U]$.

Cones

An affine monoid M generates the cone

$$\mathbb{R}_+M = \left\{ \sum a_i x_i : x_i \in M,\ a_i \in \mathbb{R}_+ \right\}$$

Since $M = \sum_{i=1}^n \mathbb{Z}_+ x_i$ is finitely generated, $\mathbb{R}_+M$ is finitely generated:

$$\mathbb{R}_+M = \left\{ \sum_{i=1}^n a_i x_i : a_1, \ldots, a_n \in \mathbb{R}_+ \right\}.$$

The structures of M and $\mathbb{R}_+M$ are connected in many ways. It is necessary to understand the geometric structure of $\mathbb{R}_+M$.

Finite generation $\Longleftrightarrow$ cut out by finitely many halfspaces:

Theorem 1.5. *Let $C \neq \emptyset$ be a subset of $\mathbb{R}^m$. Then the following are equivalent:*

- *there exist finitely many elements $y_1, \dots, y_n \in \mathbb{R}^m$ such that $C = \mathbb{R}_+ y_1 + \cdots + \mathbb{R}_+ y_n$;*
- *there exist finitely many linear forms $\lambda_1, \dots, \lambda_s$ such that C is the intersection of the half-spaces $H_i^+ = \{x : \lambda_i(x) \geq 0\}$.*

For full-dimensional cones the (essential) support hyperplanes $H_i = \{x : \lambda_i(x) = 0\}$ are unique:

Proposition 1.6. *If C generates $\mathbb{R}^m$ as a vector space and the representation $C = H_1^+ \cap \cdots \cap H_s^+$ is irredundant, then the hyperplanes H_i are uniquely determined (up to enumeration). Equivalently, the linear forms λ_i are unique up to positive scalar factors.*

rational generators $\Longleftrightarrow$ rationality of the support hyperplanes:

Proposition 1.7. *The generating elements $y_1, \dots, y_n$ can be chosen in $\mathbb{Q}^m$ (or $\mathbb{Z}^m$) if and only if the λ_i can be chosen as linear forms with rational (or integral) coefficients.*

Such cones are called rational.

Proposition 1.8. *If $Y = \{y_1, \dots, y_n\} \subset \mathbb{Q}^m$, then $\mathbb{Q}^m \cap \mathbb{R}_+ Y = \mathbb{Q}_+ Y$.*

Gordan's lemma and normality

As seen above, affine monoids define rational cones. The converse is also true.

Lemma 1.9 (Gordan's lemma). *Let $L \subset \mathbb{Z}^d$ be a subgroup and $C \subset \mathbb{R}^d$ a rational cone. Then $U \cap C$ is an affine monoid.*

Proof. Let $V = \mathbb{R}L \subset \mathbb{R}^d$. Then:

- $V \cap \mathbb{Q}^d = \mathbb{Q}L$;
- $C \cap V$ is a rational cone in V
- $\Rightarrow$ We may assume that $L = \mathbb{Z}^d$.

C is generated by elements $y_1, \dots, y_n \in M = C \cap \mathbb{Z}^d$.

$$x \in C \Rightarrow x = a_1 y_1 + \cdots + a_n y_n \qquad a_i \in \mathbb{R}_+.$$

$$x = x' + x'', \qquad x' = \lfloor a_1 \rfloor y_1 + \cdots + \lfloor a_n \rfloor y_n.$$

Clearly $x' \in M$. But

$$x \in M \Rightarrow x'' \in \mathrm{gp}(M) \cap C \Rightarrow x'' \in M.$$

x'' lies in a bounded set $B \Rightarrow$

M generated by $\quad y_1, \dots, y_n \quad$ and the finite set $\quad M \cap B$.

The monoid $M = U \cap C$ has a special property:

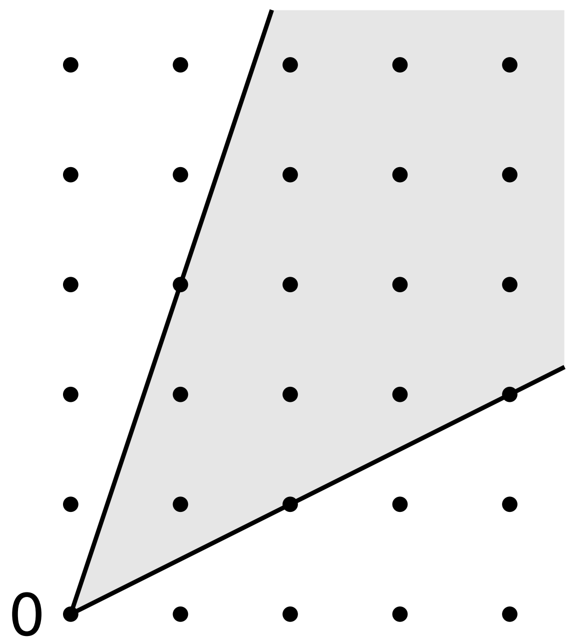
Definition 1.10. A monoid M is normal $\Longleftrightarrow$

$x \in \mathrm{gp}(M),\ kx \in M$ for some $k \in \mathbb{Z}, k > 0 \quad \Rightarrow \quad x \in M$.

Proposition 1.11.

- $M \subset \mathbb{Z}^d$ *normal affine monoid* $\iff$ *there exists a rational cone* C *such that* $M = \mathrm{gp}(M) \cap C$;
- $M \subset \mathbb{Z}^d$ *affine monoid* $\Rightarrow$ *the normalization* $\bar{M} = \mathrm{gp}(M) \cap \mathbb{R}_+ M$ *is affine.*

Briefly: **Normal affine monoids are discrete cones.**



Positivity, gradings and purity

Definition 1.12. A monoid M is positive if $x, -x \in M \Rightarrow x = 0$.

Definition 1.13. A grading on M is a homomorphism $\deg : M \to \mathbb{Z}$. It is positive if $\deg x > 0$ for $x \neq 0$.

Proposition 1.14. *For M affine the following are equivalent:*

(a) M is positive;

(b) $\mathbb{R}_+ M$ is pointed (i. e. contains no full line);

(c) M is isomorphic to a submonoid of $\mathbb{Z}^s$ for some s;

(d) M has a positive grading.

Proof. (c) $\Rightarrow$ (d) $\Rightarrow$ (a) trivial.

(a) $\Rightarrow$ (b) Set $C = \mathbb{R}_+ M$. One shows:

$\{x \in C : -x \in C\} = \mathbb{R}\{x \in M : -x \in M\}$.

Therefore: M positive $\Rightarrow$ C pointed.

(b) $\Rightarrow$ (c) Let C be positive. For each facet F of C there exists a unique linear form $\sigma_F : \mathbb{R}^d \longrightarrow \mathbb{R}$ with the following properties:

- $F = \{x \in C : \sigma_F(x) = 0\}, \quad \sigma_F(x) \geq 0$ for all $x \in C$;
- σ has integral coefficients, $\quad \sigma(\mathbb{Z}^d) = \mathbb{Z}$.

These linear forms are called the support forms of C. Let $s = \#\text{facets}(C)$ and define

$$\sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^s, \qquad \sigma(x) = \big(\sigma_F(x) : F \text{ facet}\big).$$

Then $\sigma(M) \subset \sigma(\bar{M}) \subset \mathbb{Z}^s_+$. Since C is positive, σ is injective! We call σ the standard embedding.

For M normal the standard embedding has an important property:

Proposition 1.15. *M positive affine monoid. Then the following are equivalent:*

(a) M is normal;

(b) σ maps M isomorphically onto $\mathbb{Z}^s \cap \sigma(\mathrm{gp}(M))$.

M pure submonoid of $N \iff M = N \cap \mathrm{gp}(M)$.

Corollary 1.16. *M affine, positive, normal $\iff M$ isomorphic to a pure submonoid of $\mathbb{Z}^s_+$ for some s.*

Normality and purity of $K[M]$

An integral domain R is normal if R integrally closed in $\mathrm{QF}(R)$.

R pure subring of $S \iff S = R \oplus T$ as an R-module.

Theorem 1.17. *M positive affine monoid.*

(a) M *normal* $\iff K[M]$ *normal*

(b) $M \subset \mathbb{Z}_+^s$ *pure submonoid* $\iff K[M]$ *pure subalgebra of* $K[\mathbb{Z}_+^s] = K[Y_1,\dots,Y_s]$

Proof. (a) $x \in \mathrm{gp}(M)$, $mx \in M$ for some $m > 0$

$\Rightarrow X^x \in K[\mathrm{gp}(M)] \subset \mathrm{QF}(K[M]), \quad (X^x)^m \in K[M]$.

Thus: $K[M]$ normal $\Rightarrow X^x \in K[M] \Rightarrow x \in M$.

Conversely, let M be normal, $\mathrm{gp}(M) = \mathbb{Z}^r$, $C = \mathbb{R}_+ M$

$\Rightarrow M = \mathbb{Z}^r \cap C$

$C = H_1^+ \cap \cdots \cap H_s^+, \quad H_i^+$ rational closed halfspace $\Rightarrow$

$$M = N_1 \cap \cdots \cap N_s, \qquad N_i = \mathbb{Z}^r \cap H_i^+$$

$$\Rightarrow \quad K[M] = K[N_1] \cap \cdots \cap K[N_s]$$

N_i discrete halfspace

Consider hyperplane H_i bounding H_i^+. Then $H_i \cap \mathbb{Z}^s$ direct summand of $\mathbb{Z}^s$.

$\Rightarrow \mathbb{Z}^s$ has basis $u_1, \dots, u_r$ with $u_1, \dots, u_{r-1} \in H_i, \quad u_r \in H_i^+$

$\Rightarrow K[N_i] \cong K[\mathbb{Z}^{r-1} \oplus \mathbb{Z}_+] \cong K[Y_1^{\pm 1}, \dots, Y_{r-1}^{\pm 1}, Z]$

Thus $K[M]$ intersection of factorial (hence normal) domains $\Rightarrow K[M]$ normal

(b) $T = K\{X^x : x \in \mathbb{Z}^s \setminus M\} \Rightarrow K[Y_1, \dots, Y_s] = K[M] \oplus T$ as K-vector space

M pure submonoid $\Rightarrow T$ is $K[M]$-submodule

Converse not difficult.

A grading on M induces a grading on $K[M]$:

Proposition 1.18. *Let M be an affine monoid with a grading $\deg$. Then*

$$K[M] = \bigoplus_{k \in \mathbb{Z}} K\{X^x : \deg x = k\}$$

is a grading on $K[M]$.

If $\deg$ is positive, then $K[M]$ is positively graded.

The class group

R normal Noetherian domain (or a Krull domain).

$I \subset \mathrm{QF}(R)$ fractional ideal $\iff$ there exists $x \in R$ such that xI is a non-zero ideal

I is divisorial $\iff (I^{-1})^{-1} = I$ where

$$I^{-1} = \{x \in \mathrm{QF}(R) : xI \subset R\}.$$

$(I,J) \mapsto ((IJ)^{-1})^{-1}$ defines a group structure on $\mathrm{Div}(R) = \{\text{div. ideals}\}$

Fact: $\mathrm{Div}(R)$ free abelian group with basis $\mathbb{Z}\,\mathrm{div}(\mathfrak{p})$, $\mathfrak{p}$ height 1 prime ideal ($\mathrm{div}(I)$ denotes I as an element of $\mathrm{Div}(I)$)

$\mathrm{Princ}(R) = \{xR : x \in \mathrm{QF}(R)\}$ is a subgroup

$$\mathrm{Cl}(R) = \frac{\mathrm{Div}(R)}{\mathrm{Princ}(R)}$$

is called the (divisor) class group.

It parametrizes the isomorphism classes of divisorial ideals.

R is factorial $\iff \mathrm{Cl}(R) = 0$.

Let M be a positive normal affine monoid, $R = K[M]$. Choose

$$x \in \mathrm{int}(M) = \{y \in M : \sigma_F(y) > 0 \text{ for all facets } F\}.$$

$\Rightarrow M[-x] = \mathrm{gp}(M)$

$\Rightarrow R[(X^x)^{-1}] = K[\mathrm{gp}(M)] = L$ = Laurent polynomial ring

Nagata's theorem $\Rightarrow$ exact sequence

$$0 \to U \to \mathrm{Cl}(R) \to \mathrm{Cl}(L) \to 0,$$

U generated by classes $[\mathfrak{p}]$ of minimal prime overideals of X^x

L factorial $\Rightarrow \mathrm{Cl}(L) = 0 \Rightarrow \mathrm{Cl}(R) = U$.

For a facet F of $\mathbb{R}_+M$ set

$$\mathfrak{p}_F = R\{x \in M : \sigma_F(x) \geq 1\}.$$

$\Rightarrow$ $\mathfrak{p}_F$ is a prime ideal since $R/\mathfrak{p}_F \cong K[M \cap F]$.

Evidently

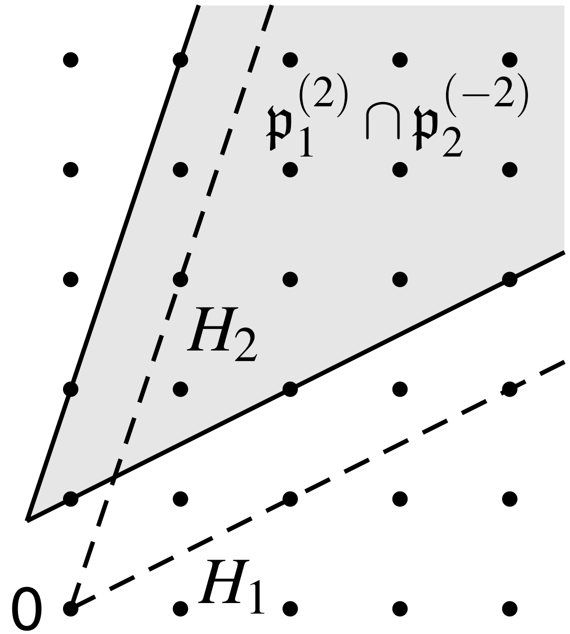
$$Rx = R\{X^y : \sigma_F(y) \geq \sigma_F(x) \text{ for all } F\} = \bigcap_F \mathfrak{p}_F^{(\sigma_F(x))}$$

$\mathfrak{p}_F^{(k)} = R\{X^y : \sigma_F(y) \geq k\}$ is the k-th symbolic power of $\mathfrak{p}_F$.

$\Rightarrow$ $\mathrm{Cl}(R) = \sum_F \mathbb{Z}[\mathfrak{p}_F]$.

$$\left[\bigcap_F \mathfrak{p}_F^{(k_F)}\right] = \sum_F k_F[\mathfrak{p}_F]$$

$\Rightarrow$ every divisorial ideal is isomorphic to an ideal $\bigcap_F \mathfrak{p}_F^{(k_F)}$



Monomial ideal I principal

$\iff$ there exists a monomial X^y with $I = X^y R$

Enumerate the facets $F_1, \dots, F_s$, $\mathfrak{p}_i = \mathfrak{p}_{F_i}$, $\sigma_i = \sigma_{F_i}$

Theorem 1.19 (Chouinard).

$$\mathrm{Cl}(R) \cong \frac{\bigoplus_{i=1}^{s} \mathbb{Z}\,\mathrm{div}(\mathfrak{p}_i)}{\{\mathrm{div}(RX^y) : y \in \mathrm{gp}(M)\}} \cong \frac{\mathbb{Z}^s}{\sigma(\mathrm{gp}(M))}$$

where σ is the standard embedding.

Lecture 2

Homological properties and combinatorial applications

Magic Squares

In 1966 H. Anand, V. C. Dumir, and H. Gupta investigated a combinatorial problem:

- Suppose that n distinct objects, each available in k identical copies, are distributed among n persons in such a way that each person receives exactly k objects.

What can be said about the number $H(n,k)$ of such distributions?

They formulated some conjectures:

(ADG-1) there exists a polynomial $P_n(k)$ of degree $(n-1)^2$ such that $H(n,k) = P_n(k)$ for all $k \gg 0$;

(ADG-2) $H(n,k) = P_r(n)$ for all $k > -n$; in particular $P_n(-k) = 0$, $k = 1, \ldots, n-1$;

(ADG-3) $P_n(-k) = (-1)^{(n-1)^2} P_n(k-n)$ for all $k \in \mathbb{Z}$.

These conjectures were proved and extended by R. P. Stanley using methods of commutative algebra.

The weaker version (ADG-1) of (ADG-2) has been included for didactical purposes.

Reformulation:

$a_{ij} =$ number of copies of object i that person j receives

$\Rightarrow A = (a_{ij}) \in \mathbb{Z}_+^{n\times n}$ such that

$$\sum_{k=1}^{n} a_{ik} = \sum_{l=1}^{n} a_{lj} = k, \qquad i, j = 1, \ldots, n.$$

$H(n,k)$ is the number of such matrices A.

The system of equations is part of the definition of magic squares. In combinatorics the matrices A are called magic squares, though the usually requires further properties for those.

Two famous magic squares:

8	1	6
3	5	7
4	9	2

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

The 3×3 square goes back to the Chinese emperor Loh-shu (about 2800 B.C.) and the 4×4 appears in Albrecht Dürer's engraving Melancholia (1514). It has remarkable symmetries and shows the year of its creation.

A step towards algebra

Let $\mathscr{M}_n$ be the set of all matrices $A = (a_{ij}) \in \mathbb{Z}_+^{n\times n}$ such that

$$\sum_{k=1}^{n} a_{ik} = \sum_{l=1}^{n} a_{lj}, \qquad i,j = 1,\dots,n.$$

By Gordan's lemma $\mathscr{M}_n$ is an affine, normal monoid, and $A \mapsto$ magic sum $k = \sum_{k=1}^{n} a_{1k}$ is a positive grading on $\mathscr{M}$.

It is even a pure submonoid of $\mathbb{Z}_+^{n\times n}$.

$$\operatorname{rank} \mathscr{M}_n = (n-1)^2 + 1$$

Theorem 2.1 (Birkhoff-von Neumann). *$\mathscr{M}_n$ is generated by the its degree 1 elements, namely the permutation matrices.*

Translation into commutative algebra

We choose a field K and form the algebra

$$R = K[\mathscr{M}_n].$$

It is a normal affine monoid algebra, graded by the “magic sum”, and generated in degree 1. $\dim R = \operatorname{rank} \mathscr{M}_n = (n-1)^2 + 1$.

$\Rightarrow$

$H(n,k) = \dim_K R_k = H(R,k)$ is the Hilbert function of R!

$\Rightarrow$ (ADG-1): there exists a polynomial P_n of degree $(n-1)^2$ such that $H(n,k) = P_n(k)$ for $k \gg 0$.

In fact, take P_n as the Hilbert polynomial of R.

A recap of Hilbert functions

Let K be a field, and $R = \oplus_{k=0}^{\infty} R_k$ a graded K-algebra generated by homogeneous elements $x_1, \dots, x_n$ of degrees $g_1, \dots, g_n > 0$.

Let M be a non-zero, finitely generated graded R-module.

Then $\quad H(M,k) = \dim_K M_k < \infty \quad$ for all $k \in \mathbb{Z}$.

$H(M,\_) : \mathbb{Z} \to \mathbb{Z}$ is the Hilbert function of M.

We form the Hilbert (or Poincaré) series

$$H_M(t) = \sum_{k \in \mathbb{Z}} H(M.k) t^k.$$

Fundamental fact:

Theorem 2.2 (Hilbert-Serre). *Then there exists a Laurent polynomial* $Q \in \mathbb{Z}[t,t^{-1}]$ *such that*

$$H_M(t) = \frac{Q(t)}{\prod_{i=1}^{n}(1-t^{g_i})}.$$

More precisely: $H_M(t)$is the Laurent expansion at 0 of the rational function on the right hand side.

Refinement: M is finitely generated over a graded Noether normalization $K[y_1,\dots,y_d]$, $d = \dim M$, of $R/\operatorname{Ann} M$.

Special case: $g_1,\dots,g_n = 1$. Then $y_1,\dots,y_d$ can be chosen of degree 1 (after an extension of K).

Theorem 2.3. *Suppose that* $g_1 = \cdots = g_n = 1$ *and let* $d = \dim M$. *Then*

$$H_M(t) = \frac{Q(t)}{(1-t)^d}.$$

Moreover:

- *There exists a polynomial* $P_M \in \mathbb{Q}[X]$ *such that*

$$H(M,i) = P_M(i), \qquad i > \deg H_M,$$
$$H(M,i) \neq P_M(i), \qquad i = \deg H_M.$$

- $e(M) = Q(1) > 0$, *and if* $d \geq 1$, *then*

$$P_M = \frac{e(M)}{(d-1)!} X^{d-1} + \text{terms of lower degree.}$$

The general case is not much worse: P_M must be allowed to be a quasi-polynomial, i. e. a “polynomial” with periodic coefficients (instead of constant ones).

Theorem 2.4. *Suppose $R/\operatorname{Ann} M$ has a graded Noether normalization generated by elements of degrees $e_1,\dots,e_d$. Then*

$$H_M(t) = \frac{Q(t)}{\prod_{i=1}^{d}(1-t^{e_i})}, \qquad Q(1) > 0.$$

Moreover there exists a quasi-polynomial P_M, whose period divides $\operatorname{lcm}(e_1,\dots,e_d)$, such that

$$H(M,i) = P_M(i), \qquad i > \deg H_M,$$
$$H(M,i) \neq P_M(i), \qquad i = \deg H_M.$$

Hochster's theorem

Theorem 2.5. *Let M be an affine normal monoid. Then $K[M]$ is Cohen-Macaulay for every field K.*

There is no easy proof of this powerful theorem. For example, it can be derived from the Hochster-Roberts theorem, using that $K[M] \subset K[X_1, \dots, X_s]$ can be chosen as a pure embedding.

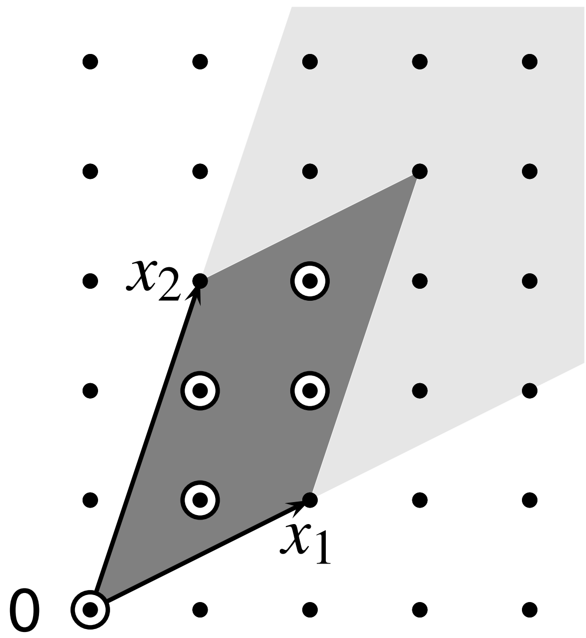
A special case is rather simple:

Definition 2.6. An affine monoid M is simplicial if the cone $\mathbb{R}_+ M$ is generated by $\operatorname{rank} M$ elements.

Proposition 2.7. *Let M be a simplical affine normal monoid. Then $K[M]$ is Cohen-Macaulay for every field K.*

Proof. Let $d = \operatorname{rank} M$. Choose elements $x_1, \ldots, x_d \in M$ generating $\mathbb{R}_+ M$, and set

$$\operatorname{par}(x_1, \ldots, x_d) = \left\{ \sum_{i=1}^{d} q_i x_i : q_i \in [0, 1) \right\}.$$



$$B = \mathrm{par}(x_1, \ldots, x_d) \cap \mathbb{Z}^d$$

$$N = \mathbb{Z}_+ x_1 \cdots + \mathbb{Z}_+ x_d.$$

The arguments in the proof Gordan's lemma and the linear independence of $x_1, \ldots, x_d$ imply:

- $M = \bigcap_{z \in B} z + N$
- N is free.
- The union is disjoint.

In commutative algebra terms:

- $K[M]$ is finite over $K[N]$.
- $K[N] \cong K[X_1, \ldots, X_d]$.
- $K[M]$ is a free module over $K[N]$.

$\Rightarrow K[M]$ is Cohen-Macaulay.

Combinatorial consequence of the Cohen-Macaulay property:

Theorem 2.8. *Let M be a graded Cohen-Macaulay module over the positively graded K-algebra R and $x_1,\dots,x_d$ a h.s.o.p. for M, $e_i = \deg x_i$. Let*

$$H_M(t) = \frac{h_a t^a + \cdots + h_b t^b}{\prod_{i=1}^{d}(1-t^{e_i})}, \qquad h_a, h_b \neq 0.$$

Then $h_i \geq 0$ for all i.

If $M = R = K[R_1]$, then $h_i > 0$ for all $i = 0,\dots,b$.

Proof. $h_a t^a + \cdots + h_b t^b$ is the Hilbert series of

$$M/(x_1 M + \cdots + x_d M).$$

Reciprocity

(ADG-3) compares values $P_R(k)$ of the Hilbert polynomial of $R = K[\mathscr{M}_n]$ for all values of k:

(ADG-3) $P_n(-k) = (-1)^{(n-1)^2} P_n(k-n)$ for all $k \in \mathbb{Z}$.

According to (ADG-2) the shift $-n$ in $P_n(k-n)$ is the degree of $H_R(t)$ (not yet proved).

- What identity for $H_R(t)$ is encoded in (ADG-3) ?

Lemma 2.9. *Let* $P : \mathbb{Z} \to \mathbb{C}$ *be a quasi-polynomial. Set*

$$H(t) = \sum_{k=0}^{\infty} P(k)t^k \quad \text{and} \quad G(t) = -\sum_{k=1}^{\infty} P(-k)t^k.$$

Then H *and* G *are rational functions. Moreover*

$$H(t) = G(t^{-1}).$$

Corollary 2.10. *Let* R *be a positively graded, finitely generated* K*-algebra,* $\dim R = d$*, with Hilbert quasi-polynomial* P*. Suppose* $\deg H_R(t) = g < 0$*. Then the following are equivalent:*

- $P(-k) = (-1)^{d-1}P(k+g)$ for all $k \in \mathbb{Z}$;
- $(-1)^d H_R(t^{-1}) = t^{-g} H_R(t)$.

Strategy:

- Find an R-module ω with $H_\omega(t) = (-1)^d H_R(t^{-1})$

This is possible for R Cohen-Macaulay: ω is the canonical module of R.

- Compute ω for $R = K[\mathscr{M}_n]$ and show that $\omega \cong R(g)$, $g = \deg H_R(t)$.

$R(g)$ free module of rank 1 with generator in degree $-g$. Thus $H_{R(g)}(t) = t^{-g} H_R(t)$.

More generally:

- Compute ω for $R = K[M]$ with M affine, normal.

The canonical module

In the following: R positively graded Cohen-Macaulay K-algebra, $x_1,\dots,x_d$ h.s.o.p., $\deg x_i = g_i$

First $R = S = K[X_1,\dots,X_d]$:

$$(-1)^d H_S(t^{-1}) = \frac{(-1)^d}{\prod_{i=1}^d (1-t^{-g_i})} = \frac{t^{g_1+\cdots+g_d}}{\prod_{i=1}^d (1-t^{-g_i})} = H_\omega(t)$$

with $\omega = \omega_S = S(-(g_1+\cdots+g_d))$

The general case: R free over $K[x_1,\dots,x_d] \cong K[X_1,\dots,X_d]$, say with homogeneous basis $y_1,\dots,y_m$:

$$R \cong \bigoplus_{j=1}^{m} Sy_j \cong \bigoplus_{i=1}^{u} S(-i)^{h_i}, \qquad h_i = \#\{j : \deg y_j = i\},$$

$$H_R(t) = (h_0 + h_1 t + \cdots + h_u t^u) H_S(t) = Q(t) H_S(t)$$

Set $\omega_R = \mathrm{Hom}_S(R, \omega_S)$. Then, with $s = g_1 + \cdots + g_d$

$$\omega_R \cong \bigoplus_{i=1}^{u} \mathrm{Hom}_S(S(-i)^{h_i}, S(-s)) \cong \bigoplus_{i=1}^{u} S(-i+s)^{h_i},$$

$$H_{\omega_R}(t) = (h_0 t^s + \cdots + h_u t^{s-u}) H_S(t) = Q(t^{-1}) t^s H_S(t)$$

$$= (-1)^d Q(t^{-1}) H_S(t^{-1}) = (-1)^d H_R(t^{-1}).$$

Multiplication in the first component makes ω_R an R-module:

$$a \cdot \varphi(\_) = \varphi(a \cdot \_).$$

- But: Is ω_R independent of S ?

Theorem 2.11. *ω_R depends only on R (up to isomorphism of graded modules).*

The proof requires homological algebra, after reduction from the graded to the local case.

Gorenstein rings

Definition 2.12. A positively graded Cohen-Macaulay K-algebra is Gorenstein if $\omega_R \cong R(h)$ for some $h \in \mathbb{Z}$.

Actually, there is no choice for h:

Theorem 2.13 (Stanley). *Let R be Gorenstein. Then*

- $\omega_R \cong R(g)$, $g = \deg H_R(t)$;
- $h_0 = h_{u-i}$ for $i = 0, \ldots, u$: the h-vector is palindromic;
- $H_R(t^{-1}) = (-1)^d t^{-g} H_R(t)$.

Conversely, if R is a Cohen-Macaulay integral domain such that $H_R(t^{-1}) = (-1)^d t^{-h} H_R(t)$ for some $h \in \mathbb{Z}$, then R is Gorenstein.

Proof.

$$(-1)^d H_R(t) = (h_0 t^s + \cdots + h_u t^{s-u}) H_S(t)$$
$$t^{-h} H_R(t) = (h_s t^{u-h} + \cdots + h_0^{-h}) H_S(t)$$

Equality holds $\Longleftrightarrow$

$$h = u - s = \deg H_R(t) \quad \text{and} \quad h_i = h_{u-i},\ i = 0, \ldots, u$$

If R is a domain, then ω_R is torsionfree. Consider $R \mapsto \omega_R$, $a \mapsto ax$, $x \in \omega_R$ homogeneous, $\deg x = -g$.

This linear map is injective: $R(g) \hookrightarrow \omega_R$. Equality of Hilbert functions implies bijectivity.

The canonical module of $K[M]$

In the following M affine, normal, positive monoid. We want to find the canonical module of $R = K[M]$ (Cohen-Macaulay by Hochster's theorem).

Theorem 2.14 (Danilov, Stanley). *The ideal I generated by the monomials in the interior of $\mathbb{R}_+M$ is the canonical module of $K[M]$ (with respect to every positive grading of M).*

Note: $x \in \operatorname{int}(\mathbb{R}_+M) \iff \sigma_F(x) > 0$ for all facets F of $\mathbb{R}_+M$.

$\mathfrak{p}_F$ is generated by all monomials X^x, $x \in M$ such that $\sigma_F(x) > 0$.

$$\Rightarrow \quad I = \bigcap_{F \text{ facet}} \mathfrak{p}_F .$$

Choose a positive grading on M and let ω be the canonical module of R with respect to this grading.

By definition ω is free over a Noether normalization $\Rightarrow$ ω is a Cohen-Macaulay R-module $\Rightarrow$ ω is (isomorphic to) a divisorial ideal $\Rightarrow$

- As discussed in Lecture 1, there exist $j_F \in \mathbb{Z}$ such that

$$\omega = \bigcap \mathfrak{p}_F^{(j_F)}.$$

Without further standardization we cannot conclude that $j_F = 1$ for all F.

We have to use the natural $\mathbb{Z}^r$-grading, $\mathbb{Z}^r = \mathrm{gp}(M)$ on R !

The homological property characterizing the canonical module is

$$\operatorname{Ext}^j_R(K,\omega_R) = \begin{cases} K, & j = d, \\ 0, & j \neq d. \end{cases} \qquad d = \dim R.$$

This is to be read as an isomorphism of graded modules: let $\mathfrak{m}$ be the irrelevant maximal ideal; then $K = R/\mathfrak{m}$ lives in degree 0.

In our case $R/\mathfrak{m}$ is a $\mathbb{Z}^r$-graded module, as is ω

$\Rightarrow \operatorname{Ext}^j_R(K,\omega_R) \cong K$ lives in exactly one multidegree $v \in \mathbb{Z}^r$

$\Rightarrow \operatorname{Ext}^j_R(K,X^{-v}\omega_R) \cong K$ in multidegree $0 \in \mathbb{Z}^r$

Replace ω_R by $X^{-v}\omega_R$.

Definition 2.15. Let R be a $\mathbb{Z}^n$-graded Cohen-Macaulay ring such that the homogeneous non-units generate a proper ideal $\mathfrak{p}$ of R.

$\Rightarrow \mathfrak{p}$ is a prime ideal; set $d = \dim R_{\mathfrak{p}}$.

One says that ω is a $\mathbb{Z}^n$-graded canonical module of R if

$$\operatorname{Ext}^j_R(R/\mathfrak{p}, \omega) = \begin{cases} R/\mathfrak{p}, & j = d, \\ 0, & j \neq d. \end{cases}$$

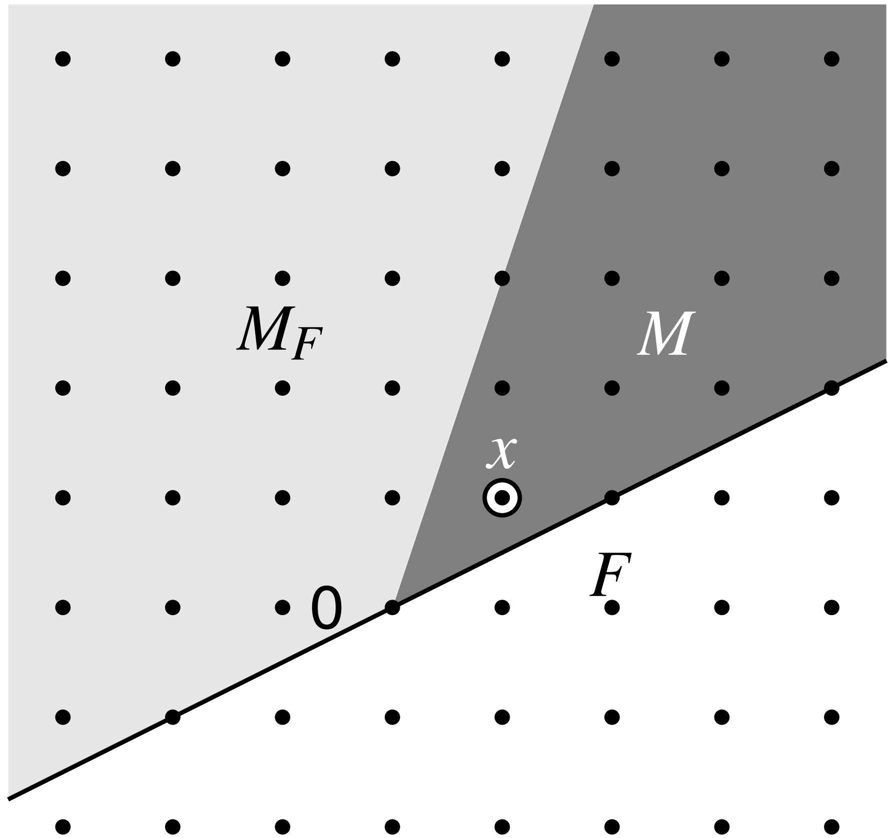
We have seen: $R = K[M]$ has a $\mathbb{Z}^r$-graded canonical module

$$\omega = \bigcap \mathfrak{p}_F^{(j_F)}$$

and it remains to show that $j_F = 1$ for all facets F.

Let $R_F = R[(M \cap F)^{-1}]$: we invert all the monomials in F.

$\Rightarrow$ R_F is the "discrete halfspace algebra" with respect to the support hyperplane through F.



$\Rightarrow \mathfrak{p}_F R_F$ is the $\mathbb{Z}^r$-graded canonical module of R_F (easy to see since $\mathfrak{p}_F R_F$ is principal generated by a monomial X^x with $\sigma_F(x) = 1$)

On the other hand: $\omega_{R_F} = (\omega_R)_F$: the $\mathbb{Z}^r$-graded canonical module "localizes" (a nontrivial fact)

$\Rightarrow j_F = 1$.

Back to the ADG conjectures

Recall that $\mathscr{M}_n$ denotes the "magic" monoid. It contains the matrix $\mathbf{1}$ with all entries 1.

Let $C = \mathbb{R}_+\mathscr{M}_n$. Then C is cut out from $\mathbb{R}\mathscr{M}_n$ by the positive orthant

$\Rightarrow \operatorname{int}(C) = \{A : a_{ij} > 0 \text{ for all } i,j\}$.

$\Rightarrow A - \mathbf{1} \in \mathscr{M}_n$ for all $A \in M \cap \operatorname{int}(C)$

$\Rightarrow$ interior ideal I is generated by $X^{\mathbf{1}}$; $\quad \mathbf{1}$ has magic sum n

$\Rightarrow I \cong R(-n)$. $\quad R = K[\mathscr{M}_n]$ is a Gorenstein ring with $\deg H_R(t) = -n$

$\deg H_R(t) = -n \Rightarrow$

(ADG-2) $H(n,r) = P_r(n)$ for all $r > -n$; in particular $P_n(-r) = 0$, $r = 1,\dots,n-1$;

(ADG-2) and R Gorenstein $\Rightarrow$

(ADG-3) $P_n(-r) = (-1)^{(n-1)^2} P_n(r-n)$ for all $r \in \mathbb{Z}$.

In terms of

$$H_R(t) = \frac{1 + h_1 t + \cdots + h_u t^u}{(1-t)^{(n-1)^2+1}}, \qquad h_u \neq 0,$$

we have seen that

- $u = (n-1)^2 + 1 - n$ (ADG-2)
- $h_i > 0$ for $i = 1, \ldots, u$ (R Cohen-Macaulay)
- $h_i = h_{u-i}$ for all i (ADG-3)

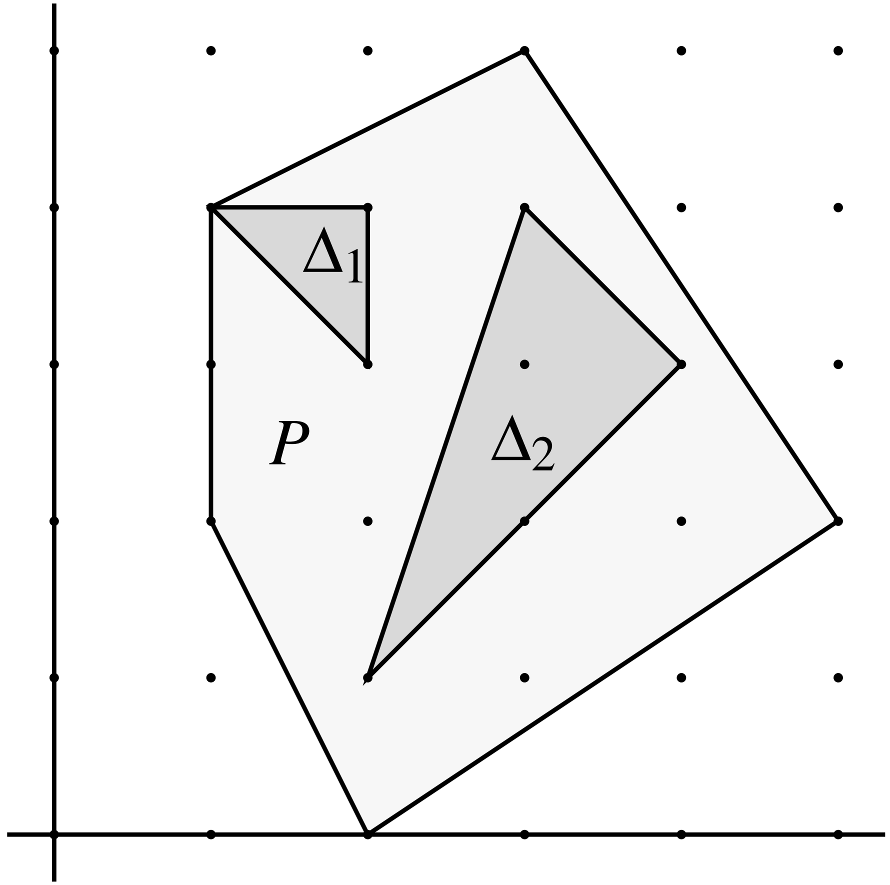
Very recent result, conjectured by Stanley and now proved by Ch. Athanasiadis:

- the sequence (h_i) is unimodal: $h_0 \le h_1 \le \cdots \le h_{\lceil u/2 \rceil}$

Lecture 3

Unimodular covers and triangulations

Recall: $P = \mathrm{conv}(x_1, \ldots, x_n) \subset \mathbb{R}^d$, $x_i \in \mathbb{Z}^d$, is called a lattice polytope.



$\Delta = \mathrm{conv}(v_0, \ldots, v_d)$, $v_0, \ldots, v_d$ affinely independent, is a simplex.

Set $U_\Delta = \sum_{i=0}^{d} \mathbb{Z}(v_i - v_0)$.

$$\mu(\Delta) = [\mathbb{Z}^d : U_\Delta] = \text{multiplicity of } \Delta$$

Δ is **unimodular** if $\mu(\Delta) = 1$.

Δ is **empty** if $\operatorname{vert}(\Delta) = \Delta \cap \mathbb{Z}^d$.

Lemma 3.1.

$$\mu(\Delta) = d!\operatorname{vol}(\Delta) = \pm\det\begin{pmatrix} v_1 - v_0 \\ \vdots \\ v_d - v_0 \end{pmatrix}$$

When is P covered by its unimodular subsimplices?

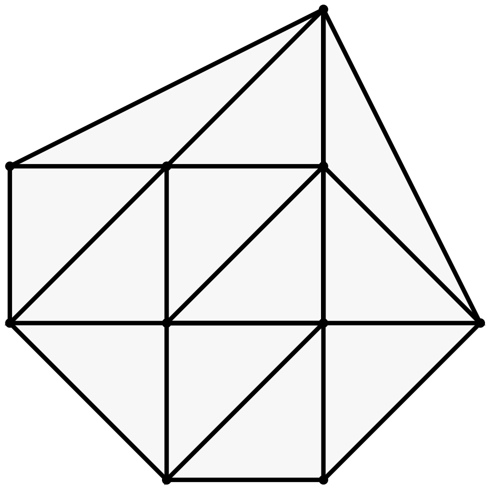
For short: P has UC.

Low Dimensions

$d = 1$: 

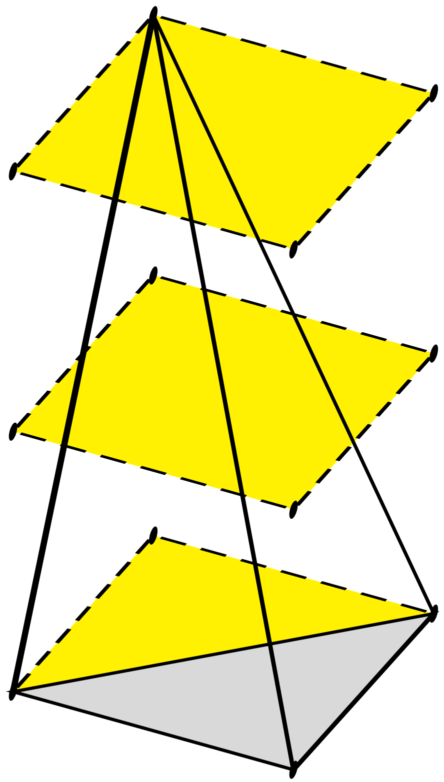
P has a unique unimodular triangulation.

$d = 2$:



Every empty lattice triangle is unimodular $\Rightarrow$ every 2-polytope has a unimodular triangulation.

$d = 3$: There exist empty simplices of arbitrary multiplicity!

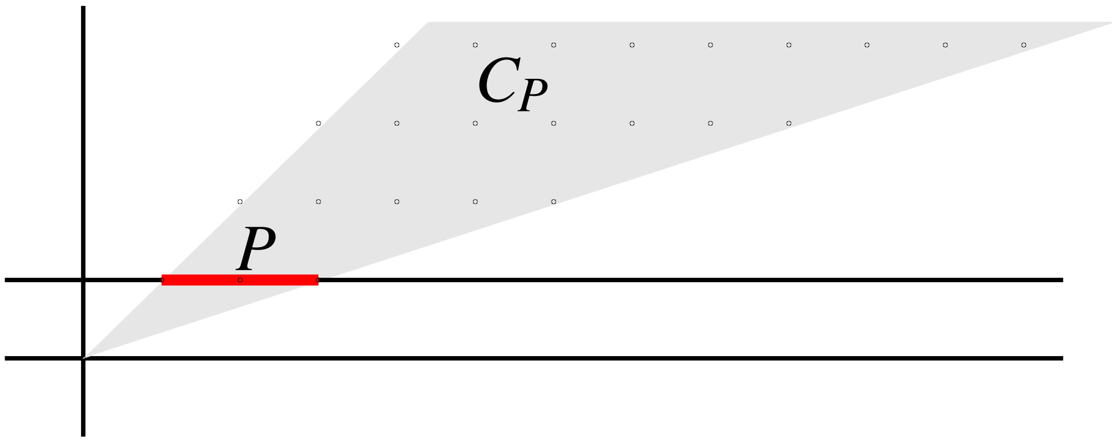


Polytopal cones and monoids

The cone over P is $C_P = \mathbb{R}_+\{(x,1) \in \mathbb{R}^{d+1} : x \in P\}$.

The monoid associated with P is $M_P = \mathbb{Z}_+\{(x,1) : x \in P \cap \mathbb{Z}^d\}$.

The integral closure of M_P is $\widehat{M}_P = C_P \cap \mathbb{Z}^{d+1}$.



Proposition 3.2. *P has UC $\Rightarrow M_P = \widehat{M}_P$ (P is integrally closed).*

P is integrally closed $\Longleftrightarrow$

(i) $\mathrm{gp}(M_P) = \mathbb{Z}^{d+1}$ and

(ii) M_P is a normal monoid ($M_P = C_P \cap \mathrm{gp}(M_P)$)

There exist non-normal 3-dimensional polytopes, for example

$$P = \{x \in \mathbb{R}^3 : x_i \geq 0, \;\; 6x_1 + 10x_2 + 15x_3 \leq 30\}.$$

Monoid algebras, toric ideals and Gröbner bases

Let K be a field. The polytopal K-algebra $K[P]$ is the monoid algebra

$$K[P] = K[M_P] = K[X_x : x \in P \cap \mathbb{Z}^d]/I_P.$$

The toric ideal I_P is generated by all binomials

$$\prod_{x \in P \cap \mathbb{Z}^d} X_x^{a_x} - \prod_{x \in P \cap \mathbb{Z}^d} X_x^{b_x},$$

$$\sum a_x x = \sum b_x x, \quad \sum a_x = \sum b_x$$

expressing the affine relations between the lattice points in P.

Sturmfels:

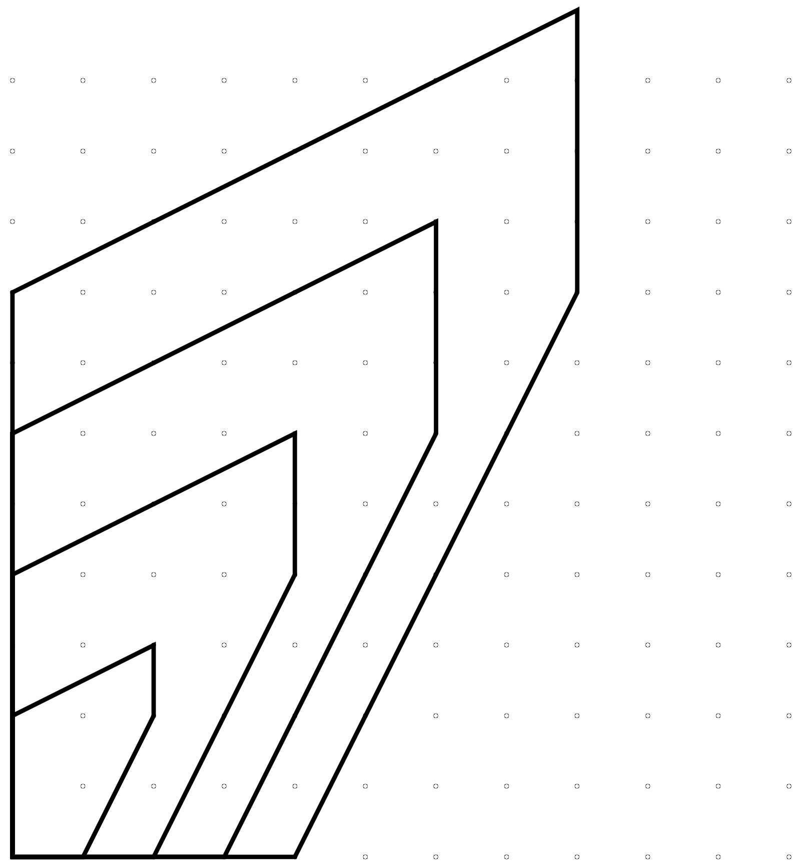
"generic" weights for $X_x \longmapsto$

$$\begin{cases} \text{(i) regular triangulation } \Sigma \text{ of } P, \quad \operatorname{vert}(\Sigma) \subset P \cap \mathbb{Z}^d \\ \text{(ii) term (pre)order on } K[X_x], \quad \operatorname{ini}(I_P) \text{ monomial ideal} \end{cases}$$

Theorem 3.3.

- $($*Stanley-Reisner ideal of* $\Sigma) = \operatorname{Rad}(\operatorname{ini}(I_P))$
- Σ *is unimodular* $\Longleftrightarrow \operatorname{ini}(I_P)$ *squarefree*

Multiples of polytopes



For $c \to \infty$ ($c \in \mathbb{N}$) the lattice points $cP \cap \mathbb{Z}^d$ approximate the continuous structure of $cP \sim P$ better and better.

Algebraic results:

Theorem 3.4.

- cP *integrally closed for* $c \geq \dim P - 1$. *Thus* $K[cP]$ *normal for* $c \geq \dim P - 1$.
- I_{cP} *has an initial ideal generated by degree* 2 *monomials for* $c \geq \dim P$. *Thus* $K[cP]$ *is Koszul for* $c \geq \dim P$.

Proof of Koszul property uses technique of Eisenbud-Reeves-Totaro.

Questions:

(i) Does cP have UC for $c \geq \dim P - 1$?

(ii) Does cP have a regular unimodular triangulation of degree 2 for $c \geq \dim P$?

Positive answers: (i) $\dim P \leq 3$, (ii) $\dim P \leq 2$.

No algebraic obstructions !

Positive rational cones and Hilbert bases

C generated by finitely many $v \in \mathbb{Z}^d$, $x, -x \in C \Rightarrow x = 0$.

Gordan's lemma: $C \cap \mathbb{Z}^d$ is a finitely generated monoid.

Its irreducible element form the Hilbert basis $\mathrm{Hilb}(C)$ of C.

C is simplicial $\iff C$ generated by linearly independent vectors $v_1, \ldots, v_d$

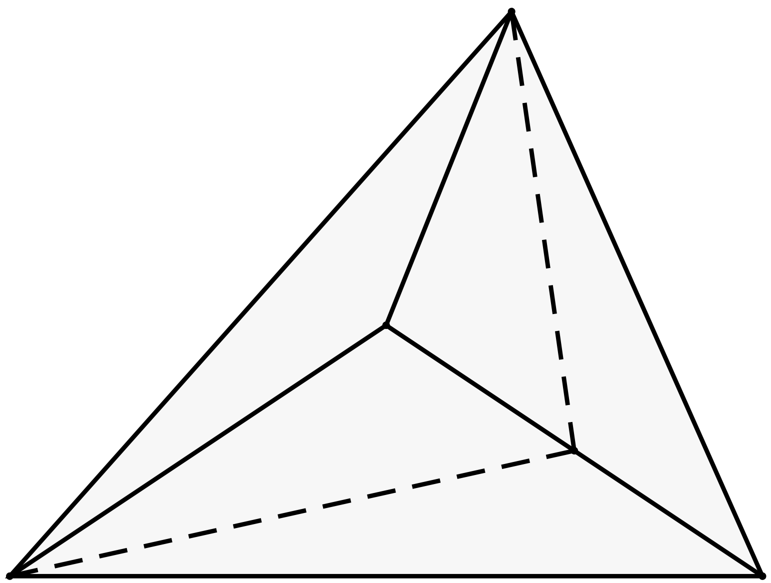
$$\mu(C) = [\mathbb{Z}^d : \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_d]$$

C is unimodular $\iff C$ generated by a $\mathbb{Z}$-basis of $\mathbb{Z}^d$

$\iff \mu(C) = 1$

Theorem 3.5. *C has a triangulation into unimodular subcones.*

Proof: Start with arbitrary triangulation. Refine by iterated stellar subdivision to reduce multiplicities.



But: P has a unimodular triangulation $\Rightarrow C_P$ satisfies UHT.

UHT: C has a Unimodular Triangulation into cones generated by subsets of $\mathrm{Hilb}(C)$.

UHC: C is Covered by its Unimodular subcones generated by subsets of $\mathrm{Hilb}(C)$.

A condition with a more algebraic flavour:

ICP: (Integral Carathéodory Property) for every $x \in C \cap \mathbb{Z}^d$ there exist $y_1, \dots, y_d \in \mathrm{Hilb}(C)$ with $x \in \mathbb{Z}_+ y_1 + \cdots + \mathbb{Z}_+ y_d$.

Dimension 3

Cones of dimension 3:

Theorem 3.6 (Sebő). $\dim C = 3 \Rightarrow C$ *has UHT*

If $C = C_P$, $\dim P = 2$, this is easy since P has UT. General case is somewhat tricky.

Polytopes of dimension 3:

First triangulate P into empty simplices and then use classification of empty simplices (White):

$$\Delta_{pq} = \operatorname{conv}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & 1 \end{pmatrix}, \qquad 0 \le q < p,\ \gcd(p,q) = 1$$

$$\mu(\Delta_{pq}) = p$$

No classification known in dimension ≥ 4. Essential difference to dimension 3: lattice width of Δ may be > 1.

Lagarias & Ziegler , Kantor & Sarkaria:

Proposition 3.7. *cP has UC for $c \geq 2$.*

Theorem 3.8.

- $2\Delta_{pq}$ *has UT* $\iff q = 1$ *or* $q = p - 1$.
- $4P$ *has UT for all* P.
- $c\Delta_{pq}$ *has UT for* $c \geq 4$.

Question: What about $3\Delta_{pq}$?

Counterexamples

$P = 2\Delta_{53}$ integrally closed 3 polytope without UT $\Rightarrow C_P$ has dimension 4 and violates UHT (first counterexample by Bouvier & Gonzalez-Sprinberg)

C_6 with Hilbert basis $z_1, \dots, z_{10}$, is of form C_{P_5}, $\dim P_5 = 5$, P_5 integrally closed, and violates UHC and ICP (B & G & Henk, Martin, Weismantel)

$$
\begin{aligned}
z_1 &= (0,1,0,0,0,0), & z_6 &= (1,0,2,1,1,2),\\
z_2 &= (0,0,1,0,0,0), & z_7 &= (1,2,0,2,1,1),\\
z_3 &= (0,0,0,1,0,0), & z_8 &= (1,1,2,0,2,1),\\
z_4 &= (0,0,0,0,1,0), & z_9 &= (1,1,1,2,0,2),\\
z_5 &= (0,0,0,0,0,1), & z_{10} &= (1,2,1,1,2,0).
\end{aligned}
$$

$\Rightarrow P_5$ violates UC

There exists a polytope of dimension 10 with UT, but without a regular unimodular triangulation (Hibi & Ohsugi)

Questions:

- Do all integrally closed polytopes P of dimensions 3 and 4 have UC ?
- Do all cones C of dimensions 4 and 5 have UHC ?
- Does there exist C with ICP, but violating UHC ?

Triangulating cP

Theorem 3.9 (Knudsen & Mumford, Toroidal embeddings). *Let P be a lattice d-polytope. Then cP has a regular unimodular triangulation for a some $c \in \mathbb{Z}_+$, $c > 0$.*

Not so hard: UC of d-simplices with non-overlapping interiors

Harder: UT

Most difficult: regularity

Questions: Does cP have UT for $c \gg 0$? Can we bound c uniformly in terms of dimension ? Is $c \geq \dim P$ enough ?

Covering cP

Theorem 3.10. *Let P be a d-polytope. Then there exists $\mathfrak{c}_d$ such that cP has UC for all $c \geq \mathfrak{c}_d^{\mathrm{pol}}$, and*

$$\mathfrak{c}_d^{\mathrm{pol}} = O\left(d^{16.5}\right)\left(\frac{9}{4}\right)^{(\operatorname{ld}\gamma(d))^2}, \qquad \gamma(d) = (d-1)\lceil\sqrt{d-1}\rceil.$$

For the proof one needs a similar theorem about cones—cones allow induction on d.

Theorem 3.11. *Let C be a rational simplicial d-cone and Δ_C the simplex spanned by O and the extreme integral generators. Then*

(a) (M. v. Thaden) *C has a triangulation into unimodular simplicial cones D_i such that $\operatorname{Hilb}(D_i) \subset c\Delta_C$ for some*

$$c \leq \frac{d^2}{4}(\mu(C))^7\left(\frac{9}{4}\right)^{(\operatorname{ld}(\mu(C)))^2}.$$

(b) C has a cover by unimodular simplicial cones D_i such that $\operatorname{Hilb}(D_i) \subset c\Delta_C$ for some

$$c \leq \frac{d^2}{4}(d+1)\big(\gamma(d)\big)^8\left(\frac{9}{4}\right)^{(\operatorname{ld}(\gamma(d)))^2}.$$

Sketch of proof of Theorem 3.11:

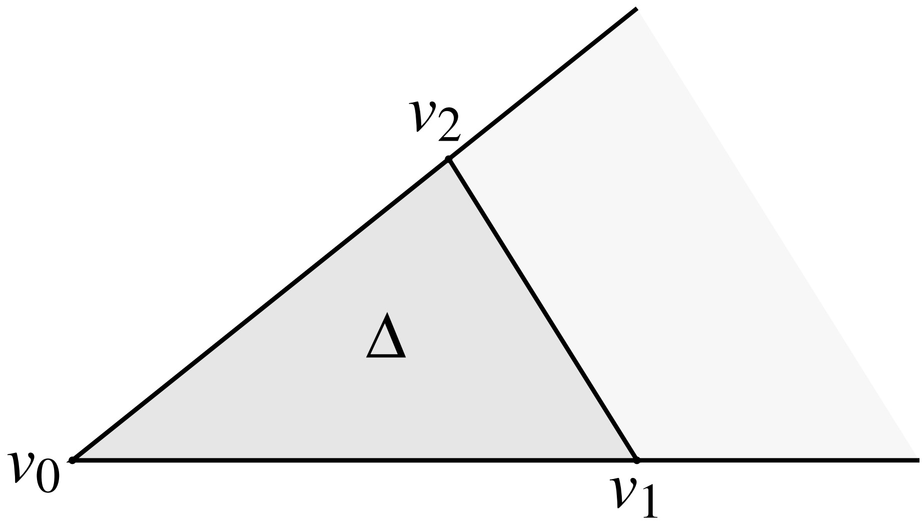
(i) $d-1 \rightarrow d$: we can cover the “corners”] of C with unimodular subcones.

(ii) Extend the corner covers far enough into C. To have enough room, we must go “further up” in the cone. We loose unimodularity, but the multiplicity remains under control: $\leq \gamma(d) = \lceil \sqrt{d-1} \rceil (d-1)$

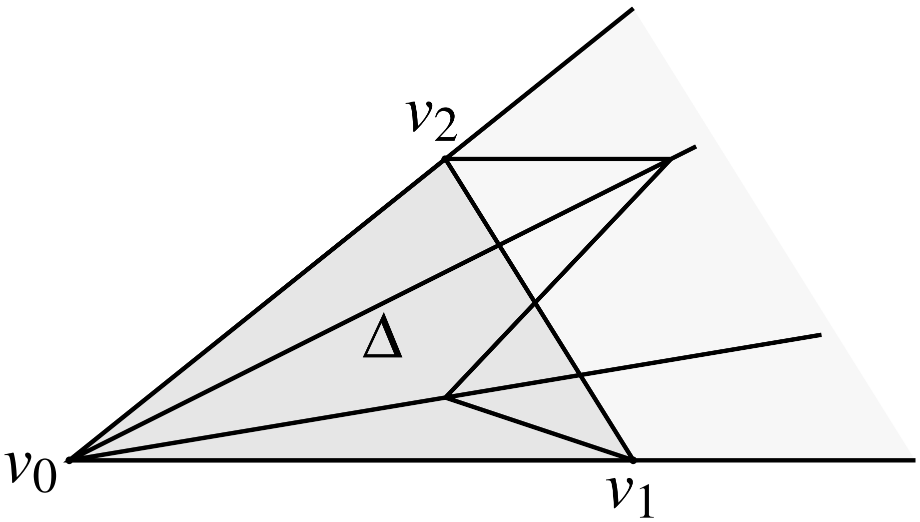
(iii) Apply part (a) of theorem to restore unimodularity.

(iv) Part (a): Control the “lengths” of the vectors in iterated stellar subdivision.

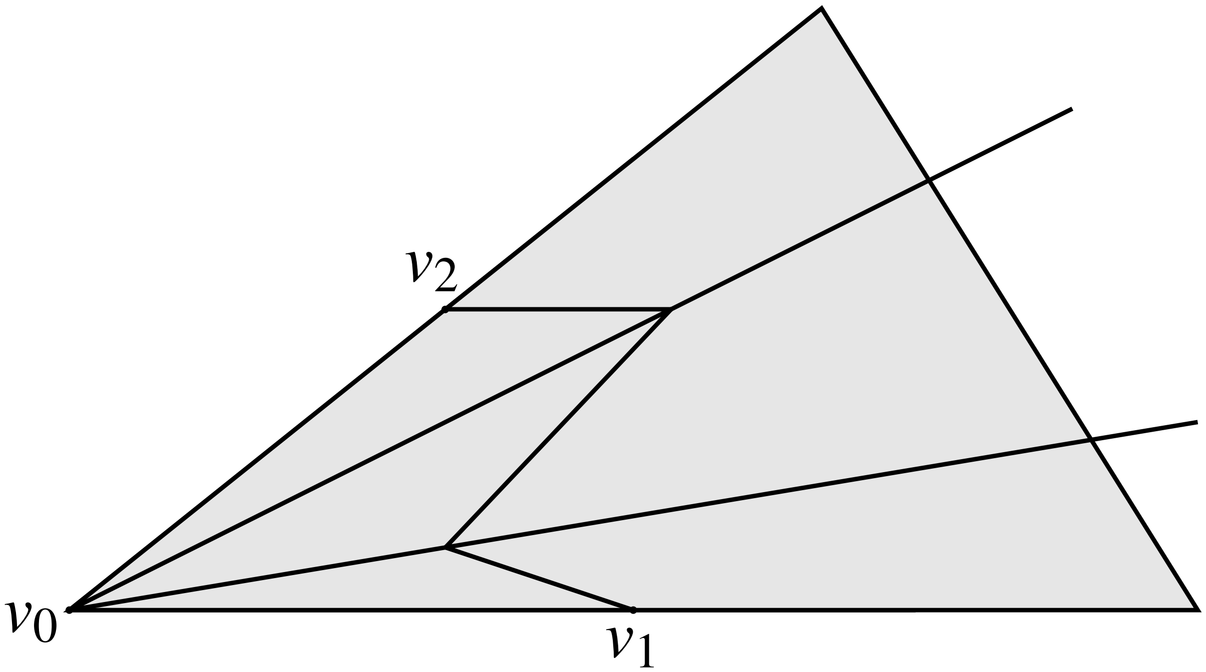
Sketch of proof of Theorem 3.10: May assume $P = \Delta$ is an (empty) simplex. Consider corner cones of Δ:



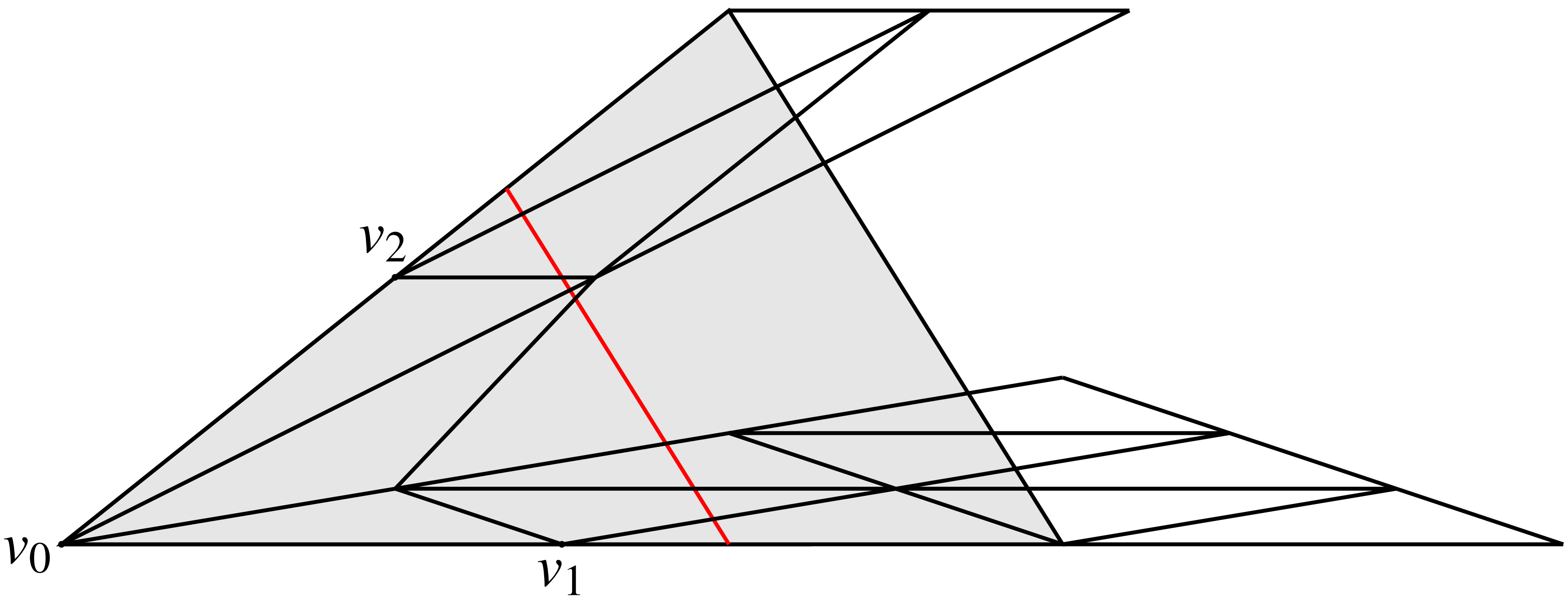
Apply Theorem 3.11 to corner cone:



Must multiply P by factor c' from Theorem 3.11 to get basic unimodular corner simplices into P



Tile corner cones:



Must multiply $c'P$ by $c'' \approx d\sqrt{d}$ to get the tiling by unimodular corner simplices close enough ($=$ beyond red line) to facet opposite of v_0.

Lecture 4

From vector spaces to polytopal algebras

Every so often you should try a damn-fool experiment —

from J. Littlewood's A MATHEMATICIAN'S MISCELLANY

The category $\mathrm{Pol}(K)$

Recall from Lecture 3 that a lattice polytope P is the convex hull of finitely many points $x_i \in \mathbb{Z}^n$.

M_P submonoid of $\mathbb{Z}^{n+1}$ generated by $(x,1)$, $x \in P \cap \mathbb{Z}^n$.

For a field K we let $\mathrm{Pol}(K)$ be the category

- with objects the graded algebras $K[P] = K[M_P]$
- with morphisms the graded K-algebra homomorphisms

Main question: To what extent is $\mathrm{Pol}(K)$ determined by combinatorial data ?

$\mathrm{Pol}(K)$ generalizes $\mathrm{Vect}(K)$, the category of finite-dimensional K-vector spaces:

Δ_n n-dimensional unit simplex

$$\Rightarrow K[\Delta_n] = K[X_1, \dots, X_{n+1}]$$

$$\begin{aligned}\mathrm{Hom}_K(K^m, K^m) &\leftrightarrow \mathrm{gr.hom}_K(S(K^m), S(K^n))\\ &\leftrightarrow \mathrm{gr.hom}_K(K[X_1, \dots, X_m], K[X_1, \dots, X_n])\\ &\leftrightarrow \mathrm{gr.hom}_K(K[\Delta_{m-1}], K[\Delta_{n-1}]))\end{aligned}$$

What properties of $\mathrm{Vect}(K)$ can be passed on $\mathrm{Pol}(K)$?

Note: $\mathrm{Pol}(K)$ not abelian

Why not graded affine monoid algebras $K[M]$ in full generality?

Proposition 4.1. *Let P, Q be lattice polytopes. Then the K-algebra homomorphisms $K[P] \to K[Q]$ correspond bijectively to K-algebra homomorphisms $\overline{K[P]} \to \overline{K[Q]}$ of the normalizations.*

In fact, $K[P]$ equals $\overline{K[P]}$ in degree 1.

In the following the base field K is often replaced by a general commutative base ring R.

Toric automorphisms and symmetries

Elementary fact of linear algebra: $\mathrm{GL}_n(K)$ is generated by matrices of 3 types:

- diagonal matrices
- permutation matrices
- elementary transformations

Actually, the permutation matrices are not needed. But their analogues in the general case cannot always be omitted.

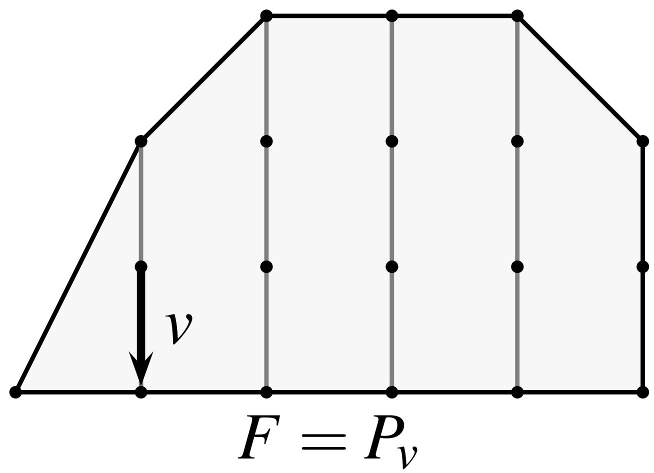
It easy to generalize diagonal matrices and permutation matrices:

- the diagonal matrices correspond to $(\lambda_1, \dots \lambda_{n+1}) \in \mathbb{T}_{n+1} = (K^*)^{n+1}$ acting on $K[P] \subset K[X_1, \dots, X_{n+1}]$ via the substitution $X_i \mapsto \lambda_i X_i$,
- the permutation matrices represent symmetries of Δ_{n-1} and correspond to the elements of the (affine!) symmetry group $\Sigma(P)$ of P.

How can we generalize elementary transformations?

Column structures

A column structure arranges the lattice points in P in columns:



More formally: $v \in \mathbb{Z}^n$ is a column vector if their exists a facet F, the base facet $P_v = F$ of v, such that

$$x + v \in P \qquad \text{for all } x \in P \setminus F.$$

A column vector $v \in \mathbb{Z}^n$ is to be identified with $(v, 0) \in \mathbb{Z}^{n+1}$.

Elementary automorphisms

To each facet F of P there corresponds a facet of the cone $\mathbb{R}_+ M_P$, also denoted by F.

Recall the support form σ_F. For $F = P_v$ set $\sigma_v = \sigma_F$.

Define a map from M_P to $R[\mathbb{Z}^{n+1}]$ by

$$e_v^\lambda : x \mapsto (1 + \lambda v)^{\sigma_v x}.$$

σ_v $\mathbb{Z}$-linear and v column vector $\Rightarrow e_v^\lambda$ homomorphism from M_P into $(R[M_P], \cdot)$

$\Rightarrow e_v^\lambda$ extends to an endomorphism of $R[M_P]$

Since $e_v^{-\lambda}$ is its inverse, e_v^λ is an automorphism.

Proposition 4.2. *$v_1, \ldots, v_s$ pairwise different column vectors for P with the same base facet $F = P_{v_i}$. Then*

$$\varphi : (R,+)^s \to \mathrm{gr.aut}_R(R[P]), \qquad (\lambda_1, \ldots, \lambda_s) \mapsto e_{v_1}^{\lambda_1} \circ \cdots \circ e_{v_s}^{\lambda_s},$$

is an embedding of groups.

$e_{v_i}^{\lambda_i}$ and $e_{v_j}^{\lambda_j}$ commute and the inverse of $e_{v_i}^{\lambda_i}$ is $e_{v_i}^{-\lambda_i}$.

R field $\Rightarrow$ φ is homomorphism of algebraic groups.

$\Rightarrow$ subgroup $\mathbb{A}(F)$ of $\mathrm{gr.aut}_R(R[P])$ generated by e_v^{λ} with $F = P_v$ is an affine space over R

$\mathrm{Col}(P)$ = set of column vectors of P.

The polytopal linear group

Theorem 4.3. *Let K a field.*

- *Every $\gamma \in \operatorname{gr.aut}_K(K[P])$ has a presentation*

$$\gamma = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_r \circ \tau \circ \sigma,$$

  $\sigma \in \Sigma(P)$, $\tau \in \mathbb{T}_{n+1}$, *and* $\alpha_i \in \mathbb{A}(F_i)$.
- $\mathbb{A}(F_i)$ *and* $\mathbb{T}_{n+1}$ *generate conn. comp. of unity* $\operatorname{gr.aut}_K(K[P])^0$.
- $= \{\gamma \in \operatorname{gr.aut}_K(K[P])$ *inducing* id *on div. class group of* $\overline{K[P]}\}$.
- $\dim \operatorname{gr.aut}_K(K[P]) = \#\operatorname{Col}(P) + n + 1$.
- $\mathbb{T}_{n+1}$ *is a maximal torus of* $\operatorname{gr.aut}_K(K[P])$.

The proof uses in a crucial way that every divisorial ideal of $K[M_P]$ is isomorphic to a monomial ideal.

This fact allows a polytopal Gaussian algorithm.

Using elementary automorphisms it corrects an arbitrary γ to an automorphism δ such that $\delta(\operatorname{int}(K[P])) = \operatorname{int}(K[P])$.

Lemma 4.4. $\delta(\operatorname{int}(K[P])) = \operatorname{int}(K[P]) \Rightarrow \delta = \tau \circ \sigma$, $\tau \in \mathbb{T}_{n+1}$, $\sigma \in \Sigma(P)$

Important fact: the divisor class group of $\overline{K[P]}\}$ is a discrete object.

To some extent one can also classify retractions of $K[P]$.

Milnor's classical K_2

Its construction is based on

- the passage to the "stable" group of elementary automorphisms
- the Steinerg relations

Construction of the stable group: $E_n(R)$ subgroup generated by of elementary matrices,

$$E \in E_n(R) \mapsto \begin{pmatrix} E & 0 \\ 0 & 1 \end{pmatrix} \in E_{n+1}(R)$$

$$\mathbb{E}(R) = \varinjlim E_n(R).$$

The Steinberg relations for elementary matrices:

$$e_{ij}^{\lambda} e_{ij}^{\mu} = e_{ij}^{\lambda+\mu}$$

$$[e_{ij}, e_{jk}^{\mu}] = e_{ik}^{\lambda\mu}, \qquad i \neq k$$

$$[e_{ij}, e_{ki}^{\mu}] = e_{kj}^{-\lambda\mu} \qquad j \neq k$$

$$[e_{ij}^{\lambda}, e_{kl}^{\mu}] = 1 \qquad i \neq l, j \neq k$$

The stable Steinberg group of K is defined by

- generators x_{ij}^{λ}, $i, j \in \mathbb{N}$, $i \neq j$, $\lambda \in K$ representing the elementary matrices
- the (formal) Steinberg relations $x_{ij}^{\lambda} x_{ij}^{\mu} = x_{ij}^{\lambda+\mu}$, $[x_{ij}, x_{jk}^{\mu}] = x_{ik}^{\lambda\mu}$ etc.

Set

$$K_2(R) = \mathrm{Ker}\big(\mathbb{S}\mathrm{t}(R) \to \mathbb{E}(R)\big), \qquad x_{ij}^{\lambda} \mapsto e_{ij}^{\lambda}.$$

Milnor's theorem:

Theorem 4.5. *The exact sequence*

$$1 \to K_2(R) \to \mathbb{S}\mathrm{t}(R) \to \mathbb{E}(R) \to 1$$

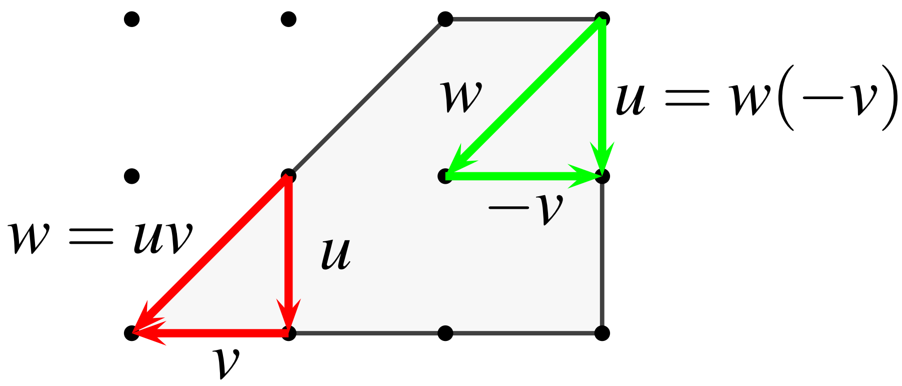
is a universal central extension and $K_2(R)$ is the center of $\mathbb{S}\mathrm{t}(R)$.

Products of column vectors

Let $u, v, w \in \mathrm{Col}(P)$. We say that

$$\color{red}{uv = w} \quad \Longleftrightarrow \quad w = u + v \quad \text{and} \quad \mathsf{P}_w = P_u$$

Examples of products of column vectors:

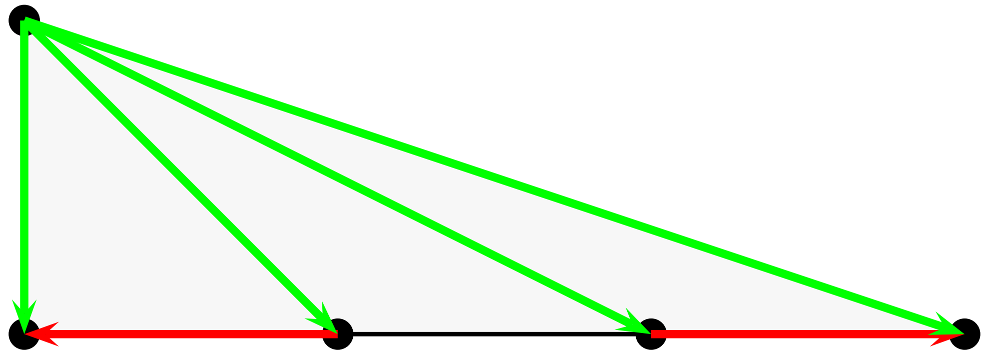


$\Rightarrow$ partial, non-commutative product structure on $\mathrm{Col}(P)$.

Balanced polytopes

A polytope is balanced if

$$\sigma_F(v) \leq 1 \quad \text{for all } v \in \mathrm{Col}(P),\ F = P_w.$$



A nonbalanced polytope

Polytopal Steinberg relations

Proposition 4.6. *P balanced, $u, v \in \mathrm{Col}(P)$, $u + v \neq 0$, $\lambda, \mu \in R$. Then*

$$e_v^{\lambda} e_v^{\mu} = e_v^{l+\mu}$$

$$[e_u^{\lambda}, e_v^{\mu}] = \begin{cases} e_{uv}^{-\lambda\mu} & \text{if } uv \text{ exists,} \\ e_{vu}^{\mu\lambda} & \text{if } vu \text{ exists,} \\ 1 & \text{if } u + v \notin \mathrm{Col}(P). \end{cases}$$

Note: we know nothing about $[e_u^{\lambda}, e_{-u}^{\mu}]$ if $u, -u \in \mathrm{Col}(P)$!

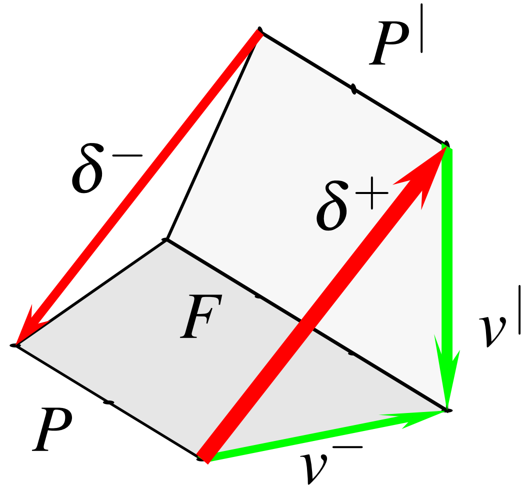
Doubling along a facet

Let $F = P_v$ be a facet of P and choose coordinates in $\mathbb{R}^n$ such that $\mathbb{R}^{n-1}$ is the affine hyperplane spanned by F.

$$P^{-} = \{(x', x_n, 0) : (x', x_n) \in P\}$$

$$P^{|} = \{(x', 0, x_n) : (x', x_n) \in P\}$$

$$P^{\lrcorner_F} = \operatorname{conv}(P, P^{|}) \subset \mathbb{R}^{n+1}.$$



$$v = v^{-} = \delta^{+} v^{|}$$

$$v^{|} = \delta^{-} v^{-}$$

Crucial facts:

$$\mathrm{Col}(P) \hookrightarrow \mathrm{Col}(P^{\lrcorner F})$$

$$G \mapsto \mathrm{conv}(G^{-}, G^{|}), \quad G \neq F$$

$$F \mapsto P^{|}$$

$$P^{-} = \text{new facet}$$

Lemma 4.7. P *balanced* $\Rightarrow P^{\lrcorner F}$ *balanced and*

$$\mathrm{Col}(P^{\lrcorner F}) = \mathrm{Col}(P)^{-} \cup \mathrm{Col}(P)^{|} \cup \{\delta^{+}, \delta^{-}\}.$$

Doubling spectra

The chain of lattice polytopes $\mathfrak{P} = (P = P_0 \subset P_1 \subset \dots)$ is called a doubling spectrum if

- for every $i \in \mathbb{Z}_+$ there exists a column vector $v \subset \mathrm{Col}(P_i)$ such that $P_{i+1} = P_i^{\lrcorner v}$,
- for every $i \in \mathbb{Z}_+$ and any $v \in \mathrm{Col}(P_i)$ there is an index $j \geq i$ such that $P_{j+1} = P_j^{\lrcorner v}$.

Associated to $\mathfrak{P}$ are the 'infinite polytopal' algebra

$$R[\mathfrak{P}] = \lim_{i \to \infty} R[P_i]$$

and the filtered union

$$\mathrm{Col}(\mathfrak{P}) = \lim_{i \to \infty} \mathrm{Col}(P_i).$$

Now we can define a stable elementary group:

$$\mathbb{E}(R,P) = \text{subgroup of } \operatorname{gr.aut}_R(R[\mathfrak{P}]) \text{ generated by } e_v^\lambda$$
$$v \in \operatorname{Col}(\mathfrak{P}),\ \lambda \in R.$$

Note: it depends only on P, not on the doubling spectrum.

Theorem 4.8. $\mathbb{E}(R,P)$ *is a perfect group with trivial center.*

Polytopal Steinberg groups

The group $\mathbb{S}\mathrm{t}(R,P)$ is defined by

- generators x_v^λ, $v \in \mathrm{Col}(fP)$, $\lambda \in R$ representing the elementary automorphisms e_v^λ
- the (formal) Steinberg relations between the x_v^λ

It depends only on the partial product structure on $\mathrm{Col}(P)$. This allows some functoriality in P.

Polytopal K_2

In analogy with Milnor's theorem we have

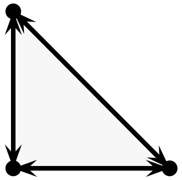
Theorem 4.9. *P balanced polytope $\Rightarrow \mathbb{S}\mathrm{t}(R,P) \to \mathbb{E}(R,P)$ is a universal central extension with kernel equal to the center of $\mathbb{S}\mathrm{t}(R,P)$.*

Definition 4.10.

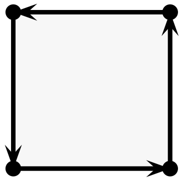
$$K_2(R,P) = \mathrm{Ker}\big(\mathbb{S}\mathrm{t}(R,P) \to \mathbb{E}(R,P)\big).$$

Balanced polygons

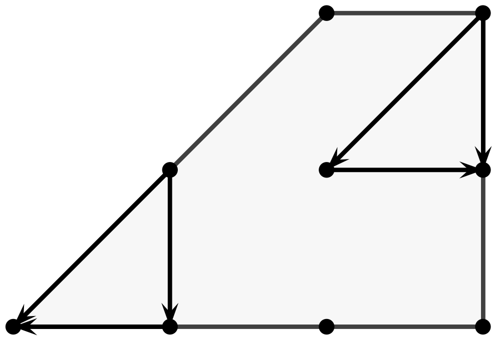
It is not difficult to classify the balanced polygons = 2-dimensional polytopes: ($K_2 =$ classical K_2)



$\{\pm u, \pm v, \pm w\}$ K_2



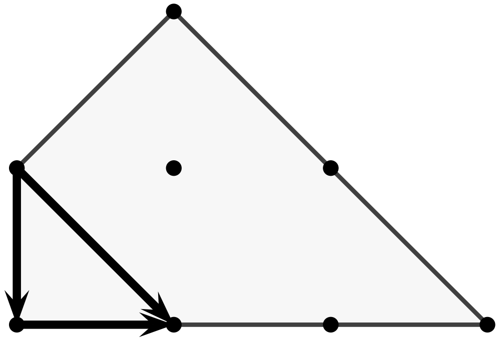
$\{\pm u, \pm v\}$ $K_2 \oplus K_2$



$\{u, \pm v, w\}$

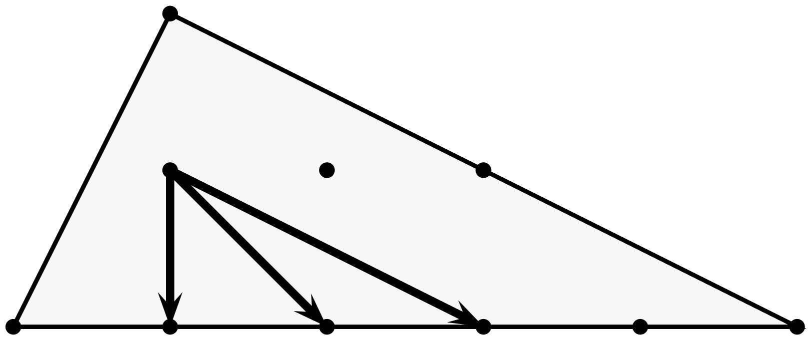
$w = uv$

$u = w(-v)$ $K_2 \oplus K_2$

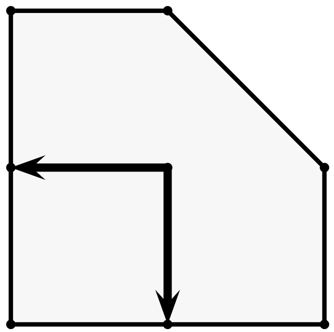


$\{u, v, w\}$

$w = uv$ $\qquad$ $K_2 \oplus K_2$



$\{v : P_v = F\}$ $\qquad$ K_2



$\{u, v\}$ $\qquad$ $K_2 \oplus K_2$

Higher K-groups

Using Quillen's $+$-construction or Volodin's construction one can define higher K-groups.

For certain well-behaved polytopes both constructions yield the same result (in the classical case proved by Suslin).

Potentially difficult polytope: